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Dagger closure

vorgelegt
von

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Contents

Introduction

L'algèbre n'est qu'une géométrie écrite;
la géométrie n'est qu'une algèbre figurée.

Sophie Germain

The main objects of study of this thesis are ideal closure operations. More specifically, we will mostly be concerned with tight closure, dagger closure and solid closure. We will prove that solid closure coincides in graded dimension two with a graded version of dagger closure in equal characteristic zero. We will also show that (graded) dagger closure agrees with tight closure in all dimensions in positive characteristic under some mild finiteness conditions.

There is no established axiomatic framework for ideal closure operations but some basic properties that they all satisfy are the following axioms:

An ideal closure operation is a mapping $^* : \mathcal{I} \to \mathcal{I}$, where $\mathcal{I}$ is the set of ideals of a commutative ring R, such that the following axioms are satisfied:

(i) $I^* = I^{**}$ for all $I \in \mathcal{I}$.

(ii) If $I, J \in \mathcal{I}$ such that $I \subseteq J$ then $I^* \subseteq J^*$.

(iii) For all $I \in \mathcal{I}$ one has $I \subseteq I^*$.

One can easily extend these axioms to submodules of a given ambient module. Two classical examples of ideal closure operations are the radical

$$\mathrm{Rad}(I) = \{f \in R \,|\, f^n \in I \text{ for some } n > 0\}$$

and the integral closure

$$\bar{I} = \{f \in R \,|\, f^n + r_1 f^{n-1} + \ldots + r_n = 0, \text{ where } r_j \in I^j\}$$

of an ideal I. One readily has the inclusion $\bar{I} \subseteq \mathrm{Rad}(I)$.

The most prominent non-classical closure operation is *tight closure*. This was invented by Hochster and Huneke more than twenty years ago. It exploits the Frobenius homomorphism and hence is a priori only defined for rings containing a field of positive characteristic. Specifically, the tight closure of an ideal I in a domain R of positive characteristic p is

$$I^* = \{f \in R | \text{there is } c \neq 0 \text{ such that } cf^q \in I^{[q]} \text{ for all } q = p^e\},$$

where $I^{[q]}$ is the image of I under the e-fold Frobenius.

However, via *reduction mod p* (see [44] and also [49] for an exhaustive treatment) one can extend tight closure to rings containing a field of characteristic zero. But this is a very technical business. The basic idea for this is that if a given problem is defined by finitely many data then one can consider the $\mathbb{Z}$-algebra generated by these data. Looking at the reduction modulo a prime number p one is then in a situation where one has a Frobenius. One then defines that an element is contained in the tight closure if and only if this holds for almost all prime reductions. There is however no known extension of tight closure to rings not containing a field. We will not deal with tight closure in equal characteristic zero in this thesis.

Tight closure is always contained in integral closure but is much smaller in general. The theory of tight closure has simplified many proofs of known theorems and also generalised the results in many cases. For example, the Briançon-Skoda theorem and homological conjectures come to mind. Ideal closure operations, especially tight closure, also have ties to the study of singularities of varieties. The heuristic is that since every ideal in a regular ring is tightly closed, tight closure measures to a certain extent the deviation of a ring from being regular. But in reality the connection is more subtle. For instance, rings where every ideal I generated by $\operatorname{ht} I$ elements is tightly closed are called *F-rational* and are a characteristic p analogue of rational singularities (see e. g. [72] for this line of thought and other characteristic p methods in algebraic geometry). In this direction, one should also mention that test ideals (which are objects that naturally arise in tight closure theory) provide a characteristic p analogue for multiplier ideals. Both rational singularities and multiplier ideals in characteristic zero are defined via a resolution of singularities.

Another close relative of tight closure theory is the *Hilbert-Kunz multiplicity* of a primary ideal which was invented by Kunz with classification of singularities in mind. Although, it is nowadays an object of study quite in its own right. Hilbert-Kunz multiplicity is related to tight closure in the same way as Hilbert-Samuel multiplicity is related to integral closure.

We refer to [20] for a survey article about tight closure and to [21, Paragraph 10] or [50] for a more detailed treatment.[1]

Thus, there are already several good reasons for studying closure operations apart from natural curiosity. Perhaps the most intriguing reason though is that a sufficiently well-behaved closure operation would settle the long standing homological conjectures in mixed characteristic (that is, for local rings not containing a field). One of these conjectures is the direct summand conjecture:

Conjecture. Let R be a noetherian regular local ring. If S is a finite R-algebra then R is a direct summand of S as an R-module. That is, there is an R-module M such that $S \cong M \oplus R$.

This is known to be true for local rings containing a field (see [45] or [21, Theorem 9.2.3]) but is open in the mixed characteristic case in dimension ≥ 4. Having an ideal closure operation which is sufficiently well-behaved would be enough to imply the direct summand conjecture. More precisely, let R be a complete local domain and assume that there exists an ideal closure operation * which has the following two properties:

[1] There is a survey article addressing the many recent developments in this theory. It has not yet been published but is available on the math archive – see [14].

(i) Let $R \subseteq S$ be a finite extension of rings. If I is an ideal in R then $(IS) \cap R \subseteq I^*$.

(ii) If R is also regular then $I^* = I$ for every ideal I in R.

Then the direct summand conjecture holds. This also spotlights the problem of finding a closure operation in all characteristics (or in mixed characteristic) that has tight closure-like properties.

The latest progress in this direction was made by Heitmann in [43] where he considered several ideal closure operations and, in particular, settled the direct summand conjecture in dimension 3. It should be noted that the closure operations he used are quite similar to dagger closure (which will be one of the main objects of interest for this thesis). Before giving the definition of one of the closure operations considered by Heitmann we recall that a *rank one valuation* is a non-trivial valuation whose value group is contained in the real numbers. A valuation of rank zero is the trivial valuation and the rank of any valuation is non-negative. Moreover, we denote by R^+ the integral closure of R in an algebraic closure of $Q(R)$. That is, R^+ is the limit of all finite extension domains of R.

Definition. Let R be a domain and I an ideal in R. Then the *full rank one closure* of I is given by the set of elements $x \in R$ such that for every valuation ν on R^+ of rank at most one, every prime number p and every positive integer n and every $\varepsilon > 0$ there exists $d \in R^+$ with $\nu(d) < \varepsilon$ such that $dx \in (I, p^n)R^+$.

We will prove that this closure operation coincides with tight closure in characteristic $p > 0$ if R is a domain essentially of finite type over an excellent local ring (see Proposition 9.7). We refer to Section 9 for further discussion of this closure operation (in particular for the peculiar inclusion of the trivial valuation) and to [42].

Dagger closure was introduced by Hochster and Huneke in 1991 in [47] as an alternative characterisation of tight closure in complete local domains of positive characteristic. Its definition is characteristic free and therefore immediately yields a closure operation also in characteristic zero. This is also motivated by the problem of finding a tight closure-like closure operation in characteristic zero that does not use reduction to positive characteristic. The main goal of this thesis is to show that two closure operations, namely *graded dagger closure* and *solid closure*, coincide in dimension two for $\mathbb{N}$-graded domains finitely generated over a field. This is not too difficult in positive characteristic and in this case holds in any dimension. The hard part is to prove the equivalence in characteristic zero.

The idea for defining dagger closure is that in positive characteristic one already has an interesting notion of a closure operation if one defines

$$I^+ = \{f \in R \,|\, f \in IS \text{ for some finite extension domain } S \supseteq R\}$$

for an ideal I in a domain R. This does not yield an interesting closure operation if R contains a field of characteristic zero (if R is normal then I^+ is trivial due to the splitting provided by field trace). Also note that this is the smallest closure that has any chance of being applicable to prove the direct summand conjecture.

Dagger closure is now defined as the set of elements which are multiplied into the extended ideal in finite extension domains by a sequence of *arbitrarily small* elements. A priori, it is not clear what arbitrarily small means. One way to make sense of this is to attach a ($\mathbb{Q}$-valued) valuation to R^+. In particular, for plus closure the unit element 1 multiplies f into the extended ideal and this is certainly small enough.

But in general there is no canonical choice of a valuation. In the complete local domain case any two valuations that yield an interesting closure operation in this way are equivalent. In the graded case the canonical choice is the valuation induced by the grading which assigns to an element the degree (in $\mathbb{Q}_{\geq 0}$) of the lowest non-zero homogeneous term (zero will have value infinity). The definition we will work with most of the time is the following

Definition. Let R be an $\mathbb{N}$-graded domain and I an ideal and ν the valuation induced by grading. Then f belongs to the graded dagger closure of I, written $f \in I^{\dagger \mathrm{GR}}$, if for every $\varepsilon > 0$ there exists a finite $\mathbb{Q}_{\geq 0}$-graded extension domain S containing an element a such that $\nu(a) < \varepsilon$ and $af \in IS$.

Surprisingly little is known about dagger closure. There has however been some recent interest due to Heitmann's proof of the direct summand conjecture in dimension 3 in [43]. Specifically, Roberts, Singh and Srinivas studied dagger closure to some extent in [66], although their main focus in this paper is the notion of *almost Cohen-Macaulay*.

Solid closure was introduced by Hochster in 1994 in [46] and was originally thought of as a possible replacement for tight closure in characteristic zero (it agrees with tight closure in positive characteristic under some mild finiteness conditions). But due to an example by Roberts it is known that solid closure is "too big" in dimension greater than two. That is, there are regular rings of dimension ≥ 3, where the solid closure of an ideal is strictly larger than the ideal itself. There is however a refinement of solid closure, called *parasolid closure*, by Brenner (see [4]) which has all the right properties if the ring contains a field.

Definition. Let R be a noetherian ring and let $f_1, \ldots, f_n, f_0 \in R$. Then f_0 belongs to the *solid closure* of $(f_1, \ldots, f_n)$, written $f_0 \in (f_1, \ldots, f_n)^\star$, if for every local complete domain $R' = \widehat{R_\mathfrak{m}}/\mathfrak{q}$ (where $\mathfrak{m}$ is a maximal ideal of R and $\mathfrak{q}$ is a minimal prime of $\widehat{R_\mathfrak{m}}$) we have $H^d_\mathfrak{m}(A') \neq 0$, where $d = \dim R'$ and $A' = R'[T_1, \ldots, T_n]/(f_1T_1 + \ldots + f_nT_n + f_0)$.

Once one has a definition of dagger closure that is valid in non-complete domains, it is not too hard to prove that dagger closure and tight closure coincide in positive characteristic using the result of Hochster and Huneke – see Corollary 9.12. But this is far from being true for characteristic zero and solid closure.

The main result of this thesis is the following theorem which appears as Theorem 8.13 in Section 8.

Theorem. Let R denote an $\mathbb{N}$-graded two-dimensional domain of finite type over a field R_0 and I a homogeneous ideal of R. Then $I^{\dagger \mathrm{GR}} = I^\star$.

Although, our main result is pure commutative algebra the techniques we employ will mostly be geometric. We will use the interpretation of solid closure (in dimension two) in terms of vector bundles over curves developed by Brenner. Of course, one might ask why one should focus on this geometric approach, especially since it is (at the moment) essentially limited to dimension two.

To begin with, the idea that dagger closure might coincide with solid closure in characteristic zero only arises through this geometric interpretation. And we will in fact see that they do not coincide in dimension ≥ 3 in characteristic zero (see Corollary 9.23). – This is due to the fact that (graded) dagger closure is not "too big" in any dimension provided that the ring contains a field.

Furthermore, it provides us with a variety[2] of tools from algebraic geometry such as vector bundles, intersection theory etc. And equally important it gives us geometric intuition.

Moreover, this geometric approach to tight closure has proved its potency in numerous ways. It yielded new insights into tight closure and improved existing theorems (notably inclusion bounds in all dimensions – see [8]) and it was used to settle some long standing open problems of tight closure theory (the localisation problem , whether plus closure is tight closure (see [18])). It also led to new results in Hilbert-Kunz theory (e. g. the rationality of the Hilbert-Kunz multiplicity in graded dimension two (see [11])).

We briefly want to discuss the idea for the connection beetween solid closure and dagger closure. To fix ideas, let R be a standard graded two-dimensional normal domain over an algebraically closed field and $f_1, \ldots, f_n$ homogeneous elements generating an R_+-primary ideal. To keep technicalities to a minimum we also restrict ourselves to characteristic zero for this discussion.

Under these conditions one can associate to $f_1, \ldots, f_n$ a vector bundle over $Y = \operatorname{Proj} R$ – the *syzygy bundle* $\operatorname{Syz}(f_1, \ldots, f_n)$. Under the additional assumption that this vector bundle is *semistable*, containment of a given (homogeneous) element f_0 of degree d_0 in $(f_1, \ldots, f_n)^\star$ is equivalent to the non-ampleness of the divisor $\mathcal{O}_{\mathbb{P}(\mathcal{S})}(1)$ inside the projective bundle $\mathbb{P}(\mathcal{S})$, where $\mathcal{S} = \operatorname{Syz}(f_0, f_1, \ldots, f_n)(d_0)^\vee$. The first heuristic evidence that dagger closure should be connected to (non-)ampleness is provided by the following criterion due to Seshadri (see [37, Theorem I.7.1]):

Theorem. Let X be a complete scheme over an algebraically closed field k. A divisor D is ample if and only if there exists $\varepsilon > 0$ such that for all integral curves $C \subseteq X$ the inequality $\frac{D.C}{\operatorname{mult} C} \geq \varepsilon$ holds.

Here $\operatorname{mult} C$ is the maximum of the Hilbert-Samuel multiplicities of the stalks of C. If f is already contained in the extended ideal for some finite extension domain then one has a disjoint section, that is, a curve Y' finite over Y contained in $\mathbb{P}(\mathcal{S})$ but disjoint to $\mathcal{O}_{\mathbb{P}(\mathcal{S})}(1)$ – in particular, $\mathcal{O}_{\mathbb{P}(\mathcal{S})}(1).Y' = 0$. This illustrates that containment in dagger closure should be connected to existence of curves intersecting the divisor $\mathcal{O}_{\mathbb{P}(\mathcal{S})}(1)$ in a "small" way. This characterisation can be made precise as we shall see in Proposition 7.13 and Theorem 7.23.

It is worth noting that we will not "directly" prove that solid closure and graded dagger closure coincide in this setting. The strategy will be to establish that containment

[2] It is still an open problem whether it is quasi-compact.

in graded dagger closure is equivalent to the non ampleness of $\mathrm{Syz}(f_0, f_1, \dots, f_n)(d_0)^\vee$ under the additional assumption that $\mathrm{Syz}(f_1, \dots, f_n)$ is semistable. For the general case one looks at a certain filtration of $\mathrm{Syz}(f_1, \dots, f_n)$ with semistable quotients (the so-called *Harder-Narasimhan filtration* of $\mathcal{S}$) and establishes the same numerical criterion that characterises containment in solid closure. These reduction steps are carried out using geometric criteria that characterise these ideal closure operations.

We now give a short summary of the following sections. In Section 1 we define the closure operations of interest for this thesis and prove some basic facts. Section 2 will briefly introduce vector bundles. Then we will recall the notion of (strong) semistability for vector bundles on a smooth projective curve and collect the results of interest for us. The next section will deal with Brenner's interpretation of solid closure for primary ideals over graded rings in terms of syzygy bundles. Furthermore, we will give a rough outline of the ideas for proving the desired identity $I^\star = I^{\dagger\mathrm{GR}}$. Then in Section 4 we will prove that dagger closure is contained in solid closure for homogeneous primary ideals under the additional assumption that the syzygy bundle is (strongly) semistable. Section 5 will collect some necessary facts about section rings. We will then return to discussing some ideas in Section 6 and also prove one crucial step for the inclusion $I^\star \subseteq I^{\dagger\mathrm{GR}}$ in two special cases.

Beginning with Section 7 we will develop the notion of *almost zero* for vector bundles which will be crucial both for the other inclusion as well as for the reduction step along the (strong) Harder-Narasimhan filtration. In Section 8 we remove the condition of (strong) semistability and also relax the conditions on the ring. The main result of this thesis is Theorem 8.13 which asserts that solid closure and dagger closure of homogeneous ideals coincide for two-dimensional $\mathbb{N}$-graded domains finitely generated over a field R_0.

Next, we will discuss a possible definition of dagger closure in a (not necessarily graded) domain and prove some basic properties in Section 9. In particular, we will show that dagger closure is trivial on ideals in regular $\mathbb{N}$-graded rings of finite type over a field R_0 (see Corollary 9.23). Hence, showing that solid closure does not coincide with dagger closure in higher dimensions in characteristic zero. In particular, dagger closure is not "too big" in any dimension.

In Section 10 we will show an inclusion result for dagger closure for certain section rings of abelian varieties. Finally, in Section 11 we collect several open questions related to dagger closure or algebraic geometry that arose in the context of this thesis.

We note that all rings considered in this thesis will be commutative with identity.

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1 Graded dagger closure

In this section we introduce graded dagger closure and solid closure. We also discuss some other closure operations and prove basic facts about graded dagger closure. The main focus in the following sections will be the geometric study of graded dagger closure. We will continue studying graded dagger closure with algebraic methods in Section 9.

1.1 Graded dagger closure

1.1 Definition. Let R be a domain. The integral closure R^+ of R in an algebraic closure of its field of fractions $Q(R)$ is called the *absolute integral closure* of R.

This was first studied by M. Artin in [1]. In [47], Hochster and Huneke gave a characterisation of tight closure in complete local domains of characteristic $p > 0$ in terms of multipliers of small order with respect to a $\mathbb{Q}$-valued valuation ν on R^+. In this section we will recall some of their results and then propose a similar definition for $\mathbb{N}$-graded domains of arbitrary characteristic.

Before presenting a first definition of dagger closure we introduce the plus closure and the tight closure of an ideal.

1.2 Definition. Let R be a noetherian domain and I an ideal. We define the *plus closure* I^+ of I as the set of elements $f \in R$ such that $f \in IR^+$.

1.3 Definition. Let R be a ring of characteristic $p > 0$. The *tight closure* of an ideal I is given by

$$I^* = \{f \in R : \text{there exists } t \in R, t \notin \text{minimal prime}, tf^q \in I^{[q]} \text{ for almost all } q = p^e\},$$

where $I^{[q]}$ is the image of I under the e-fold iteration of the Frobenius $F : R \to R, x \mapsto x^p$ for $q = p^e$.

Note that if R is a domain then the condition on t just means that t is nonzero. Moreover, in this case we may replace "for almost all q" by "for all q". Indeed, assume that $tf^q \in I^{[q]}$ holds for all $e > e_0$ for some nonzero t (in particular, $I \neq 0$). Then we may fix nonzero elements $u_i \in I^{[p^i]}$ for $i \leq e_0$ and replace t by $u_1 \cdots u_{e_0} t$. If I is given by generators $f_1, \ldots, f_n$ then $I^{[q]}$ is generated by $f_1^q, \ldots, f_n^q$.

1.4 Remark. Plus closure only yields a useful closure operation in positive characteristic. More precisely, if R is a normal integral domain containing the rational numbers and I an ideal then $I^+ = I$. This can be seen using a field trace argument. First of all, note that $f \in I^+$ if and only if there is a finite extension domain S of R such that $f \in IS$. Denote the degree of $Q(R) \subseteq Q(S)$ by d. Then $\frac{1}{d}Tr : S \to R$ splits the inclusion $i : R \to S$.

It is also known that plus closure does not have all the desired properties in mixed characteristic in general (see [41]).

It is not difficult to show that $I^+ \subseteq I^*$ in positive characteristic (see e. g. [50, Theorem 1.7]). Brenner showed (see [12, Theorem 4.2]) that equality holds for homogeneous ideals if R is an $\mathbb{N}$-graded two-dimensional domain of finite type over k, where k is an algebraic closure of a finite field. He also proved that equality holds if R is a normal homogeneous coordinate ring of an elliptic curve and I is R_+-primary and homogeneous (see [9, Theorem 4.3]). Brenner and Monsky have shown that the inclusion is strict in general if k contains transcendental elements (see [18]).

1.5 Definition. Let R be a domain and Γ a totally ordered abelian group. A *valuation* on R is a mapping $\nu : Q(R) \to \Gamma \cup \{\infty\}$ such that

(i) $\nu(0) = \infty$.

(ii) $\nu : Q(R)^\times \to \Gamma$ is a group homomorphism.

(iii) For all $x, y \in Q(R)$: $\nu(x + y) \geq \min\{\nu(x), \nu(y)\}$, where $\infty \geq \gamma$ for all $\gamma \in \Gamma$.

We now come to a first definition of dagger closure due to Hochster and Huneke (see [47]).

1.6 Definition. Let $(R, \mathfrak{m})$ be a complete local domain. Fix a valuation ν on R with values in $\mathbb{Z}$ such that $\nu(R) \geq 0$ and $\nu(\mathfrak{m}) > 0$. Extend ν to R^+ so that it takes values in $\mathbb{Q}$. Then an element $x \in R$ belongs to the *dagger closure* $I^\dagger$ of an ideal $I \subseteq R$ if there exists a sequence of elements $a_n \in R^+$ such that $\nu(a_n) \to 0$ as $n \to \infty$ and $a_n x \in IR^+$ for all n.

1.7 Remark. There is always an extension for a given valuation ν on a domain R to a field $R \subseteq K$ (see [76, vol. 2, VI §4, Theorem 5'] and [ibid., vol. 2, VI §11]) (with possibly larger value group).

Note that ν in Definition 1.6 is non-negative on R^+. For if $x \in R^+$, we have an equation $x^d = \sum_{i=0}^{d-1} a_i x^i$ with $a_i \in R$. Hence $d\nu(x) \geq \min_i\{\nu(a_i) + i\nu(x)\} \geq i\nu(x)$ and therefore $(d - i)\nu(x) \geq 0$.

We also remark that in the situation of Definition 1.6 two valuations on R which are positive on $\mathfrak{m}$ and non-negative on R are equivalent by a theorem of Izumi (see [53]).

1.8 Remark. One can think of dagger closure as an "asymptotic" version of plus closure since for plus closure 1 multiplies the given element into the ideal and $\nu(1) = 0$.

The main motivation for introducing dagger closure is that it immediately gives a notion of an ideal closure operation in all characteristics.

In search of a good ideal closure operation in all characteristics a good start is that it should agree with tight closure in positive characteristic (despite tight closure not commuting with localisation).

1.9 Theorem. Let $(R, \mathfrak{m})$ be a complete local noetherian domain of characteristic $p > 0$. Then dagger closure coincides with tight closure.

Proof. See [47, Theorem 3.1]. Assume that R is a domain of characteristic $p > 0$ and $I \subseteq R$ an ideal and ν is any valuation on R. For the inclusion $I^* \subseteq I^\dagger$ consider the ring R^∞, where we adjoin all p^nth roots of elements of R to R and extend the valuation.

If one has $cx^q \in I^{[q]}$ then one has $c^{1/q}x \in IR^\infty$ and $\nu(c^{1/q}) = \frac{1}{q}\nu(c)$ goes to zero as q tends to infinity.

The key ingredient for the other inclusion (which is much harder to prove) is a generalisation of the direct summand theorem ([47, Theorem 3.2]) which asserts that a map $Au \to A^+$ splits over a complete regular local ring (A, m) of equal characteristic if the order of u is "small enough" (that is, $\nu(u) < \nu(m)$). Here ν is a $\mathbb{Z}$-valued valuation non-negative on A and positive on m. $\square$

We will now introduce a graded version of dagger closure. First of all, we need a graded version of R^+ – this is provided by a result of Hochster and Huneke which we now recall.

Let R be a domain graded by a sub-semigroup H of $\mathbb{Q}^r$. Write $H^\#$ for the set $\{h \in \mathbb{Q}^r : dh \in H \text{ for some integer } d > 0\}$ and denote by $H - H$ the group generated by H, then $\mathbb{Q}H = (H-H)^\#$ is the $\mathbb{Q}$ vector space spanned by H. Finally H^- denotes the set $(H - H) \cap H^\#$.

1.10 Lemma. Let R be a domain graded by $H \subseteq \mathbb{Q}^r$. Let Ω be an algebraic closure of the field of fractions of R. Let X denote an indeterminate over Ω.

(a) The following conditions on an element $\theta \in \Omega \setminus \{0\}$ are equivalent:

  (1) θ is a root of a nonzero polynomial $F \in R[X]$ such that X can be assigned a degree in $\mathbb{Q}H$ that makes F homogeneous.

  (2) X can be assigned a degree in $\mathbb{Q}H$ such that the ideal P of polynomials in $R[X]$ satisfied by θ is homogeneous with respect to the corresponding grading in $R[X]$.

  (3) The grading on R extends to a grading on $R[\theta]$ indexed by $\mathbb{Q}H$.

  When these conditions hold, the degree of X in parts (1) and (2) and the degree of θ in part (3) are the same and unique: this element of $\mathbb{Q}H$ shall be referred to as $\deg\theta$, and θ referred to as a homogeneous element of Ω.

(b) The homogeneous elements in Ω span a domain T graded by $\mathbb{Q}H$; this extends the H-grading on R.

(c) The elements of T integral over R form a subring S of T containing R, which is graded by $H^\#$. S is a direct limit of module-finite, $\mathbb{Q}H$-graded extension domains of R and it contains an isomorphic copy of every such extension domain; write $S = R^{+\mathrm{GR}}$.

(d) The homogeneous elements of S with degrees in H^- span a subring of S graded by H^-: denote this ring by $R^{+\mathrm{gr}}$. Thus $R^{+\mathrm{gr}} \subseteq R^{+\mathrm{GR}} = R^+ \cap T$. Moreover, $R^{+\mathrm{gr}}$ is a direct summand of $R^{+\mathrm{GR}}$ as a module over $R^{+\mathrm{gr}}$.

Proof. See [48, Lemma 4.1] $\square$

We will mostly use this result for $H = \mathbb{N}$. Then $H - H = \mathbb{Z}$ and $H^\# = \mathbb{Q}_{\geq 0}$. Thus $R^{+\mathrm{GR}}$ will be $\mathbb{Q}_{\geq 0}$-graded and $R^{+\mathrm{gr}}$ will be $\mathbb{N}$-graded.

1.11 Definition. Let R be a noetherian $\mathbb{N}$-graded domain and I an ideal. We define the *graded plus closure* $I^{+\mathrm{gr}}$ of I as the set of elements $f \in R$ such that $f \in IR^{+\mathrm{gr}}$.

1.12 Proposition. Let R be a $\mathbb{Q}$-graded domain. The map $\nu : R \setminus \{0\} \to \mathbb{Q}$ sending $f \in R \setminus \{0\}$ to $\deg f_i$, where f_i is the minimal homogeneous component of f induces a valuation on R with values in $\mathbb{Q}$. This valuation will be referred to as the *valuation induced by the grading*.

Proof. Standard. $\square$

This proposition applies in particular to $R^{+\mathrm{GR}}$.

1.13 Definition. Let R denote an $\mathbb{N}$-graded domain and let I be an ideal of R. Let ν be the valuation induced by the grading on R. Then we define the *graded dagger closure* $I^{\dagger\mathrm{GR}}$ of an ideal I as the set of elements f in R such that for all positive ε there exists an element $a \in R^{+\mathrm{GR}}$ with $\nu(a) < \varepsilon$ and such that af lies in the extended ideal $IR^{+\mathrm{GR}}$. If R is not a domain we say that $f \in I^{\dagger\mathrm{GR}}$ if $f \in (IR/P)^{\dagger\mathrm{GR}}$ for all minimal primes P of R.[3]

1.14 Remark. We note that one trivially has an inclusion $I^{+\mathrm{gr}} \subseteq I^{\dagger\mathrm{GR}}$. If one defined graded dagger closure for an $\mathbb{N}$-graded domain R, where R_0 is a field, allowing only elements of $R^{+\mathrm{gr}}$ as "multipliers of small order" with the valuation induced by the grading then one would obtain graded plus closure.

For the rest of this section we fix the following

1.15 Situation. Let R denote an $\mathbb{N}$-graded ring and I a homogeneous ideal. Let ν_P be the valuation on $Q((R/P)^{+\mathrm{GR}})$ induced by the grading after killing a minimal prime P.

1.16 Proposition. Assume Situation 1.15. Then $I^* \subseteq I^{\dagger\mathrm{GR}}$.

Proof. First of all, we may assume that R is a domain. Now if R is a domain of characteristic $p > 0$ then qth roots of an element $x \in R$ are contained in $R^{+\mathrm{GR}}$. To see this we look at the homogeneous decomposition of x and take into account that the Frobenius is a homomorphism. If $x \in I^*$ then one has $cx^q \in I^{[q]}$ for all $q > 0$. Hence, $c^{1/q}x \in IR^{+\mathrm{GR}}$ and $\nu(c^{1/q}) = \frac{1}{q}\nu(c)$ goes to zero as q tends to infinity and $c^{1/q} \in R^{+\mathrm{GR}}$. $\square$

Furthermore, we will see in Section 9 that graded dagger closure agrees with tight closure for ideals in an $\mathbb{N}$-graded ring R of finite type over a field $k = R_0$ of positive characteristic (see Corollary 9.12). There we will also propose a definition of dagger closure which is valid for arbitrary domains.

1.17 Lemma. The graded dagger closure of a homogeneous ideal is again a homogeneous ideal.

[3]Note that this is well-defined since the minimal primes are homogeneous (see [21, Lemma 1.5.6 (a)]).

Proof. Assume that R is a domain and I a homogeneous ideal. It is clear that $I^{\dagger \mathrm{GR}}$ is an ideal. Note that for a graded ring extension $R \subseteq S$ and a homogeneous ideal I the extended ideal IS is again homogeneous. Assume now that $f \in I^{\dagger \mathrm{GR}}$. This means that for every $\varepsilon > 0$ there exists $a \in R^{+\mathrm{GR}}$ with $\nu(a) < \varepsilon$ such that $af \in IR^{+\mathrm{GR}}$. Write f_1 for the minimal homogeneous component of f and similarly for a and af. We then have that $(af)_1 = a_1 f_1 \in IR^{+\mathrm{GR}}$ because $IR^{+\mathrm{GR}}$ is homogeneous. Since $\nu(a_1) = \nu(a)$ this implies that $f_1 \in I^{\dagger \mathrm{GR}}$. Repeating the argument for $f - f_1$ inductively shows that every homogeneous component of f is contained in $I^{\dagger \mathrm{GR}}$.

If R is not a domain then $I^{\dagger \mathrm{GR}}$ is the intersection of homogeneous ideals and hence homogeneous. □

This result also implies that one may choose the multipliers of small order in Definition 1.13 to be homogeneous. As another immediate consequence we note:

1.18 Corollary. Assume Situation 1.15. An element f is contained in $I^{\dagger \mathrm{GR}}$ if and only if all of its homogeneous components are contained in $I^{\dagger \mathrm{GR}}$.

In light of this corollary we will restrict our attention to homogeneous elements.

1.19 Lemma. Let R be an $\mathbb{N}$-graded domain and assume that the grading is non-trivial. An element f belongs to $I^{\dagger \mathrm{GR}}$ if and only if there is a sequence a_n of elements in $R^{+\mathrm{GR}}$ with $\nu(a_n) = 1/n$ such that $a_n f \in IR^{+\mathrm{GR}}$.

Proof. Only the only if part is non-trivial. So let a be an element with $\deg(a) \leq \frac{1}{n}$ such that $af \in IR^{+\mathrm{GR}}$. We only need to find an element x of degree $\frac{1}{n} - \deg(a)$, because we have $xaf \in IR^{+\mathrm{GR}}$ since this is an ideal. Let b be a homogeneous element of $\deg b = l > 0$. Fix $m \in \mathbb{N}$ and consider the polynomial $X^{ln} - b^m$ – the zeros of this polynomial are homogeneous elements of degree $\frac{m}{n}$. Thus we can construct elements of arbitrary positive degree in $R^{+\mathrm{GR}}$. □

1.20 Remark. If we want to include the non-domain case here then we have to suppose that the grading is non-trivial for every minimal prime reduction R/P. In the main case of interest, where R is finitely generated over a field R_0, this may fail to be true if and only if $R_+ = \bigoplus_{d \geq 1} R_d$ is the only minimal prime. And in this case $0^{\dagger \mathrm{GR}} = R_+$.

1.2 A review of solid closure

We want to compare graded dagger closure to another closure operation, namely to solid closure. We recall this notion.

1.21 Definition. Let R be a noetherian ring and let $f_1, \ldots, f_n, f_0 \in R$. Then f_0 belongs to the *solid closure* of $(f_1, \ldots, f_n)$, written $f_0 \in (f_1, \ldots, f_n)^\star$, if for every local complete domain $R' = \widehat{R_\mathfrak{m}}/\mathfrak{q}$ (where $\mathfrak{m}$ is a maximal ideal of R and $\mathfrak{q}$ is a minimal prime of $\widehat{R_\mathfrak{m}}$) we have $H^d_\mathfrak{m}(A') \neq 0$, where $d = \dim R'$ and $A' = R'[T_1, \ldots, T_n]/(f_1 T_1 + \ldots + f_n T_n + f_0)$ is the *forcing algebra* for $(f_0; f_1, \ldots, f_n)$ over R'.

See [46] for a detailed discussion of this closure operation. In particular, we refer to Section 4 of [46] to see that this notion only depends on the ideal I and not on the chosen set of generators of I. If R contains a field of characteristic $p > 0$ then solid closure coincides with tight closure under mild conditions on the ring. This is in particular true if R is essentially of finite type over an excellent local ring (cf. [46, Paragraph 8] for details). We note, however, that solid closure is strictly larger than tight closure in equal characteristic zero and $\dim R \geq 2$ (see [46, Theorem 11.4] for the inclusion $I^* \subseteq I^\star$ and [17, Remark 4.7] for an example in dimension two, where $f \in I^\star$ but $f \notin I^*$).

In fact, solid closure in equal characteristic zero only yields a good closure operation in dimension less than three. This is due to an example by Roberts (cf. [65] or [46, 7.22 and 7.23]) showing that the solid closure of the ideal (x^3, y^3, z^3) in the polynomial ring $k[x, y, z]$, where k is a field of characteristic zero, is not the ideal itself (it contains $x^2y^2z^2$). We also note that there is a refinement of solid closure called parasolid closure which agrees with solid closure in all dimensions in positive characteristic and also has the right properties in equal characteristic zero in all dimensions (see [4]).

It is quite obvious from the definition that the tight closure of a homogeneous ideal is again homogeneous. This is not immediate for solid closure in characteristic zero. Thus the following

1.22 Proposition. Let R be an $\mathbb{N}$-graded ring, where R_0 is a field of characteristic zero which contains all roots of unity, and let I be a homogeneous ideal. Then $I^\star$ is homogeneous.

Proof. Consider the ring automorphisms φ_λ for $\lambda \in R_0^\times$ which map a homogeneous element x to $\lambda^{\deg x} x$. Since I is homogeneous we have $\varphi_\lambda(I) = I$ and consequently $I^\star = \varphi_\lambda(I)^\star$. Assume that $f \in I^\star$. We have to show that each homogeneous component of f is contained in $I^\star$. We will induct on the number r of nonzero components. The assertion is clear for $r = 0, 1$. So let $r > 1$.

Write $f = \sum_{i=0}^n f_i$ where $\deg f_i = i$ and assume that $f_n \neq 0$. Let λ be a primitive nth root of unity. Then $\varphi_\lambda(f) - f \in I^\star$ and this has $r-1$ homogeneous components. $\square$

1.23 Remark. (a) Proposition 1.22 applies more generally to any ideal closure operation, even when extended to graded modules. Moreover, the condition that R_0 contains all roots of unity is not very restrictive. Indeed, assume that * is an ideal closure operation, $\varphi : R \to S$ a ring homomorphism, $I \subseteq R$ an ideal and $f \in (\varphi(I)S)^*$. Suppose furthermore that * satisfies one of the following properties:

(i) If $R \subseteq S$ is an integral ring extension then $(IS)^* \cap R \subseteq I^*$.

(ii) If S is faithfully flat over R then $f \in I^*$.

Then we can drop the assumption that R_0 contains all roots of unity. Indeed, if k is an algebraic extension which contains all roots of unity of R_0 then $R_0 \subseteq k$ is integral and $R \otimes_{R_0} k$ is faithfully flat. Also note that solid closure satisfies this second condition (see [46, Theorem 5.9]).

(b) The proof of Proposition 1.22 does not work in positive characteristic $p > 0$. If for example (in the notation of the proof) the degrees of the homogeneous components of f are powers of p then there are no non-trivial roots of unity. We do not know whether there is a direct argument in positive characteristic showing that $I^\star$ is homogeneous for a homogeneous ideal I which avoids the reinterpretation as tight closure.

For ease of reference we thus have

1.24 Corollary. Let R be an $\mathbb{N}$-graded domain of finite type over a field R_0 and let I be a homogeneous ideal. Then $I^\star$ is again homogeneous.

Proof. If the characteristic of R is zero then the result is due to Proposition 1.22 and Remark 1.23. If $\operatorname{char} R > 0$ then $I^\star = I^*$ by virtue of [46, Theorems 8.5 and 8.6]. □

2 A survey of vector bundles on curves

In what follows we shall assume familiarity with Hartshorne's textbook [39] including most of the exercises. We now fix several geometric conventions that shall be in force throughout this thesis. A *variety* is assumed to be a separated integral scheme of finite type over a field. A *curve* is a 1-dimensional variety. In particular, it need not be smooth. A *surface* is a variety of dimension two. All locally free sheaves we deal with will tacitly assumed to be coherent (in other words of finite rank).

2.1 Vector bundles

2.1 Definition. Let Y be a scheme. A (*geometric*) *vector bundle* of rank n over Y is a scheme $f : X \to Y$ together with an open covering $(U_i)_{i \in I}$ of Y, and isomorphisms $\psi_i : f^{-1}(U_i) \to \mathbb{A}^n_{U_i}$, such that for any i, j, and for any open affine $V = \operatorname{Spec} A \subseteq U_i \cap U_j$, the automorphism $\psi = \psi_j|_{f^{-1}(V)} \psi_i^{-1}|_{\mathbb{A}^n_V}$ of $\mathbb{A}^n_V$ is given by a linear A-algebra automorphism θ of $A[x_1, \ldots, x_n]$, that is, $\theta(x_i) = \sum_j a_{ij} x_j$ for suitable $a_{ij} \in A$.

If $f : X \to Y$ is a morphism of schemes then one can consider the *sheaf of sections* $\mathcal{S}(X/Y)$ on Y which is given by $U \mapsto \{\text{sections of } f \text{ over } U\}$. If $f : X \to Y$ is a vector bundle of rank n then $\mathcal{S}(X/Y)$ has a natural structure of an $\mathcal{O}_Y$-module which makes $\mathcal{S}(X/Y)$ into a locally free sheaf of rank n.

If $\mathcal{E}$ is any locally free sheaf over a scheme Y then one can show that the relative Spec of Sym $\mathcal{E}$ is a vector bundle over Y (here Sym denotes the symmetric algebra – see [31, 1.7.4] or [39, Ex. II.5.16]). Moreover, its associated sheaf of sections is naturally isomorphic to the dual $\mathcal{E}^\vee = \mathcal{H}om(\mathcal{E}, \mathcal{O}_Y)$ of $\mathcal{E}$.

It turns out that these constructions yield an equivalence of categories:

2.2 Theorem. Let Y be a scheme and $n \in \mathbb{N}$. The constructions above induce an equivalence of categories between the category of vector bundles of rank n over Y and the category of locally free sheaves of rank n on Y.

Proof. See [39, Ex. II.5.18]. □

Due to this equivalence of locally free sheaves and vector bundles we will use these terms interchangeably. We will sometimes also say bundle if we mean vector bundle.

One should note though that an inclusion $0 \to \mathcal{E} \to \mathcal{F}$ of locally free sheaves on Y does not induce an inclusion of associated vector bundles in general.

2.3 Proposition. Let Y be a noetherian integral scheme and $0 \to \mathcal{E} \to \mathcal{F}$ an exact sequence of locally free sheaves. Then the morphism $\mathcal{F}^\vee \to \mathcal{E}^\vee$ induces a closed immersion of vector bundles if and only if it is surjective.

Proof. If the morphism is surjective then the claim follows by [31, Proposition 1.7.11 (v)]. So assume that the morphism is not surjective. In particular, we have that on some open affine $\operatorname{Spec} A \subseteq Y$ which trivialises the morphism $\mathcal{F}_U^\vee \to \mathcal{E}_U^\vee$ is not surjective. It follows that the induced map of (graded) algebras is not surjective since it is not surjective in degree 1. $\square$

2.4 Proposition. Let Y be a noetherian integral scheme and $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$ a short exact sequence of coherent sheaves, where $\mathcal{E}$, $\mathcal{F}$ are locally free. Then the following conditions are equivalent:

(i) $\mathcal{G}$ is locally free.

(ii) The induced map $\mathcal{E}_y \otimes_{\mathcal{O}_y} \mathcal{O}_y/m_y \to \mathcal{F}_y \otimes_{\mathcal{O}_y} \mathcal{O}_y/m_y$ is injective for all $y \in Y$. In this case we say that the map $\mathcal{E} \to \mathcal{F}$ is *fibrewise injective.*

In this case $0 \to \mathcal{E} \to \mathcal{F}$ induces a closed immersion of vector bundles.

Proof. The dualised sequence is exact if $\mathcal{E}xt^1(\mathcal{G}, \mathcal{O}_Y)$ vanishes. This latter condition holds if $\mathcal{G}$ is projective or $\mathcal{O}_Y$ is injective. Furthermore, $\mathcal{G}$ is projective if and only if it is locally free (use [39, Proposition III.6.8]). Hence, if (i) is satisfied $0 \to \mathcal{E} \to \mathcal{F}$ induces the desired closed immersion in light of Proposition 2.3.

We prove the equivalence of (i) and (ii) (alternatively, this is also covered in [30, Chapitre 0, 5.5.5]). Assume that (i) is satisfied. Hence, the $\mathcal{G}_y$ are free. Therefore, $\operatorname{Tor}_1(\mathcal{G}_y, \mathcal{O}_y/m_y) = 0$ and this implies that the maps in (ii) are injective.

Conversely, assume that the map is fibrewise injective. By Nakayama's lemma $\mathcal{G}_y$ is generated by $n = \operatorname{rk} \mathcal{G}$ elements (this follows since the rank is additive on short exact sequences and since the map $\mathcal{E}_y \to \mathcal{F}_y$ is injective on the fibres). Hence, we have a short exact sequence $0 \to K \to \mathcal{O}_y^n \to \mathcal{G}_y \to 0$. Tensoring with the function field $K(Y)$ we obtain that $K \otimes_{\mathcal{O}_y} K(Y) = 0$ since $K(Y)$ is flat. Again by flatness and using [58, Theorem 1.2.4] this implies $K = 0$. $\square$

2.5 Remark. We claim that if Y is a noetherian integral scheme then $\mathcal{O}_Y$ is injective if and only if $Y = \operatorname{Spec} k$ is the spectrum of a field (in this case any coherent sheaf on Y is in fact free). Indeed, assume that $\mathcal{O}_Y$ is injective and consider an open affine subset $U = \operatorname{Spec} R$. It follows that R is injective as well (cf. [39, Lemma III.6.1]). Assume to the contrary that R contains a regular element t which is not a unit. Then the map $(t) \to R, t \mapsto 1$ does not lift to a module homomorphism $R \to R$. Hence, R is not injective and so we must have $R = k$ a field. Since Y is quasicompact it follows that Y consists of finitely many $U_i = \operatorname{Spec} k_i$, where k_i is a field. As we assumed that Y is irreducible we have $Y = \operatorname{Spec} k$.

Finally, we briefly recall the notion of projective space bundle.

2.6 Definition. Let Y be a noetherian scheme. A *projective n-space bundle* over Y (or *$\mathbb{P}^n$-bundle* for short) is a scheme $\pi : P \to Y$ together with an open covering $(U_i)_{i\in I}$ of Y and isomorphisms $\psi_i : f^{-1}(U_i) \to \mathbb{P}^n_{U_i}$, such that for any i, j, and for any open affine $V = \operatorname{Spec} A \subseteq U_i \cap U_j$, the automorphism $\psi = \psi_j|_{f^{-1}(V)}\psi_i^{-1}|_{\mathbb{P}^n_V}$ of $\mathbb{P}^n_V$ is given by a linear A-algebra automorphism θ of the homogeneous coordinate ring $A[x_0, \ldots, x_n]$, that is, $\theta(x_i) = \sum_j a_{ij}x_j$ for suitable $a_{ij} \in A$.

If $\mathcal{E}$ is a locally free sheaf of rank $n+1$ on a noetherian scheme Y then $\mathbb{P}(\mathcal{E})$ is a $\mathbb{P}^n$-bundle over Y (see [39, II.7] or [31, §4] for the construction of $\mathbb{P}(\mathcal{E})$ and basic properties). Similarly to the affine case one has

2.7 Theorem. Let X be a regular noetherian scheme. Then there is a natural 1-1 correspondence between $\mathbb{P}^n$-bundles over X and equivalence classes of locally free sheaves of rank $n+1$ under the equivalence relation $\mathcal{E} \sim \mathcal{E}'$ if and only if $\mathcal{E} \cong \mathcal{E}' \otimes \mathcal{L}$, where $\mathcal{L}$ is an invertible sheaf on X.

Proof. See [39, Ex. II.7.10]. □

We note that a surjection $\mathcal{S}' \to \mathcal{S} \to 0$ of locally free sheaves on a noetherian scheme X induces a closed immersion $\mathbb{P}(\mathcal{S}) \to \mathbb{P}(\mathcal{S}')$ of schemes.

2.8 Lemma. Let Y be a noetherian scheme and $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$ a short exact sequence of coherent sheaves, where $\mathcal{F}$ and $\mathcal{G}$ are locally free. Then $\mathcal{E}$ is also locally free.

Proof. This is a local question so that we may reduce to a short exact sequence $0 \to E \to F \to G \to 0$ of finitely generated modules over a local noetherian ring R where F, G are free.

In particular, G is projective and the sequence splits. Therefore, E is a direct summand of the free module F hence projective. But over a local noetherian ring projective and free are equivalent conditions for finitely generated modules (see e. g. [23, Theorem A3.2]). Hence, E is free. □

2.9 Definition. Let X be a projective variety with a fixed ample invertible sheaf $\mathcal{O}_X(1)$ and let $\mathcal{S}$ be a locally free sheaf of rank r on X. We call $\mathcal{S}$ a *syzygy bundle* with respect to $\mathcal{O}_X(1)$ if $\mathcal{S}$ fits into a short exact sequence of the form

$$0 \to \mathcal{S} \to \bigoplus_{i=0}^{r} \mathcal{O}_X(-a_i) \to \mathcal{O}_X \to 0,$$

where the a_i are integers. If $\mathcal{S}$ is a syzygy bundle then a global section of $\mathcal{S}(m)$ is called a *syzygy of total degree m* of $\mathcal{S}$.

Note that the condition that $\mathcal{S}$ be locally free may be replaced by $\mathcal{S}$ being coherent due to Lemma 2.8.

2.10 Remark. Dualising the morphisms $\mathcal{O}_X(-a_i) \to \mathcal{O}_X$ in Definition 2.9 we obtain morphisms $\mathcal{O}_X \to \mathcal{O}_X(a_i)$. Every such morphism corresponds to a choice of a section $H^0(X, \mathcal{O}_X(a_i))$. Calling these sections f_i one usually denotes $\mathcal{S}$ by $\operatorname{Syz}(f_0, \ldots, f_r)$.

In order to relate the notion of syzygy bundle to the classical notion of syzygies of an ideal consider the following

2.11 Example. Let R be an $\mathbb{N}$-graded domain generated by finitely many elements of degree 1 over a field R_0. Then $Y = \operatorname{Proj} R$ is a variety over R_0 and the $R(n)^\sim$ are invertible (cf. [39, Proposition II.5.12]). Let I be an R_+-primary homogeneous ideal with homogeneous generators $f_0, \ldots, f_r$ of degrees d_i and consider the following exact sequence (a part of a free resolution of I):

$$\bigoplus_{i=0}^{r} R(-d_i) \xrightarrow{f_0,\ldots,f_r} R \longrightarrow R/I \longrightarrow 0.$$

This induces the following exact sequence of sheaves on Y:

$$0 \longrightarrow \mathcal{S} \longrightarrow \bigoplus_{i=0}^{r} \mathcal{O}_Y(-d_i) \xrightarrow{f_0,\ldots,f_r} \mathcal{O}_Y \longrightarrow 0,$$

where $\mathcal{S}$ denotes the kernel of the sheaf morphism $f_0, \ldots, f_r$. Note that R/I is zero on $\operatorname{Proj} R$ since I is R_+-primary.

It is in fact true that every syzygy bundle is essentially obtained in this way. For a fixed ample invertible sheaf $\mathcal{O}_Y(1)$ on a projective variety Y the so-called section ring $\bigoplus_{i \geq 0} H^0(Y, \mathcal{O}_Y(i))$ plays the role of R (see Section 5 for details on section rings). One also only needs a complex that is exact for localisations at homogeneous prime ideals $\neq R_+$.

2.2 Semistability

For the rest of this section Y denotes a smooth projective curve over an algebraically closed field k of arbitrary characteristic. In what follows we shall review the notion of (strong) semistability for a locally free sheaf on a smooth projective curve. In dimension ≥ 2 there are several different concepts of semistability – we will come back to this in Section 10. The main sources for this subsection are [39] and [57].

2.12 Lemma. Let $\mathcal{F}$ be a coherent sheaf on Y. Then $\mathcal{F} = \mathcal{T} \oplus \mathcal{E}$, where $\mathcal{E}$ is locally free and $\mathcal{T}$ a torsion sheaf.

Proof. See [57, Lemma 5.2.2]. □

2.13 Definition. Let $\mathcal{E} \subseteq \mathcal{F}$ be locally free sheaves on Y and denote the cokernel by $\mathcal{Q}$. The *saturation* $\mathcal{E}^s$ of $\mathcal{E}$ in $\mathcal{F}$ is the kernel of the map $\mathcal{F} \to \mathcal{Q} \to \mathcal{Q}/\mathcal{T}$, where $\mathcal{T}$ is the torsion subsheaf of $\mathcal{Q}$. Note that by construction $\mathcal{F}/\mathcal{E}^s$ is torsion-free.

2.14 Proposition. There is a unique notion of a *degree* for any coherent sheaf $\mathcal{F}$ on Y such that $\deg \mathcal{F} \in \mathbb{Z}$ and such that:

(i) If D is a divisor then $\deg \mathcal{L}(D) = \deg D$, where $\mathcal{L}(D)$ is the element of $\operatorname{Pic} Y$ corresponding to D.

(ii) If $\mathcal{F}$ is a torsion sheaf then $\deg \mathcal{F} = \sum_{P \in Y} \operatorname{length} \mathcal{F}_P$.

(iii) If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence then $\deg \mathcal{F} = \deg \mathcal{F}' + \deg \mathcal{F}''$.

Proof. Clearly, the zero sheaf must have degree 0. Any locally free sheaf of positive rank is a successive extension of invertible sheaves and any coherent sheaf fits into an exact sequence $0 \to \mathcal{T} \to \mathcal{F} \to \mathcal{G} \to 0$, where $\mathcal{G}$ is locally free and $\mathcal{T}$ is the torsion subsheaf of $\mathcal{F}$. So deg is uniquely determined by these conditions.

For any coherent sheaf $\mathcal{F}$ we set $\deg \mathcal{F} = \chi(\mathcal{F}) - \operatorname{rk} \mathcal{F}\, \chi(\mathcal{O}_Y)$, where χ is the Euler characteristic. Since χ and rk are additive on short exact sequences (iii) is satisfied. The Theorem of Riemann-Roch for curves (see [39, Theorem IV.1.3]) yields that (i) holds. Let $\mathcal{T}$ be a torsion sheaf and let $\{y_1, \ldots, y_n\}$ be the support of $\mathcal{T}$. Note that $\operatorname{rk} \mathcal{T} = 0$. We have $H^1(Y, \mathcal{T}) = 0$ since $\operatorname{Supp} \mathcal{T}$ is a closed affine subscheme of Y (then use [39, Lemma III.2.10, Theorem III.3.5]). Finally, $\mathcal{T}$ is a direct sum of skyscraper sheaves $\mathcal{T}_{y_i}$, hence the global sections are exactly $\bigoplus_i \mathcal{T}_{y_i}$. This proves (ii). $\square$

2.15 Definition. Let $\mathcal{S}$ denote a locally free sheaf of $\operatorname{rk} \geq 1$ on Y. The *slope* of $\mathcal{S}$ is defined as $\mu(\mathcal{S}) = \deg \mathcal{S} / \operatorname{rk} \mathcal{S}$. The bundle $\mathcal{S}$ is called *semistable* if for every nonzero coherent subsheaf $\mathcal{F} \subset \mathcal{S}$ (which is then locally free since Y is a smooth curve) we have $\mu(\mathcal{F}) \leq \mu(\mathcal{S})$. Strict inequality defines the notion of *stability.*

If $\mathcal{E} \subseteq \mathcal{S}$ and $\mu(\mathcal{E}) > \mu(\mathcal{S})$ then we say that $\mathcal{E}$ is a *destabilising subsheaf* of $\mathcal{E}$ or that $\mathcal{E}$ *destabilises* $\mathcal{S}$ and similarly for quotients.

2.16 Lemma. (a) Let $\mathcal{S}$ be a locally free sheaf on Y. Then $\mathcal{S}$ is semistable if and only if $\mu(\mathcal{F}) \leq \mu(\mathcal{S})$ for subsheaves $\mathcal{F}$, such that $\mathcal{S}/\mathcal{F}$ is locally free.

(b) A locally free sheaf $\mathcal{S}$ on Y is semistable if and only if for every nonzero quotient locally free sheaf $\mathcal{S} \to \mathcal{Q} \to 0$ one has $\mu(\mathcal{Q}) \geq \mu(\mathcal{S})$.

(c) If $\mathcal{E}, \mathcal{F}$ are locally free sheaves on Y with respective slopes $\mu(\mathcal{E}), \mu(\mathcal{F})$ then

$$\mu(\mathcal{E} \otimes \mathcal{F}) = \frac{\deg(\mathcal{E}) \operatorname{rk} \mathcal{F} + \deg(\mathcal{F}) \operatorname{rk} \mathcal{E}}{\operatorname{rk} \mathcal{E} \operatorname{rk} \mathcal{F}} = \mu(\mathcal{E}) + \mu(\mathcal{F}).$$

Proof. (a) One may always pass to the saturation of $\mathcal{F}$ in $\mathcal{S}$ (this can only increase the slope). And the cokernel is then torsion-free, hence locally free since Y is a smooth curve.

(b) Consider the exact sequence $0 \to \mathcal{K} \to \mathcal{S} \to \mathcal{Q} \to 0$ of locally free sheaves and use the additivity of the rank and the degree (see the discussion after [57, Definition 5.3.1]).

(c) The claim about the degree follows via induction on the ranks taking into account that a vector bundle is a successive extension of bundles of lower rank and that the degree is additive on short exact sequences. The rank is equal to the rank of

$\mathcal{E}_y \otimes \mathcal{F}_y$ for any $y \in Y$ so that the assertion about the denominator is a well-known fact from algebra.

□

It is convenient to define the *minimal slope* $\mu_{\min}(\mathcal{S}) = \min\{\mu(\mathcal{Q}) : \mathcal{S} \to \mathcal{Q} \to 0\}$ and the *maximal slope* $\mu_{\max}(\mathcal{S}) = \max\{\mu(\mathcal{F}) : 0 \to \mathcal{F} \to \mathcal{S}\}$. With these definitions $\mathcal{S}$ is semistable if and only if $\mu_{\min}(\mathcal{S}) = \mu_{\max}(\mathcal{S}) = \mu(\mathcal{S})$. Moreover, one has $\mu_{\max}(\mathcal{S}^\vee) = -\mu_{\min}(\mathcal{S})$.

2.17 Theorem. Let $\mathcal{S}$ be a locally free sheaf on Y. Then there exists a unique filtration

$$0 = \mathcal{S}_0 \subset \mathcal{S}_1 \subset \ldots \subset \mathcal{S}_t = \mathcal{S}$$

of subbundles such that the quotients $\mathcal{S}_i/\mathcal{S}_{i-1}$ are locally free and semistable with slopes μ_i and such that $\mu_1 > \ldots > \mu_t$. Moreover, one has $\mu_{\max}(\mathcal{S}) = \mu_1$ and $\mu_{\min}(\mathcal{S}) = \mu_t$. This filtration is called the *Harder-Narasimhan filtration* of $\mathcal{S}$.

Proof. See [34, Proposition 1.3.9] or [57, Proposition 5.4.2]. □

2.18 Lemma. Let $f : Y' \to Y$ be a finite separable morphism of smooth projective curves. Then a locally free sheaf $\mathcal{S}$ on Y is semistable if and only if $f^*\mathcal{S}$ is semistable.

Proof. See [61, Proposition 3.2]. □

It is in general *not* true that the pullback along a non-separable morphism of a semistable bundle is again semistable. See for example [28, Theorem 1]. It is true if the genus of Y is zero or one (cf. [60, Theorem 2.1]) but not for higher genus (for $g = 0$ this also follows in a more elementary fashion from a theorem of Grothendieck (see [51, Theorem 1.3.1] or Example 4.12 (a) below)). More generally, Lange and Pauly proved the following

2.19 Proposition. Let Y be a smooth projective curve of genus $g \geq 2$ over an algebraically closed field of positive characteristic. Then there is a stable vector bundle $\mathcal{E}$ of rank 2 such that $F^*\mathcal{E}$ is not semistable, where F denotes the k-linear Frobenius morphism.

Proof. See [54, Theorem 1]. □

To remedy this situation one makes the following

2.20 Definition. Let $\mathcal{S}$ be a locally free sheaf on a smooth projective curve Y over an algebraically closed field k of characteristic $p > 0$. The bundle $\mathcal{S}$ is called *strongly semistable* if $F^{e*}\mathcal{S}$ is semistable for every $e \geq 0$, where $F : Y' \to Y$ denotes the k-linear Frobenius morphism (see [39, Remark 2.4.1]).

2.21 Remark. We remark that strong semistability is usually defined using the absolute Frobenius rather than the k-linear Frobenius. The reason for deviating from this definition is that we want our morphisms to be k-linear since we will frequently refer to Theorem 3.14 below which only deals with k-linear morphisms.

Also note that any finite k-morphism factors as a power of the k-linear Frobenius and a separable morphism. Hence, one may restrict ones attention to Frobenius pullbacks in light of Lemma 2.18.

2.22 Proposition. If $\mathcal{S}$ is strongly semistable on Y then $f^*\mathcal{S}$ is semistable for every finite morphism $f : Y' \to Y$ of smooth projective curves.

Proof. See [61, Proposition 5.1]. This follows since f factors into a Frobenius power and a separable morphism. □

If $\mathcal{S}$ is a locally free sheaf on Y then the pullback along a Frobenius of the Harder-Narasimhan filtration is in general not the Harder-Narasimhan filtration of $F^*\mathcal{S}$. However, there is the following result due to Langer:

2.23 Theorem. Let Y smooth projective curve over an algebraically closed field k of positive characteristic and $\mathcal{S}$ a locally free sheaf on Y. Then there exists a Frobenius power F^e such that the quotients of the Harder-Narasimhan filtration of $F^{e*}\mathcal{S}$ are all strongly semistable. We call such a filtration a *strong Harder-Narasimhan filtration* of $\mathcal{S}$.

Proof. See [55, Theorem 2.7]. □

Note that Langer's theorem holds for smooth projective varieties of any dimension. We simply stated a version that is sufficient for our setting.

Also note that the strong Harder-Narasimhan filtration is not unique insofar as one may always consider higher Frobenius pullbacks. Of course, one can fix the minimal e such that the quotients of the Harder-Narasimhan filtration of $F^{e*}\mathcal{S}$ are strongly semistable and in this case one may speak of *the* strong Harder-Narasimhan filtration.

2.24 Convention. If we assume that some locally free sheaf $\mathcal{S}$ is strongly semistable without having fixed a characteristic then this just means that we suppose it to be semistable if the characteristic is zero and strongly semistable in positive characteristic.

Likewise, if some statement about strong semistability involves a pullback along the k-linear Frobenius F without having explicitly fixed positive characteristic then this statement also holds if F is replaced by the identity for $\operatorname{char} k = 0$.

2.25 Definition. Let Y denote a smooth projective curve over an algebraically closed field k and let $\mathcal{S}$ denote a locally free sheaf. Then we define

$$\bar{\mu}_{\max}(\mathcal{S}) = \sup\{\frac{\mu_{\max}(\varphi^*\mathcal{S})}{\deg\varphi} \,|\, \varphi : Z \to Y \text{ finite dominant } k\text{-morphism}\}$$

and

$$\bar{\mu}_{\min}(\mathcal{S}) = \inf\{\frac{\mu_{\min}(\varphi^*\mathcal{S})}{\deg\varphi} \,|\, \varphi : Z \to Y \text{ finite dominant } k\text{-morphism}\}.$$

2.26 Remark. It is sufficient to consider in the above definition only morphisms which are not separable, since $\mu_{\min}$ resp. $\mu_{\max}$ are well-behaved with respect to separable morphisms by Lemma 2.18 and by Theorem 2.17. Moreover, any non-separable morphism factors as some power of the k-linear Frobenius and a separable morphism. Hence, it is sufficient to consider only powers of the k-linear Frobenius morphism.

As the Harder-Narasimhan filtration of $F^{e*}\mathcal{S}$ has strongly semistable quotients for e sufficiently large the supremum and the infimum in the above definition are actually attained. That is, we have a maximum and a minimum.

Finally, we collect some properties of bundles related to (strong) semistability which we will need in the sequel.

2.27 Proposition. Let Y be a smooth projective curve over an algebraically closed field.

(a) Any line bundle is strongly semistable.

(b) If a vector bundle $\mathcal{S}$ is (strongly) semistable then so is its dual $\mathcal{S}^\vee$.

(c) If $\mathcal{S}$ is a (strongly) semistable vector bundle and $\mathcal{L}$ a line bundle then $\mathcal{S}\otimes\mathcal{L}$ is also (strongly) semistable.

(d) If $\mathcal{E}$ and $\mathcal{F}$ are vector bundles such that $\mu_{\min}(\mathcal{E}) > \mu_{\max}(\mathcal{F})$ then $\operatorname{Hom}(\mathcal{E},\mathcal{F}) = 0$.

Proof. (a) is clear. To proof (b) observe that one has a canonical isomorphism $\mathcal{S} \to \mathcal{S}^{\vee\vee}$. So if $\mathcal{E}$ is a destabilising subbundle of $\mathcal{S}^\vee$, then $\mathcal{E}^\vee$ is a destabilising quotient for $\mathcal{S}^{\vee\vee}$. For (c) note that if $\mathcal{E}$ is a destabilising bundle for $\mathcal{S}\otimes\mathcal{L}$ then $\mathcal{E}\otimes\mathcal{L}^{-1}$ destabilises $\mathcal{S}$. For (d), see the proof of [57, Proposition 5.3.3]. □

2.28 Example. (a) Let $0 \to \mathcal{S} \to \mathcal{S}' \to \mathcal{O}_Y \to 0$ be an exact sequence of locally free sheaves on Y. Assume that $\mathcal{S}$ is strongly semistable with $\deg\mathcal{S} = 0$ (e. g. $\mathcal{S} \in \operatorname{Pic}^0 Y$). Then $\mathcal{S}'$ is also strongly semistable.

To see this choose a subsheaf $\mathcal{G}$ of $\mathcal{S}'$ of maximal rank of all subsheaves with maximal slope. Assume that $\mathcal{S}'$ is not semistable. It follows that $\mu(\mathcal{G}) > 0$. In particular, $\mathcal{G}$ is not contained in $\mathcal{S}$. Hence, we must have a nontrivial morphism $\mathcal{G} \to \mathcal{O}$ induced by the inclusion. But by construction $\mathcal{G}$ is semistable so we get a contradiction with Proposition 2.27 (d). The same argument applied to $F^{e*}\mathcal{S}'$ shows strong semistability.

(b) Let $\mathcal{E} = \mathcal{L}\oplus\mathcal{G}$ be a direct sum of line bundles $\mathcal{L},\mathcal{G}$. Then $\mathcal{E}$ is (strongly) semistable if and only $\deg\mathcal{L} = \deg\mathcal{G}$.

Assume that the degrees are equal. Then $\mathcal{E}$ is (strongly) semistable if and only if $\mathcal{E}\otimes\mathcal{L}^\vee$ is so by Proposition 2.27 (c). But now we are in situation (a) since $\mathcal{G}\otimes\mathcal{L}^\vee$ is strongly semistable of degree zero. Alternatively, it is sufficient to verify that $\mu(\mathcal{M}) \leq \mu(\mathcal{E})$ for line bundles $\mathcal{M}$ contained in $\mathcal{E}$. The map $\mathcal{M} \to \mathcal{L}$ induced by projection is either injective or zero. In the latter case we have $\mathcal{M} \subseteq \mathcal{G}$ and hence $\mu(\mathcal{M}) \leq \mu(\mathcal{G}) = \mu(\mathcal{E})$. The other case is similar. Since line bundles are strongly semistable we also have $\mu(\mathcal{M}) \leq \mu(F^{e*}\mathcal{G})$ for any line bundle $\mathcal{M}$ contained in some $F^{e*}\mathcal{E}$.

Assume now that the degrees do not coincide and that $\deg\mathcal{L} > \deg\mathcal{G}$. Then $\mathcal{L}$ is a destabilising subsheaf for $\mathcal{E}$.

3 Geometric interpretation of solid closure

In the following we will review the geometric interpretation of solid closure developed by Brenner (cf. [5], [7] and [12]). We will then use this to prove that graded dagger closure coincides with solid closure in the following sections. We now recall the main theorems and definitions.

3.1 Solid closure and vector bundles

3.1 Definition. Let G denote a group scheme over a scheme X. A scheme $B \to X$ together with an operation $\sigma : G \times_X B \to B$ is called a *geometric torsor* (or *torsor* for short) for G if there exist an open covering $(U_i)_{i \in I}$ of X and isomorphisms $\varphi_i : B|_{U_i} \to G|_{U_i}$ such that the diagrams (with $U = U_i, \varphi = \varphi_i$)

$$\begin{tikzcd} G|_U \times_U B|_U \arrow[r, "\sigma"] \arrow[d, "id \times \varphi"] & B|_U \arrow[d, "\varphi"] \\ G|_U \times_U G|_U \arrow[r, "\mu"] & G|_U \end{tikzcd}$$

commute, where μ is the multiplication morphism.

3.2 Remark. A vector bundle V of rank n over a scheme Y is in particular a commutative group scheme. The multiplication $\mu : V \times_Y V \to V$ is locally given by the identification with $\mathbb{A}^n_U$ for suitable open subsets $U \subseteq Y$.

3.3 Proposition. Let X denote a noetherian separated scheme and let $p : V \to X$ denote a vector bundle with sheaf of sections $\mathcal{S}$. Then there is a 1-1 correspondence between cohomology classes $c \in H^1(X, \mathcal{S})$ and geometric V-torsors.

Proof. See [13, Proposition 3.2] □

The following proposition is well-known. But due to lack of a suitable reference we shall provide a proof.

3.4 Proposition. Let R be a noetherian $\mathbb{N}$-graded normal domain of dimension $d = \dim R \geq 2$, $X = \operatorname{Proj} R$ and $\mathcal{O}_X(1) = R(1)^{\sim}$. Then $H^0(X, \mathcal{O}_X(n)) = R_n$.

Proof. See also [39, Proposition II.5.13] for the case of a polynomial ring in finitely many variables over a (not necessarily noetherian) ring. Since R is normal and of dimension at least two we have $\operatorname{depth} R \geq 2$. Hence, we find elements $x_1, \ldots, x_m$ such that x_1, x_2 form a regular sequence and such that the $D_+(x_i)$ cover X. Then to give a global section s of $\mathcal{O}_X(n)$ is equivalent to giving sections $s_i \in \Gamma(D_+(x_i), \mathcal{O}_X(n))$ whose images agree on the intersections $D_+(x_i x_j)$. Each s_i is a homogeneous element of degree n in R_{x_i} and its restriction to $D_+(x_i x_j)$ is just the image of s_i in $R_{x_i x_j}$.

Since we assume R to be integral the localisation maps $R \to R_{x_i}$ and $R_{x_i} \to R_{x_i x_j}$ are injective. Hence, $\Gamma(X, \mathcal{O}_X(n))$ is the intersection of the $(R_{x_i})_n$ in $(R_{x_1 \cdots x_m})_n$. We clearly have $R_n \subseteq (R_{x_i})_n$ for all i. So let s_i be sections of $\Gamma(D_+(x_i), \mathcal{O}_X(n))$

which agree on the $D_+(x_ix_j)$. In particular, $s_1 = a_1/x_1^k = a_2/x_2^l = s_2$ on $D(x_1x_2)$. Hence, $a_1x_2^l = a_2x_1^k$ in R. Note that since x_2 is a regular element of R (x_2, x_1) is also a regular sequence. This kind of reasoning easily implies that (x_1^k, x_2^l) is a regular sequence. Killing x_1^k we see that $a_1x_2^l = 0$ in $R/(x_1^k)$. Hence, we must have $a_1 = 0$ and therefore $a_1 = a_1'x_1^k$. Similarly $a_2 = a_2'x_2^l$. Therefore, $a_1/x_1^k, a_2/x_2^l \in R$. □

3.5 Remark. If R is a noetherian ring of depth $R = d$ then there is always an R-sequence of length d that is stable under permutation (see [23, Ex. 17.6]). In particular, we do not need the hypothesis that R is a domain since the localisation at a regular element is always injective.

Also note that we could replace R being normal with depth $R \geq 2$ which is a weaker condition provided that $\dim R \geq 2$.

Let R be a normal standard graded domain of dimension $d \geq 2$ over an algebraically closed field k and write $Y = \operatorname{Proj} R$. Let $f_1, \ldots, f_n$ denote homogeneous generators of degrees $d_1, \ldots, d_n$ of an R_+-primary homogeneous ideal and fix a homogeneous element f_0 of degree m. We may identify $H^0(Y, \mathcal{O}_Y(m))$ with R_m by virtue of Proposition 3.4.

These data yield the following short exact sequence of locally free sheaves on Y which we call the *presenting sequence* for the twisted syzygy bundle $\mathcal{S} = \operatorname{Syz}(f_1, \ldots, f_n)(m)$ with forcing data $(f_1, \ldots, f_n)$ (cf. Example 2.11):

$$0 \longrightarrow \mathcal{S} \longrightarrow \bigoplus_{i=1}^n \mathcal{O}_Y(m - d_i) \xrightarrow{f_1, \ldots, f_n} \mathcal{O}_Y(m) \longrightarrow 0.$$

The element $f_0 \in R_m$ defines via the connecting homomorphism the cohomology class

$$c = \delta(f_0) \in H^1(Y, \operatorname{Syz}(f_1, \ldots, f_n)(m)) = \operatorname{Ext}^1(\mathcal{O}, \operatorname{Syz}(f_1, \ldots, f_n)(m)).$$

This class corresponds to the extension $\mathcal{S}' = \operatorname{Syz}(f_0, f_1, \ldots, f_n)(m)$ (see [39, Ex. III.6.1] for this correspondence). The complement $T = \mathbb{P}(\mathcal{S}'^\vee) \setminus \mathbb{P}(\mathcal{S}^\vee)$ is a geometric $\mathcal{S}$-torsor, which also corresponds to c. We will not use any of the machinery attached to torsors. For our purposes it will mostly be a convenient way to reference the complement $\mathbb{P}(S'^\vee) \setminus \mathbb{P}(\mathcal{S}^\vee)$ attached to the cohomology class $c \in H^1(Y, \mathcal{S})$.

Also note that in this situation $\mathbb{P}(\mathcal{S}^\vee)$ is a closed subvariety of $\mathbb{P}(\mathcal{S}'^\vee)$ and the (effective) Weil divisor corresponding to $s \in H^0(\mathbb{P}(\mathcal{S}'^\vee), \mathcal{O}(1)_{\mathbb{P}(\mathcal{S}'^\vee)}) = H^0(X, \mathcal{S}^\vee)$ given by the dualised presenting sequence is precisely $\mathbb{P}(\mathcal{S}^\vee)$ (cf. [5, Proposition 3.4 (iii)]) – we call this the *forcing divisor*.

The element f_0 is contained in the solid closure of $(f_1, \ldots, f_n)$ if and only if the cohomological dimension of T is $d - 1$(see [39, Ex. III.4.8] or [36] for the notion of cohomological dimension). If $d = 2$ this means that $f_0 \in (f_1, \ldots, f_n)^\star$ if and only if T is not an affine scheme (cf. [5, Proposition 3.9]).

For future reference we fix the following

3.6 Situation. Let R be a normal standard graded two-dimensional integral k-algebra where k is an algebraically closed field of arbitrary characteristic. Moreover, let $I = (f_1, \ldots, f_n)$ be an R_+-primary homogeneous ideal, where the f_i are homogeneous of degrees d_i.

The next result is also covered in [5, Lemma 3.7] – but we will give a different proof.

3.7 Proposition. Assume Situation 3.6 and let f_0 be homogeneous of degree m. Then $f_0 \in (f_1, \ldots, f_n)$ if and only if $\delta(f_0) = c$ in $H^1(\operatorname{Proj} R, \operatorname{Syz}(f_1, \ldots, f_n)(m))$ is zero, that is, the extension defined by c splits.

Proof. Write $Y = \operatorname{Proj} R$ and $\mathcal{S} = \operatorname{Syz}(f_1, \ldots, f_n)$. Twist the exact sequence defining $\mathcal{S}$ by m and take cohomology. We look at the following part of the long exact sequence:

$$0 \longrightarrow H^0(Y, \mathcal{S}(m)) \longrightarrow \bigoplus_{i=1}^n H^0(Y, \mathcal{O}(m - d_i)) \xrightarrow{f_1, \ldots, f_n} R_m \xrightarrow{\delta} H^1(Y, \mathcal{S}(m)).$$

From this we can extract the exact sequence

$$0 \longrightarrow (f_1, \ldots, f_n)_m \longrightarrow R_m \longrightarrow H^1(Y, \operatorname{Syz}(f_1, \ldots, f_n)(m)) \ ,$$

and here we see that $\delta(f_0) = 0$ if and only if $f_0 \in (f_1, \ldots, f_n)_m$. $\square$

3.8 Proposition. Let Y be a noetherian scheme and $\mathcal{E}$ a locally free sheaf on Y. Let $g : Y' \to Y$ be any morphism. Then to give a morphism $Y' \to \mathbb{P}(\mathcal{E})$ over Y, it is equivalent to give an invertible sheaf $\mathcal{L}$ on Y' and a surjective map of sheaves on Y', $g^*\mathcal{E} \to \mathcal{L}$.

Proof. See [39, Proposition II.7.12]. $\square$

3.9 Corollary. Let Y be a noetherian scheme, $\mathcal{E}$ a locally free sheaf on Y and consider $\pi : \mathbb{P}(\mathcal{E}) \to Y$. There is a natural 1-1 correspondence between sections $s : Y \to \mathbb{P}(\mathcal{E})$ of π and quotient invertible sheaves $\mathcal{E} \to \mathcal{L} \to 0$ of $\mathcal{E}$.

Proof. Apply Proposition 3.8 with $g = id$. $\square$

Next we want to show that if f_0 is homogeneous of degree m in Situation 3.6 then $\delta(f_0) = c$ in $H^1(\operatorname{Proj} R, \operatorname{Syz}(f_1, \ldots, f_n)(m))$ is zero if and only if there is a section of $\mathbb{P}(\operatorname{Syz}(f_0, \ldots, f_n)(m)^\vee) \to \operatorname{Proj} R$ that is disjoint to $\mathbb{P}(\operatorname{Syz}(f_1, \ldots, f_n)(m)^\vee)$ (see [5, Lemma 3.7]). This fact holds in the following more general form:

3.10 Proposition. Let Y be a noetherian scheme and let $0 \to \mathcal{S} \to \mathcal{S}' \to \mathcal{O}_Y \to 0$ be an exact sequence of locally free sheaves. Then the sequence splits if and only if there is a section s of $\mathbb{P}(\mathcal{S}'^\vee) \to Y$ which is disjoint to $\mathbb{P}(\mathcal{S}^\vee)$.

Proof. We use the correspondence between sections of $\mathbb{P}(\mathcal{S}'^\vee)$ and surjections $\mathcal{S}'^\vee \to \mathcal{L} \to 0$, where $\mathcal{L}$ is a line bundle on Y, as outlined in Corollary 3.9. Denote the rank of $\mathcal{S}$ by n.

First of all, we deal with the case $n = 0$. This means that $\mathcal{S} = 0$ and $\mathcal{S}' = \mathcal{O}_Y$. Then we have the identical section which is disjoint to $\mathbb{P}(0) = \varnothing$. So we may suppose that $n \geq 1$.

Assume now that the sequence splits. This implies that we have a surjection $\mathcal{S}'^\vee = \mathcal{S}^\vee \oplus \mathcal{O} \to \mathcal{O} \to 0$ which induces a section $s : Y \to \mathbb{P}(S'^\vee)$. Locally on an open affine $U = \operatorname{Spec} A \subseteq Y$ which trivialises $\mathcal{S}^\vee$ the section s is given by $\mathcal{O}_U^n \oplus \mathcal{O}_U \to \mathcal{O}_U \to 0$

where the standard basis vectors e_i map to zero for $i = 1, \dots, n$ and $e_{n+1} \mapsto 1$ (see [39, Theorem II.7.1]). In order to get the closed immersions on the (relative) Proj we have to consider the symmetric (graded) algebras of $\mathcal{O}_U^{n+1}$, $\mathcal{O}_U^n$ and $\mathcal{O}_U$ where the $e_i = x_i$ are generators in degree 1 and the maps are induced via the morphisms of sheaves. These ring homomorphisms then induce the morphisms on the Proj in the usual way (see [39, Ex. II.2.14 and II.3.12]). Hence, we see that the image of s is contained in $D_+(x_{n+1})$. On the other hand, the closed immersion $\mathbb{P}(\mathcal{S}^\vee) \to \mathbb{P}(\mathcal{S}'^\vee)$ is given on U by

$$A[x_1, \dots, x_{n+1}] \to A[x_1, \dots, x_n], x_{n+1} \mapsto 0 \text{ and } x_i \mapsto x_i \text{ otherwise.}$$

That is, its image is contained in $V_+(x_{n+1})$ and this is disjoint to the image of s.

Conversely, assume that there is a disjoint section s induced by $\mathcal{S}'^\vee \to \mathcal{L}^\vee \to 0$. We have a morphism $\mathcal{S}'^\vee \to \mathcal{S}^\vee \oplus \mathcal{L}^\vee$ and we have to prove that it is an isomorphism. This is a local property so we may assume that $Y = \operatorname{Spec} A$ is affine and that all sheaves are free. We therefore have a surjective morphism $\varphi : A^{n+1} \to A$, where $e'_i \mapsto a_i$ which is induced by s. And we also have a surjective morphism $\psi : A^{n+1} \to A^n$, $e'_i \mapsto \sum_j a_{ij} e_j$ induced by $\mathcal{S}'^\vee \to \mathcal{S}^\vee$. These maps correspond to surjective maps on the symmetric algebras which then induce the closed immersions on the Proj. Denote the kernel ideal for the ring homomorphisms induced by φ resp. ψ by I resp. J and note that they are generated in degree 1.

Since the section is disjoint to $\mathbb{P}(\mathcal{S}^\vee)$ we must have that $V_+(I + J) = \varnothing$. Hence, $\operatorname{Rad}(I + J) = R_+$. This implies that $A^{n+1}/(\ker \varphi + \ker \psi)$ is a torsion module. Indeed, this is the first graded piece of the ring $A[x_0, \dots, x_n]/(I + J)$ and since we only have linear relations we must have that each component is torsion.

It is enough to show that we have an isomorphism $\varphi \times \psi : A^{n+1} \to A \times A^n$. This is local on the stalks so that we may furthermore assume that (A, m, k) is a local ring. This implies that $\ker \varphi$ is free. Indeed, the sequence

$$0 \to \ker \varphi \to A^{n+1} \to A \to 0$$

splits and hence $\ker \varphi \oplus A = A^{n+1}$. Therefore, $\ker \varphi$ is projective thus free since A is local. Similarly, we see that $\ker \psi$ is free.

Next we claim that $\ker \varphi \oplus \ker \psi = A^{n+1}$. By Nakayama's Lemma $\ker \varphi + \ker \psi = A^{n+1}$. Fix a generator f of $\ker \psi$. If there is a nonzero $a \in A$ such that $af \in \ker \varphi$ then $A^{n+1}/\ker \varphi$ has torsion which is a contradiction.

But this implies that we have an isomorphism $\varphi \times \psi$ as desired. □

3.11 Remark. At this point we make an observation that shall be crucial later on (see the discussion in Section 6). Let Y be a smooth projective curve over an algebraically closed field and $0 \to \mathcal{O}_Y \to \mathcal{S}' \to \mathcal{S} \to 0$ a short exact sequence of locally free sheaves. We thus have an inclusion $\mathbb{P}(\mathcal{S}^\vee) \subseteq \mathbb{P}(\mathcal{S}'^\vee)$ of varieties. Suppose furthermore that we have a section $s : Y \to \mathbb{P}(\mathcal{S}'^\vee)$ corresponding to $\mathcal{S}'^\vee \to \mathcal{L} \to 0$. When is the image of s contained in $\mathbb{P}(\mathcal{S}^\vee)$?

Recalling that the morphism is given on a suitable open affine subset $U \subset Y$ by mapping a basis of $\mathcal{S}'^\vee|_U$ to $\mathcal{L}|_U$ it is clear that this is the case if and only if the surjection factors through $\mathcal{S}^\vee$. That is, if and only if we have a commutative diagram:

$$\begin{tikzcd} \mathcal{S}'^\vee \arrow[r] \arrow[d] & \mathcal{L} \\ \mathcal{S}^\vee \arrow[ur] & \end{tikzcd}$$

3.2 Solid closure and ampleness

Recall that the ampleness of a vector bundle $\mathcal{G}$ is by definition the ampleness of the invertible sheaf $\mathcal{O}_{\mathbb{P}(\mathcal{G})}(1)$ on $\mathbb{P}(\mathcal{G})$. If in Situation 3.6 one has that $\operatorname{Syz}(f_1, \ldots, f_n)$ is semistable in characteristic zero resp. strongly semistable in positive characteristic then one has the following criterion relating solid closure and ampleness:

3.12 Theorem. Let Y denote a smooth projective curve over an algebraically closed field k. Let $\mathcal{S}$ denote a locally free sheaf and let $c \in H^1(Y, \mathcal{S})$ be a cohomology class with corresponding torsor $T \to Y$. If $\mathcal{S}$ is strongly semistable then the following are equivalent:

(i) The torsor T is an affine scheme.

(ii) $\bar{\mu}_{\max}(\mathcal{S}) < 0$ and $c \neq 0$ (in positive characteristic $F^{e*}(c) \neq 0$ for all Frobenius powers F^e).

(iii) The dual extension $\mathcal{S}'^\vee$ given by c is an ample bundle.

The implications (ii) $\Leftrightarrow$ (iii) $\Rightarrow$ (i) hold for any locally free sheaf $\mathcal{S}$.

Proof. See [12, Proposition 2.1]. □

This entails

3.13 Corollary. Assume Situation 3.6 and let f_0 be a homogeneous element of degree m. Write $Y = \operatorname{Proj} R$ and $\mathcal{S} = \operatorname{Syz}(f_1, \ldots, f_n)(m)$ and let $\mathcal{S}'$ be the extension of $\mathcal{O}_Y$ by $\mathcal{S}$ given by $c = \delta(f_0)$. If $\mathcal{S}$ is strongly semistable then $f_0 \in I^\star$ if and only if $\mathcal{S}'^\vee$ is not ample.

Proof. Since containment in solid closure is equivalent to the non-affineness of the torsor T corresponding to c, this follows from the previous theorem. □

In the case where $\mathcal{S} = \operatorname{Syz}(f_1, \ldots, f_n)$ is not strongly semistable one has to look at a strong Harder-Narasimhan filtration of $\mathcal{S}$. We will come back to this in Section 8.

Therefore, containment in solid closure in the strongly semistable case is controlled by the ampleness of the dualised extension bundle (or equivalently – of the forcing divisor). In particular, we need a suitable ampleness criterion that provides a bridge to dagger closure. For our purposes this will be the following

3.14 Theorem. Let Y denote a smooth projective curve over an algebraically closed field k. Let $\mathcal{G}$ denote a locally free sheaf on Y. Then the following are equivalent:

(i) The sheaf $\mathcal{G}$ is ample.

(ii) $\bar{\mu}_{\min}(\mathcal{G}) > 0$.

(iii) There exists $\varepsilon > 0$ such that for every finite k-morphism $\varphi : Y' \to Y$ of smooth projective curves and every invertible quotient sheaf $\varphi^*\mathcal{G} \to \mathcal{M} \to 0$ the inequality $\frac{\deg \mathcal{M}}{\deg \varphi} \geq \varepsilon > 0$ holds.

Proof. See [7, Theorem 2.3]. □

In characteristic zero the equivalence of (i) and (ii) is a well-known theorem of Hartshorne (see [38, Theorem 2.4]). We will not reproduce the proof of Theorem 3.14 here but we will discuss some aspects that serve as a useful heuristic for establishing a geometric criterion for dagger closure in the non-semistable case. First of all, we recall what plus closure looks like in the geometric setting:

3.15 Theorem. Assume Situation 3.6 and write $Y = \operatorname{Proj} R$, let $f_0 \in R$ be homogeneous of degree d_0 and denote $\operatorname{Syz}(f_1, \ldots, f_n)(d_0)$ by $\mathcal{S}$. Denote the extension of $\mathcal{O}_Y$ by $\mathcal{S}$ defined by $\delta(f_0)$ by $\mathcal{S}'$. Then the following are equivalent:

(i) $f_0 \in (f_1, \ldots, f_n)^{+\mathrm{gr}}$.

(ii) There exists a finite surjective morphism of smooth projective curves $\varphi : Y' \to Y$ such that $\mathbb{P}(\varphi^*\mathcal{S}'^\vee)$ has a section which is disjoint to $\mathbb{P}(\varphi^*\mathcal{S}^\vee)$.

Proof. See [5, Lemma 3.10]. □

Theorem 3.14 is a variant of Seshadri's ampleness criterion. Before stating Seshadri's criterion we recall that the ***multiplicity*** $\operatorname{mult}(C)$ of a curve C is given by $\max_{c \in C} \operatorname{mult}_c(C)$, where $\operatorname{mult}_c(C)$ is the Hilbert-Samuel multiplicity of the local ring $\mathcal{O}_{C,c}$ (see [69, Chapter V]).

3.16 Theorem. Let X be a complete variety over an algebraically closed field and D a Cartier divisor on X. Then D is ample if and only if there exists $\varepsilon > 0$ such that $D.C \geq \varepsilon \operatorname{mult}(C)$ for every integral curve C in X.

Proof. See [37, Theorem I.7.1]. □

The key to link Theorem 3.14 with 3.16 is the correspondence described in Proposition 3.8. Theorem 3.14 asserts that it is enough to look at the curves in $\mathbb{P}(\mathcal{G})$ that are induced by finite morphisms of smooth projective curves $g : Y' \to Y$. We note that the images of the Y' in $\mathbb{P}(\mathcal{G})$ may very well have singularities. The morphism $g : Y' \to Y$ corresponds to a surjection $g^*\mathcal{G} \to \mathcal{L} \to 0$, where $\mathcal{L}$ is a line bundle on Y'. Denoting the image of Y' in $\mathbb{P}(\mathcal{G})$ by C one then has (still assuming that Y' is finite over Y) $\deg \mathcal{L} = \mathcal{O}_{\mathbb{P}(\mathcal{G})}(1).C$.

Another important point is that the multiplicity is replaced by the relative degree which is easier to control. If $Y = \operatorname{Proj} R$ as in Situation 3.6 then the curves Y' are obtained from finite ring extensions of R. Also, thinking of dagger closure as an asymptotic version of plus closure Theorem 3.15 suggests that there should be some connection between Theorem 3.14 and dagger closure. This is the heuristic starting point why dagger closure should coincide with solid closure in this setting.

4 The first inclusion

The first step in establishing the desired identity $I^\star = I^{\dagger\mathrm{GR}}$ is to prove that if in Situation 3.6 a homogeneous element f_0 of degree d_0 is contained in the graded dagger closure then the dualised extension of $\mathcal{O}_{\operatorname{Proj} R}$ by $\operatorname{Syz}(f_1, \ldots, f_n)(d_0)$ defined by $\delta(f_0)$ (i. e. $\operatorname{Syz}(f_0, \ldots, f_n)(d_0)^\vee)$) is not ample. This will be done in this section, where we will make some of the ideas discussed so far precise.

For the other direction we will first need to establish a geometric criterion for dagger closure (this will be accomplished in Theorem 7.23). First, we will establish the equivalence of dagger closure and solid closure for the case where the associated syzygy bundle is strongly semistable (Theorem 7.46). Then, we will look at a strong Harder-Narasimhan filtration to deduce the result for an arbitrary syzygy bundle (this will be done in Section 8). We will continue this discussion of heuristics and ideas in Section 6.

4.1 Remark. There is no extra effort in developing the relevant notions for arbitrary vector bundles and we will do so. Moreover, if one looks at a (strong) Harder-Narasimhan filtration the quotients are not necessarily syzygy bundles with respect to the fixed embedding (see Remark 5.10). On the other hand, since we are working over a smooth curve, the notion of a syzygy bundle is not very restrictive – see Proposition 5.9 and also Proposition 7.25.

4.2 Lemma. *Let R be an $\mathbb{N}$-graded domain such that $\operatorname{Proj} R$ is covered by open sets $D_+(f)$, where $f \in R_1$, and let $R \subseteq S$ be a finite $\mathbb{Q}$-graded extension of domains. Consider S as an $\mathbb{N}$-graded domain by multiplying with a common denominator n. Then the inclusion induces a morphism $\varphi : \operatorname{Proj} S \to \operatorname{Proj} R$ and we have $\varphi^* \mathcal{O}_{\operatorname{Proj} R}(1) = \mathcal{O}_{\operatorname{Proj} S}(n)$.*

Proof. For the construction of the induced morphism confer [31, 2.8.1]. To see that the morphism is defined on all of $\operatorname{Proj} S$ we have to prove that $V_+(\varphi(R_+)) = \varnothing$. Write $S = R[x_1, \ldots, x_m]$. Each $x = x_j$ satisfies an equation $x^d = \sum_{i=0}^{d-1} a_i x^i$, where the $a_i \in R$ are homogeneous, and the degree of x in S is $\frac{\deg a_i}{d-i}$. If $\deg x = 0$ then $x \notin S_+$. We have $\deg x > 0$ if and only if $\deg a_i > 0$ for each i. Hence $x^d \in SR_+$. This shows that $\operatorname{Rad}(SR_+) = S_+$ and hence $V_+(\varphi(R_+)) = \varnothing$.

We can cover $\operatorname{Proj} R$ by open subsets of the form $D_+(f)$, where f has degree 1. Hence, $\operatorname{Proj} S$ is covered by $\varphi^{-1}(D_+(f)) = D_+(i(f))$ where $i : R \to S$ is the inclusion (cf. [31, 2.8.1]). The covering property also implies that $\mathcal{O}_{\operatorname{Proj} R}(1)$ is invertible (cf. [39, proof of Proposition II.5.12 (a)]). It is enough to show that $\varphi^* \mathcal{O}_{\operatorname{Proj} R}(1)$ is isomorphic to $\mathcal{O}_{\operatorname{Proj} S}(n)$ if we restrict to open subsets of the form $D_+(i(f))$ and that these isomorphisms are consistent on twofold intersections. Indeed, locally we have $R(1)_{(f)} = f R_{(f)}$ since $\deg f = 1$ and f is invertible in R_f. Moreover,

$$\varphi^* \mathcal{O}_{\operatorname{Proj} R}(1)|_{D_+(i(f))} = (f R_{(f)} \otimes_{R_{(f)}} S_{(f)})^\sim \text{ and } f R_{(f)} \otimes_{R_{(f)}} S_{(f)} \cong f S_{(f)}$$

which is immediately verified using the universal property of the tensor product. And these are precisely the elements of degree n in S_f.

We now show that this is consistent on twofold intersections. If g is another element of degree 1 then we have $\varphi^*\mathcal{O}(1) = (R_{(fg)}(1) \otimes_{R_{(fg)}} S_{(fg)})^\sim$ on $D_+(fg)$ and $R_{(fg)} = fR(1)_{(fg)}$. Arguing as above thus implies $R_{(fg)}(1) \otimes_{R_{(fg)}} S_{(fg)} \cong S(n)_{(fg)}$. □

4.3 Definition. Let $f : Y' \to Y$ be a finite dominant morphism of projective varieties and $\mathcal{L}$ a line bundle on Y. We call a line bundle $\mathcal{M}$ on Y' an *nth root* (or a *root*) of $\mathcal{L}$ if $\mathcal{M}^n$ is isomorphic to $f^*\mathcal{L}$, where $n \in \mathbb{N}$.

At this point we should probably note that there is a more geometric (and more general) version of the previous lemma. Namely the following

4.4 Lemma. Let X be a projective variety. Fix an ample line bundle $\mathcal{O}(1)$ and a line bundle $\mathcal{L}$. Let m be such that $\mathcal{L}(m)$ is generated by global sections and choose $N \in \mathbb{N}$. Then there exists a finite surjective morphism $f : Y \to X$ where Y is a normal projective variety such that $f^*\mathcal{L}(m) = \mathcal{M}^{\otimes N}$, where $\mathcal{M}$ is a line bundle which is generated by global sections.

Proof. This is covered in [66, proof of Theorem 3.4]. The idea is to consider the morphism $\psi : X \to \mathbb{P}^n$ defined by $\mathcal{L}(m)$ and a morphism $\alpha : \mathbb{P}^n \to \mathbb{P}^n$ such that $\alpha^*\mathcal{O}_{\mathbb{P}^n}(1) = \mathcal{O}_{\mathbb{P}^n}(N)$, e. g. $x_i \mapsto x_i^N$. Then take the fibre product of X and $\mathbb{P}^n$ over $\mathbb{P}^n$ and pull back to an irreducible component of the normalisation of the reduced fibre product that surjects on X. We illustrate the situation with the following commutative diagram.

$$\begin{tikzcd} X \times_{\mathbb{P}^n} \mathbb{P}^n \arrow[r] \arrow[d] & \mathbb{P}^n \arrow[d, "\alpha"] \\ X \arrow[r, "\psi"] & \mathbb{P}^n \end{tikzcd}$$

□

With stronger hypothesis one can also obtain a stronger result. But the proof is more difficult in this case.

4.5 Lemma. Let X be a normal projective variety over an algebraically closed field and let $\mathcal{O}_X(1)$ be a very ample invertible sheaf on X and $d \in \mathbb{N}$. Then there exists a morphism $f : X' \to X$ such that f is finite, flat (hence surjective) and $f^*\mathcal{O}_X(1) = \mathcal{M}^d$, where $\mathcal{M}$ is a very ample invertible sheaf. Moreover, if X is smooth then X' can be chosen to be smooth as well.

Proof. See [59, Corollary 1.15.1]. □

4.6 Corollary. Let X be a smooth projective variety over an algebraically closed field, $\mathcal{L}$ an invertible sheaf on X and $d \in \mathbb{N}$. Then there exists a covering $f : X' \to X$ such that X' is smooth and f is finite, flat (hence surjective) and $f^*\mathcal{L} = \mathcal{M}^d$, where $\mathcal{M}$ is an invertible sheaf.

Proof. We claim that there are very ample invertible sheaves $\mathcal{S}, \mathcal{T}$ such that $\mathcal{L} = \mathcal{S} \otimes \mathcal{T}^\vee$. Then we may apply the previous Lemma to $\mathcal{S}$ and $\mathcal{T}$ separately and deduce the result since pullback and dual commute. As to the claim, fix an ample line bundle $\mathcal{N}$ on X. Then $\mathcal{N}^n \otimes \mathcal{L}$ is globally generated for $n \gg 0$. Moreover, we find an m such that $\mathcal{N}^m$ is very ample (see [39, Theorem II.7.6]). Hence, $\mathcal{N}^{m+n} \otimes \mathcal{L}$ is very ample by [39, Ex. II.7.5 (d)]. Then we can set $\mathcal{S} = \mathcal{N}^{n+m} \otimes \mathcal{L}$ and $\mathcal{T} = \mathcal{N}^{n+m}$. Both are very ample and $\mathcal{T}^\vee \otimes \mathcal{S} = \mathcal{L}$ as desired. □

4.7 Remark. The trick needed to proof Corollary 4.6 may be found in the proof of [56, Theorem 4.1.10]. We do note that we no longer have that $\mathcal{M}$ is globally generated.

4.8 Lemma. Let R be an $\mathbb{N}$-graded domain finitely generated over a field $k = R_0$. Then there exists a finite ring extension $R \subseteq S$ such that S is a standard graded domain.

Proof. We may assume R to be normal. Fix homogeneous algebra generators $r_1, \ldots, r_n$ of R and write $d_i = \deg r_i$. Consider the polynomials $f_i = X^{d_i} - r_i$. Fix an irreducible polynomial g_1 dividing f_1. By [23, Corollary 4.12], g_1 is prime in $R[X]$ and $S_1 := R[X]/(g_1)$ is a finite extension domain such that f_1 has a root, i. e. a (homogeneous) element $x \in S_1$ such that $x^{d_1} = r_1$. Normalising S_1 and repeating this process we obtain a finite $\mathbb{N}$-graded extension domain S' where each r_i has a d_ith root which we call x_i. Then $S = k[x_1, \ldots, x_n]$ is of the desired form. □

4.9 Proposition. Assume Situation 3.6 and write $Y = \operatorname{Proj} R$. Assume furthermore that $f_0 \in R$ of degree d_0 is contained in $(f_1, \ldots, f_n)^{\dagger \mathrm{GR}}$. Then $\operatorname{Syz}(f_0, \ldots, f_n)(d_0)^\vee$ is not ample.

Proof. The condition $f_0 \in I^{\dagger \mathrm{GR}}$ means that f_0 is multiplied into $IR^{+\mathrm{GR}}$ by a sequence of homogeneous nonzero elements $\theta_m \in R^{+\mathrm{GR}}$ of degrees $\frac{1}{m}$, that is, $\theta_m f_0 = \sum_i a_{im} f_i$. Since we necessarily have $d_0 \geq \min\{d_1, \ldots, d_n\}$ we may assume that all relevant data for a given m are contained in a finite ring extension $S_m = R[\theta_m, a_{1m}, \ldots, a_{nm}] \subseteq R^{+\mathrm{GR}}$ whose minimal nonzero degree is $\frac{1}{m}$. Furthermore, applying Lemma 4.8 we may assume that the S_m are generated in degree $\frac{1}{m}$. Regrade S_m by multiplying the grading by m. We therefore have standard graded k-domains and consider $\operatorname{Proj} S_m = X_m$.

We will construct a sequence of line bundles $\mathcal{L}_m$ on smooth curves X'_m finite over Y that contradict the ampleness of $\operatorname{Syz}(f_0, \ldots, f_n)(d_0)^\vee$ using Theorem 3.14 (the X'_m will be the normalisations of the X_m).

Fix $m \in \mathbb{N}$ and omit the index. As we have $\theta f_0 \in (f_1, \ldots, f_n)S$ we obtain a relation $\theta f_0 = \sum_i a_i f_i$ of total degree $md_0 + 1$. This yields a syzygy $(\theta, a_1, \ldots, a_n)$ of $g^*(\operatorname{Syz}(f_0, \ldots, f_n)(d_0))(1)$, where $g : X \to Y$ is the morphism induced by the inclusion. Hence, we have an exact sequence

$$0 \longrightarrow \mathcal{O}_X(-md_0 - 1) \longrightarrow g^* \operatorname{Syz}(f_0, \ldots, f_n).$$

We tensor with $\mathcal{O}_X(md_0)$ and pull back to the normalisation $h : X' \to X$ and denote the composition gh by φ. We have $\deg h^*\mathcal{O}_X(-m) = \deg \varphi \deg \mathcal{O}_Y(-1)$. Passing to the saturation $\mathcal{M}$ of $h^*\mathcal{O}_X(-1)$ inside $(\varphi^* \operatorname{Syz}(f_0, \ldots, f_n))(md_0)$ we have that the degree of $\mathcal{M}$ is

$$\deg \mathcal{M} = \deg h^*\mathcal{O}_X(-1) + \deg \mathcal{T} = \frac{1}{m} \deg \varphi \deg \mathcal{O}_Y(-1) + \deg \mathcal{T},$$

where $\mathcal{T} = \mathcal{M}/h^*\mathcal{O}_X(-1)$ is a torsion sheaf. In particular, $\deg \mathcal{T} \geq 0$.

Now we can dualise to obtain an exact sequence

$$\varphi^* \operatorname{Syz}(f_0, \ldots, f_n)^\vee(-md_0) \longrightarrow \mathcal{M}^\vee \longrightarrow 0,$$

where $\deg \mathcal{M}^\vee \leq \frac{1}{m} \deg \mathcal{O}_Y(1) \deg \varphi$. Applying Theorem 3.14 therefore yields that $\operatorname{Syz}(f_0, \ldots, f_n)(d_0)^\vee$ is not ample. □

This entails

4.10 Corollary. Assume Situation 3.6 and assume moreover that $\operatorname{Syz}(f_1, \ldots, f_n)$ on $\operatorname{Proj} R$ is strongly semistable. Then we have

$$(f_1, \ldots, f_n)^{\dagger\mathrm{GR}} \subseteq (f_1, \ldots, f_n)^\star.$$

Proof. Combine Proposition 4.9 and Corollary 3.13. □

4.11 Corollary. Assume Situation 3.6 and assume that $n = 2$. Then we have the inclusion

$$(f_1, f_2)^{\dagger\mathrm{GR}} \subseteq (f_1, f_2)^\star.$$

Proof. Since $n = 2$ we have that $\operatorname{Syz}(f_1, f_2)(d_0)$ is a line bundle. In particular, it is strongly semistable and the result follows from Corollary 4.10. □

The converse inclusion of Corollary 4.10 is much harder to prove. We will come back to this in Theorem 7.46.

4.12 Example. Assume Situation 3.6 and assume that $R = k[x, y]$ so that $\operatorname{Proj} R = \mathbb{P}^1_k = \mathbb{P}^1$. Note that since $I^\star = I$ for any ideal $I \subseteq k[x, y]$ Corollary 4.10 implies that graded dagger closure is trivial in $k[x, y]$ for R_+-primary ideals with (strongly) semistable syzygy bundle.

(a) Let $f \in R$ be a homogeneous element of degree m. By a well-known theorem of Grothendieck (see e. g. [51, Theorem 1.3.1]) $\mathcal{S} = \operatorname{Syz}(f_1, \ldots, f_n)(m)$ is a direct sum of line bundles. It follows in positive characteristic that $\bar{\mu}_{\min}(\mathcal{S}) = \mu_{\min}(\mathcal{S})$. Indeed, the iterated Frobenius F^e has degree p^e and $F^{e*}(\mathcal{O}_{\mathbb{P}^1}(l)) = \mathcal{O}_{\mathbb{P}^1}(lp^e)$ for any line bundle on $\mathbb{P}^1$.

Write $\mathcal{S} = \bigoplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1}(a_i)$. If $\mu_{\min}(\mathcal{S}) \geq 0$ (e. g. if $\mathcal{S}$ is semistable of $\deg \mathcal{S} \geq 0$) we have $a_i \geq 0$ for all i. Hence, $H^1(\mathbb{P}^1, \mathcal{S}) = 0$ by Riemann-Roch (and since we know that $\dim H^0(\mathbb{P}^1, \mathcal{O}(a)) = a + 1$ for $a \geq 0$) or by Serre duality. It follows that every extension of $\mathcal{O}_Y$ by $\mathcal{S}$ splits and we have $f \in (f_1, \ldots, f_n)$ by Proposition 3.7.

Assume that $\mathcal{S}$ is semistable of $\deg \mathcal{S} < 0$ and $f \notin (f_1, \ldots, f_n)$. Then $\mathcal{S}'^\vee$ is ample (see [27, Proposition 2.1]) hence $f \notin (f_1, \ldots, f_n)^{\dagger\mathrm{GR}}$ by Proposition 4.9.

In the remaining case one would look at the Harder-Narasimhan filtration of $\mathcal{S}(m)$. But we have not developed the necessary machinery for this yet. On the other hand, since every vector bundle on $\mathbb{P}^1$ splits as a direct sum of line bundles this could also be achieved via an ad hoc argument. We also remark that one can employ purely algebraic arguments in the regular case (see Corollary 9.23).

(b) Let $I = (x^5, xy^4, y^5)$ and $\mathcal{S} = \mathrm{Syz}(x^5, xy^4, y^5)$ which splits as $\mathcal{O}_{\mathbb{P}^1}(-6) \oplus \mathcal{O}_{\mathbb{P}^1}(-9)$. To see this consider the presenting sequence

$$0 \to \mathcal{S} \to \mathcal{O}_{\mathbb{P}^1}(-5)^3 \to \mathcal{O}_{\mathbb{P}^1} \to 0.$$

Looking at the degrees and employing a gcd argument one sees that the relations of minimal degree are (up to multiplication by a unit) $y \cdot xy^4 - x \cdot y^5 = 0$ and $y^4 \cdot x^5 - x^4 \cdot xy^4 = 0$. Hence, $\mathcal{S}$ is generated (as a module) by the syzygies $(0, y, -x)$ and $(y^4, -x^4, 0)$ in degree 6 and 9 respectively.

We have $f = x^4y^3 \notin I$ since if $x^4y^3 = ax^5 + bxy^4 + cy^5$ then $a = 0$ since the x-degree of f is 4 and then $b = c = 0$ since the y-degree of f is 3. Thus the extension defined by $\delta(f) = c \in H^1(\mathbb{P}^1, \mathcal{S}(7))$ does not split by virtue of Proposition 3.7. The extension bundle $\mathcal{S}'$ given by the exact sequence

$$0 \to \mathcal{S}(7) \to \mathcal{S}' \to \mathcal{O}_{\mathbb{P}^1} \to 0$$

is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)^2$. Indeed, taking into account that $\mathcal{S}' = \mathrm{Syz}(x^4y^3, x^5, xy^4, y^5)$ one verifies that $(0, 0, y, -x)$, $(-y, 0, x^3, 0)$ and $(x, -y^3, 0, 0)$ are three R-linearly independent relations of minimal degree[4]. Looking at the dual of this exact sequence we see that we have a surjection $\mathcal{S}'^\vee \to \mathcal{S}^\vee(-7) \to \mathcal{O}_{\mathbb{P}^1}(-1)$, hence $\mathcal{S}'^\vee$ is not ample by Theorem 3.14 (or since it has the non-ample direct summand $\mathcal{O}_{\mathbb{P}^1}(-1)$)). Dualising again and twisting by $\mathcal{O}_{\mathbb{P}^1}(6)$ we get $0 \to \mathcal{O}_{\mathbb{P}^1} \to \mathcal{S}'(-1)$ which corresponds to the the global section $(0, 0, y, -x)$ of $\mathcal{S}'(-1)$. On the ring level this yields the relation $0 \cdot x^4y^3 = y \cdot xy^4 - x \cdot y^5$ and of course $\nu(0) = \infty$ so that we do not get a relation of small order.

Note that already $\mathcal{S}^\vee$ surjects onto the line bundle $\mathcal{O}_{\mathbb{P}^1}(-1)$ which contradicts the ampleness of $\mathcal{S}'^\vee$. This in turn means that the curve to which $\mathcal{S}'^\vee \to \mathcal{O}_{\mathbb{P}^1}(-1) \to 0$ corresponds lies in $\mathbb{P}(\mathcal{S}^\vee)$.

Once we know that $f \notin I^{\dagger\mathrm{GR}}$ we may ask what the infimum over the $\varepsilon > 0$ is such that there is $a \in R^{+\mathrm{GR}}$ with $\nu(a) < \varepsilon$ which multiplies f into $IR^{+\mathrm{GR}}$. We will compute ε for this example in Example 7.50.

5 Some results on section rings

In order to prove the inclusion $I^\star \subseteq I^{\dagger\mathrm{GR}}$ we will need, among other things, a way to construct ring extensions and syzygies of small order starting from line bundles on which the dual of a syzygy bundle surjects. For this we need some results on section rings which we collect in this section.

5.1 Definition. Let X be a scheme and $\mathcal{L}$ an invertible sheaf. The graded ring $\bigoplus_{n \geq 0} \Gamma(X, \mathcal{L}^n)$ is called the *section ring* of $\mathcal{L}$. (We refer to [30, Chapitre 0, 5.4.6] for a proof that this actually is a ring.)

[4] Of course, this is exactly the problem of determining the syzygies of the ideal (x^4y^3, x^5, xy^4, y^5) in $k[x, y]$ and can also be accomplished by the use of a computer algebra system.

Section rings are much better behaved then arbitrary graded rings. In particular, if S is the section ring of an ample invertible sheaf on a projective variety then $S(1)^\sim$ is invertible. Moreover if the variety is normal then so is S (see Proposition 5.7 below).

Another issue in working with arbitrary graded extension domains $R \subseteq S$ is that we cannot ensure that S is standard graded and normal – the normalisation of a standard graded ring may no longer be standard graded. However, if S is the section ring of a globally generated ample line bundle $\mathcal{L}$ then $\operatorname{Proj} S$ is covered by standard open sets $D_+(f)$, where the $f \in S$ are of degree 1. This slightly weaker condition is stable under finite pullbacks and is a good enough replacement for standard graded in virtually every situation.

5.2 Lemma. Let Y be a projective variety with a fixed ample line bundle $\mathcal{O}_Y(1)$. Then $\operatorname{Proj} \bigoplus_{n\geq 0} \Gamma(Y, \mathcal{O}_Y(n)) = Y$.

Proof. We have an open immersion $Y \to \operatorname{Proj} R$ (cf. [31, Théorème 4.5.2]) which is actually an isomorphism. Indeed, the morphism is proper and hence surjective. $\square$

The following Lemma is contained in [5, Lemma 3.10] but Brenner does not provide a proof.

5.3 Lemma. Let $f : Y' \to Y$ be a finite dominant morphism of projective varieties over a field k. Fix an ample line bundle $\mathcal{O}(1)$ on Y. Then

$$\bigoplus_{n\geq 0} \Gamma(Y, \mathcal{O}(n)) \subseteq \bigoplus_{n\geq 0} \Gamma(Y', f^*\mathcal{O}(n))$$

is a finite extension of graded domains.

Proof. Call these rings R and S. To begin with, R and S are domains. For if $s, t \in S$ are nonzero and homogeneous, where $s \in \Gamma(Y', f^*\mathcal{O}(m)), t \in \Gamma(Y', f^*\mathcal{O}(n))$, then multiplication $s \cdot t$ corresponds to a morphism $f^*\mathcal{O}(n) \to f^*\mathcal{O}(n) \otimes f^*\mathcal{O}(m) = f^*\mathcal{O}(n+m)$ induced by $\mathcal{O} \to f^*\mathcal{O}(m)$. The latter morphism is injective and since tensoring with $f^*\mathcal{O}(n)$ is exact we have that S is a domain. One shows similarly that R is a domain.

Note that $f^*\mathcal{O}(1)$ is ample by [39, Ex. III.5.7 (d)] or [37, Proposition 4.4] (the morphism is dominant hence surjective since it is proper). As the line bundles are ample, R and S are k-algebras of finite type (see e. g. [22, Proposition 9.2]). We have injective morphisms $\mathcal{O}(n) \to f_*f^*\mathcal{O}(n)$ (this follows since $\mathcal{O}_{f(y)} \to f_*\mathcal{O}_y$ is injective for $y \in Y'$ cf. [39, Ex. I.3.3 (c)]). Moreover, we have a commutative diagram

$$\begin{tikzcd}
\mathcal{O}(n) \otimes \mathcal{O}(m) \arrow[r] \arrow[d] & \mathcal{O}(n+m) \arrow[d] \\
f_*f^*\mathcal{O}(n) \otimes f_*f^*\mathcal{O}(m) \arrow[r] & f_*f^*\mathcal{O}(n+m).
\end{tikzcd}$$

Taking global sections shows that we have an injective k-linear ringhomomorphism as desired. This is also covered in [30, Chapitre 0, 5.4.6].

By homogeneous Noether normalisation ([21, Theorem 1.5.17]) we find homogeneous elements $x_0, \ldots, x_d$ of R such that the $D_+(x_i)$ cover $\operatorname{Proj} R = Y$. Then we have that the $f^{-1}(D_+(x_0)) = D_+(i(x_0)), \ldots, f^{-1}(D_+(x_d)) = D_+(i(x_d))$ cover $\operatorname{Proj} S$ where $i : R \to S$ denotes the inclusion. Thus, again by homogeneous Noether normalisation, $R \subseteq S$ is finite. □

5.4 Proposition. Let $f : Y' \to Y$ be a finite dominant morphism of projective varieties and let $\mathcal{O}_Y(1)$ be an ample line bundle on Y. Fix an ample line bundle $\mathcal{L}$ on Y' such that $\mathcal{L}^k = f^*\mathcal{O}_Y(l)$ for some k, l in $\mathbb{N}$. Choose a minimal k such that $\mathcal{L}^k = f^*\mathcal{O}_Y(l)$ and identify these two line bundles along a fixed isomorphism. Then

$$S = \bigoplus_{n,m \geq 0} \Gamma(Y', \mathcal{L}^m \otimes f^*\mathcal{O}_Y(n)) / \sim$$

is a finite extension domain of $R = \bigoplus_{n \geq 0} \Gamma(Y, \mathcal{O}_Y(n))$, where $\sim$ denotes the identification made above. Moreover, it is a graded ring extension if we assign elements of $\mathcal{L}^m(n)$ the degree $\frac{ml}{k} + n$.

Proof. By Lemma 5.3 we have that $\bigoplus_{n \geq 0} \Gamma(Y, \mathcal{O}_Y(n)) \subseteq \bigoplus_{n \geq 0} \Gamma(Y', f^*\mathcal{O}_Y(n))$ is a finite extension of graded domains. In particular, the original extension $R \subseteq S$ is integral. To prove that the extension is finite it remains to show that

$$\bigoplus_{t \geq 0} \Gamma(Y', \mathcal{L}^s(t)) \text{ is finite over } \bigoplus_{n \geq 0} \Gamma(Y', f^*\mathcal{O}_Y(n)) =: T \text{ for } s = 1, \ldots, k-1.$$

To this end, we may replace Y' by its normalisation. By [39, Ex. II.5.14], a d-uple embedding $T^{(d)}$ of T is then projectively normal for $d \gg 0$. Note that T is finite over $T^{(d)}$ (use [23, Corollary 4.5]) and that $T^{(d)}$ is standard graded. This in turn allows us to invoke [39, Ex. II.5.9]. Thus for each s we have finitely many $\Gamma(Y', \mathcal{L}^s(t))$ which are finite dimensional vector spaces and a finite module. Hence the extension is finite. □

5.5 Remark. Let R be an $\mathbb{N}$-graded domain finitely generated over a field $R_0 = k$ of dimension $d \geq 2$ and write $Y = \operatorname{Proj} R$. Passing to a finite ring extension one may assume that R is normal and that $\operatorname{Proj} R$ is covered by standard open sets coming from elements of degree 1. The latter condition implies that the $\mathcal{O}_Y(n)$ are invertible. Since R is normal and $d \geq 2$ one has $\Gamma(Y, \mathcal{O}_Y(n)) = R_n$. Hence, the ring extensions obtained in Proposition 5.4 are all contained in $R^{+\mathrm{GR}}$ since they are finite over R and extend the grading.

Conversely, looking at the normalisations of the rings S_m defined in the proof of Proposition 4.9 we see that every element of $R^{+\mathrm{GR}}$ is obtained in this way. Also with a similar argument one sees that one can obtain every element of $R^{+\mathrm{gr}}$ if one looks at the ring extensions $\bigoplus_{n \geq 0} \Gamma(Y', f^*\mathcal{O}_Y(n))$ for finite dominant morphisms $f : Y' \to Y$.

5.6 Proposition. Let Y be a projective variety with an ample line bundle $\mathcal{L}$. Then $\operatorname{Proj} \bigoplus_{n \geq 0} \Gamma(Y, \mathcal{L}^n)$ can be covered by a finite number of $D_+(f)$ where $f \in \Gamma(Y, \mathcal{L})$ if and only if $\mathcal{L}$ is generated by global sections.

Proof. By Lemma 5.2 there is a canonical isomorphism $Y \to \operatorname{Proj} \bigoplus_{n\geq 0} \Gamma(Y, \mathcal{L}^n)$ which we will use to identify Y and $\operatorname{Proj} \bigoplus_{n\geq 0} \Gamma(Y, \mathcal{L}^n)$.

First of all, we show that $Y_f = \{y \in Y : f \notin \mathfrak{m}_y \mathcal{L}_y\}$ and $D_+(f) = \{y \in Y : f \notin y\}$ are equal for $f \in \Gamma(Y, \mathcal{L})$. See also [31, Proposition 2.6.3] (especially for the second inclusion).

Fix a homogeneous prime P in $S = \bigoplus_{n\geq 0} \Gamma(Y, \mathcal{L}^n)$ and assume that $f \notin P$, where $\deg f = 1$. Then f is a generator of $\Gamma(D_+(f), \mathcal{L}) = S(1)_{(f)}$. And hence f is a generator of $\mathcal{L}_P$. If $f \in \mathfrak{m}_P \mathcal{L}_P$ then $\mathfrak{m}_P \mathcal{L}_P = \mathcal{L}_P$ and we get a contradiction by Nakayama's lemma.

For the other inclusion observe that $D_+(g)$ for finitely many $g \in S_d$ cover Y for some d since $\mathcal{L}$ is ample (cf. [31, Théorème 4.5.2]) and Y is quasi-compact. Consider $P \in D_+(g) \cap Y_f$ where $\deg f = 1$ – in particular, f generates the stalk $\mathcal{L}_P$. Hence, both g and f^d are generators of $\mathcal{L}_P^d$ and we therefore have $\frac{g}{f^d} f^d = g$. Consequently, f^d (and then f) is a unit in S_P since g is. And this means $f \notin P$.

We thus have $D_+(f) = Y_f$. Moreover, the Y_f cover Y if and only if $\mathcal{L}$ is generated by global sections. And since Y is quasi-compact a finite number of them will do. □

5.7 Proposition. Let X be a projective variety over a field k, $\mathcal{L}$ an ample line bundle on X and S its corresponding section ring.

(a) If $\dim S \geq 2$ then $\operatorname{depth} S \geq 2$.

(b) There exist homogeneous elements $x_1, \dots, x_r$ generating an S_+-primary ideal such that the $\mathbb{Z}$-graded ring S_{x_i} is isomorphic to $S_{(x_i)}[T, T^{-1}]$ for each i, where T is an indeterminate of degree 1.

(c) If X is normal of dimension ≥ 1 then S is normal as well.

(d) If X is non-singular then S is regular except possibly at S_+.

Proof. We follow [52, Proposition 2.1]. Denote the dimension of S by d. For (a) consider the extended Čech complex (cf. [21, Paragraph before Proposition 3.5.5])

$$0 \to S \to \bigoplus_i S_{x_i} \to \bigoplus_{i<j} S_{x_i x_j} \to \dots,$$

where $x_1, \dots, x_d$ is a homogeneous system of parameters (see [21, Proposition 1.5.11] for existence of such systems). Note that we have $\bigoplus_{n\in\mathbb{Z}} H^0(X, \mathcal{L}^n) = S$ by [39, Ex. III.7.1]. And we also have $S(n)^\sim = \mathcal{L}^n$ by a careful inspection of the proof of [ibid., Proposition II.5.15]. From the extended Čech complex one can extract the exact sequence relating local and global cohomology (cf. [23, Theorem A4.1]):

$$0 \to H^0_{S_+}(S) \to S \to \bigoplus_{n\in\mathbb{Z}} H^0(X, \mathcal{L}^n) \to H^1_{S_+}(S) \to 0.$$

We therefore have that both $H^0_{S_+}(S)$ and $H^1_{S_+}(S)$ must vanish. Hence, $\operatorname{depth} S \geq 2$ (cf. [21, Propositions 3.5.4, 3.6.4 and Theorem 3.6.3]).

To prove (b) choose homogeneous elements $x_1, \dots, x_r$ such that the $D_+(x_i)$ cover X and such that the $\mathcal{L}|_{D_+(x_i)}$ are free. It follows that the $\mathcal{L}^n$ are free over the $D_+(x_i)$ and if T is a generator for $\Gamma(D_+(x_i), \mathcal{L})$ then T^n is a generator for $\Gamma(D_+(x_i), \mathcal{L}^n)$. Since $\Gamma(D_+(x_i), \mathcal{L}^n) = S(n)_{(x_i)}$ we have the desired isomorphism.

In order to prove (c) note that since we assume X to be normal all $S_{(x_i)}$ are normal. It follows that the $S_{(x_i)}[T, T^{-1}] = S_{x_i}$ are normal as well. Moreover, the $D(x_i)$ cover $\operatorname{Spec} S \setminus S_+$ (to see this consider the affine cone $\operatorname{Spec} S$ over $\operatorname{Proj} S$ – cf. [64, Example 5.A.14]). Hence, S localised at a prime is normal except possibly for S_+. But since S_{S_+} has depth ≥ 2 by part (a) the conditions (R1) and (S2) are satisfied for S_{S_+}. We conclude that S is normal.

Finally, for (d) note that $S_{(x_i)}[T, T^{-1}] = S_{(x_i)}[T]_T$ is regular if $S_{(x_i)}$ is regular. As the $D(x_i)$ cover $\operatorname{Spec} S \setminus S_+$ the result follows. □

5.8 Lemma. Let Y denote a smooth projective curve over an algebraically closed field. Let $\mathcal{S}$ denote a locally free sheaf of rank r which is globally generated by $r + k$ sections, $k \geq 1$. Then it is also globally generated by $r + 1$ global sections.

Proof. See [10, Lemma 2.3]. □

5.9 Proposition. Let Y denote a smooth projective curve over an algebraically closed field and let $\mathcal{S}$ denote a locally free sheaf of rank r. Assume furthermore that $\det \mathcal{S} = \mathcal{O}_Y(n)$ for some integer n, where $\mathcal{O}_Y(1)$ is an ample invertible sheaf. Then some twist of $\mathcal{S}$ is a syzygy bundle with respect to $R = \bigoplus_{l \geq 0} \Gamma(Y, \mathcal{O}_Y(l))$.[5]

Proof. Since $\mathcal{O}_Y(1)$ is ample, we have that $\mathcal{S}^\vee(m)$ is generated by global sections for some $m \geq 0$. Hence, by Lemma 5.8 we obtain an exact sequence $0 \to \mathcal{L} \to \mathcal{O}_Y^{r+1} \to \mathcal{S}^\vee(m) \to 0$, where $\mathcal{L}$ is a line bundle. By [39, Ex. II.5.16 (d)] we have an isomorphism $\mathcal{L} = \mathcal{O}_Y(-mr) \otimes (\det \mathcal{S}) = \mathcal{O}_Y(n - mr)$. Therefore, we get an exact sequence $0 \to \mathcal{S} \to \mathcal{O}_Y(m)^{r+1} \to \mathcal{O}_Y(m(r+1) - n) \to 0$. Twisting by $\mathcal{O}_Y(n - m(r+1))$ yields that $\mathcal{S}(n - m(r+1))$ is a syzygy bundle. □

5.10 Remark. We mentioned in Remark 4.1 that the quotients of the Harder-Narasimhan filtration of a given syzygy bundle may not be syzygy bundles with respect to the fixed embedding. We now want to show that it may even occur that the quotients are non-torsion elements of $\operatorname{Pic}^0 Y$, where Y is the smooth base curve. So let Y be a smooth projective curve over an algebraically closed field k.

To see that such line bundles do exist assume that k is uncountable and the genus g of Y is 1. We then have an isomorphism $Y \to \operatorname{Pic}^0 Y$ (see [39, Theorem IV.4.11]). Let $N_Y : Y \to Y$ be the multiplication by N map. Then torsion elements of $\operatorname{Pic}^0 Y$ correspond to elements of the kernels of the morphisms N_Y, which are finite for each N (see [39, Remark IV.4.8.1]). In particular, the torsion elements are countable while Y is uncountable (see [39, Ex. I.4.8 (a)]).

Likewise, the existence of non-torsion elements of $\operatorname{Pic}^0 Y$ for curves of genus $g \geq 2$ depends on the base field. But since the canonical morphism of Y to its Jacobian variety is no longer an isomorphism the situation is more complicated. If $\operatorname{char} k \nmid n$

[5] I am indebted to Almar Kaid for pointing out this proposition to me.

then the group of n-torsion points on $\operatorname{Pic}^0 Y$ consists of n^{2g} points, where g is the genus of Y and the group of n-torsion points is always finite (see [63, II.6 Proposition at the end of §6]). In particular, such $\mathcal{L}$ exist if k is uncountable.

Fix a line bundle $\mathcal{M}$ of positive degree and $\mathcal{L} \in \operatorname{Pic}^0 Y$ a non-torsion element. Let $0 \to \mathcal{M} \to \mathcal{E} \to \mathcal{L} \to 0$ be an extension. Now take a line bundle $\mathcal{N}$ of negative degree such that $\deg \mathcal{M} \neq -\deg \mathcal{N}$. Consider an extension $0 \to \mathcal{E} \to \mathcal{S} \to \mathcal{N} \to 0$. Then $\deg \det \mathcal{S} = \deg \mathcal{M} + \deg \mathcal{L} + \deg \mathcal{N} \neq 0$ so that $\mathcal{S}$ is a twisted syzygy bundle by virtue of Proposition 5.9 – $\mathcal{M} \otimes \mathcal{L} \otimes \mathcal{N}$ has nonzero degree, hence is ample or the dual of an ample line bundle. Moreover, $\mathcal{M} \subset \mathcal{E} \subset \mathcal{S}$ is a strong Harder-Narasimhan filtration of $\mathcal{S}$. Indeed, $\mathcal{E}/\mathcal{M} = \mathcal{L}$ and $\mathcal{S}/\mathcal{E} = \mathcal{N}$ are both strongly semistable by Proposition 2.27 (a) and the slopes are strictly decreasing.

We also note that the first extension splits if $g = 1$ (this follows from Riemann-Roch – any line bundle of positive degree is non-special in this case).

It is actually more interesting to have an example of a vector bundle $\mathcal{S}$ such that $\mathcal{L}$ is the first nonzero term in a strong Harder-Narasimhan filtration (as we will see in the proof of Theorem 8.3). This can e. g. be accomplished by considering an extension $0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{N} \to 0$, where $\mathcal{N}$ is a line bundle of negative degree.

6 First steps towards the converse inclusion

6.1 Heuristics

We will now give an outline of the ideas for proving the inclusion $I^\star \subseteq I^{\dagger \mathrm{GR}}$ in Situation 3.6 under the additional assumption that $\operatorname{Syz}(f_1, \ldots, f_n)$ on $Y = \operatorname{Proj} R$ is strongly semistable. So let f_0 be homogeneous of degree d_0 and assume that $f_0 \in I^\star$. That is, the torsor T defined by $\delta(f_0)$ is not affine. By Theorem 3.12, this is equivalent to the fact that the forcing divisor is not ample. This yields, by Theorem 3.14, a sequence of line bundles $\mathcal{L}_m$ on smooth projective curves Y_m with finite morphisms $\varphi_m : Y_m \to Y$ such that

$$\varphi_m^* \operatorname{Syz}(f_0, \ldots, f_n)^\vee(-d_0) \to \mathcal{L}_m \to 0$$

is exact and such that $\frac{\deg \mathcal{L}_m}{\deg \varphi_m}$ is arbitrarily small for m sufficiently large (possibly negative). Dualising this sequence one obtains subbundles $\mathcal{L}_m^\vee$ of $\varphi_m^* \operatorname{Syz}(f_0, \ldots, f_n)(d_0)$ which should play the role of the saturation of some dualised $\mathcal{O}(1)$. Thus yielding syzygies $(a_0, \ldots, a_n)$ after a twist which then correspond to relations $-a_0 f_0 = \sum_i a_i f_i$. But there are some difficulties which we will have to handle.

First of all, the degrees of the $\mathcal{L}_m$ should all be non-negative. Moreover, one needs to make sure that $a_0 \neq 0$ – the heuristic here is that if it were zero then this would be a syzygy which came from $\operatorname{Syz}(f_1, \ldots, f_n)(d_0)$. The case $\deg \mathcal{L}_m < 0$ for some m also only happens if the image in $\mathbb{P}(\operatorname{Syz}(f_0, \ldots, f_n)(d_0)^\vee)$ of the curve corresponding to the surjection onto $\mathcal{L}_m$ lies in the support of the forcing divisor (see also Example 4.12 (b)). Also note that $\deg \mathcal{L}_m = 0$ for some m means that we are in a situation where f_0 is contained in the (graded) plus closure (cf. Theorem 3.15 or [5, Lemma 3.10]). We remark that this support condition really is an issue a priori, since if f_0 is contained in the solid closure (and not in the ideal itself and in positive characteristic $F^{e*}(\delta(f_0)) \neq 0$

for all $e > 0$) then the degree of $\mathrm{Syz}(f_1, \ldots, f_n)(d_0)$ is ≥ 0. Hence, its dual is not ample and this means that there are sequences of line bundles which contradict the ampleness of $\mathrm{Syz}(f_0, \ldots, f_n)(d_0)^\vee$ that factor through $\mathrm{Syz}(f_1, \ldots, f_n)(d_0)^\vee$.

Therefore, we need to make sure that we can choose the $\mathcal{L}_m$ such that the images in $\mathbb{P}(\mathrm{Syz}(f_0, \ldots, f_n)(d_0)^\vee)$ of the curves to which they correspond[6] do not lie in the support of the forcing divisor. There are more technical difficulties with which we will have to deal. For example, how to obtain global sections while controlling the degree of the twisting bundle. More precisely, $\mathcal{L}_m^\vee$ is defined on some smooth curve Y_m finite over Y. The morphism $Y_m \to Y$ induces a finite ring extension $R \subseteq S$ via $\varphi_m^* \mathcal{O}_Y(1)$ (cf. Proposition 5.3). Hence, $\mathcal{L}_m^\vee \otimes \mathcal{O}_{\mathrm{Proj}\, S}(t)$ for t sufficiently large yields syzygies. But we have to find syzygies of small order and hence need to bound t. What is more, we need that $\mathcal{O}_{\mathrm{Proj}\, S}(t)$ induces a $\mathbb{Q}$-graded ring extension. Hence, we need that there is a suitable root of $\mathcal{O}_Y(1)$ on Y_m which yields a $\mathbb{Q}$-graded extension domain via Proposition 5.4.

This is the geometric idea of the proof. As it turns out it will be better to work with the cohomology class c that defines the extension $\mathcal{S} \subset \mathcal{S}'$ in order to obtain suitable line bundles $\mathcal{L}_m$.

6.1 Remark. Note that the support condition is not so much of an issue in the parameter case. In this case the forcing divisor is a projective curve on a ruled surface and the e-invariant of this ruled surface is ≥ 0 (cf. [5, Lemma 5.9 (iii)] – again assuming that $f_0 \notin I$ and $F^{e*}(\delta(f_0)) \neq 0$ for all $e > 0$). Then one can use the moving lemma [39, Lemma V.1.2] together with [39, Proposition V.2.20] to construct bundles $\mathcal{L}_m$ with $\deg \mathcal{L}_m / \deg \varphi_m$ arbitrarily small for m large which do not lie in the support of the forcing divisor. See Proposition 7.29 for an elaboration of this idea.

6.2 Remark. We remark that one cannot settle the support condition by just looking at numerical equivalence classes if $n \geq 3$. To see this assume Situation 3.6 write $Y = \mathrm{Proj}\, R$ and assume that $\mathcal{S} = \mathrm{Syz}(f_1, \ldots, f_n)(d_0)$ is strongly semistable. Write $\mathcal{S}' = \mathrm{Syz}(f_0, \ldots, f_n)(d_0)$, where f_0 is a homogeneous element of degree d_0 and assume that $\deg \mathcal{S} > 0$. The dual of $\mathcal{S}'$ is then given by the exact sequence

$$0 \to \mathcal{O}_Y \to \mathcal{S}'^\vee \to \mathcal{S}^\vee \to 0.$$

Since both $\mathcal{S}^\vee$ and $\mathcal{O}_Y$ are strongly semistable and since $\deg \mathcal{S}^\vee < 0$, the inclusion $\mathcal{O}_Y \subset \mathcal{S}'^\vee$ is a strong Harder-Narasimhan filtration of $\mathcal{S}'^\vee$.

Now, in [25, Lemma 2.3] Fulger shows that one has an isomorphism of the closure of the effective cones of 1-cycles of $\mathbb{P}(\mathcal{S}^\vee)$ and $\mathbb{P}(\mathcal{S}'^\vee)$. Hence, one cannot discern by numerical properties whether a given curve lying on $\mathbb{P}(\mathcal{S}'^\vee)$ is already contained in $\mathbb{P}(\mathcal{S}^\vee)$. Note that Fulger proves this only in characteristic zero but this result is probably true more generally in arbitrary characteristic if one looks at a strong Harder-Narasimhan filtration.

If $\deg \mathcal{S} = 0$ then $\mathcal{S}'^\vee$ itself is strongly semistable by Proposition 2.27 (b) and Example 2.28 (a). By [25, Lemma 2.2], which also holds in characteristic $p > 0$

[6] Here and elsewhere we employ slight abuse of notation. Of course, the curve corresponds to a surjection onto $\mathcal{L}_m$.

if one assumes strong semistability, the closure of the effective cone of 1-cycles is spanned by $\xi'^{n-1}, \xi'^{n-2}f'$, where f' is the class of a fibre and ξ' is the class of $\mathcal{O}_{\mathbb{P}(\mathcal{S}'^\vee)}(1)$. Furthermore, we have a closed immersion $i : \mathbb{P}(\mathcal{S}^\vee) \to \mathbb{P}(\mathcal{S}'^\vee)$ and denoting $\mathcal{O}_{\mathbb{P}(\mathcal{S}^\vee)}(1)$ by ξ one has that $i^*\xi' = \xi$ and $i^*f' = f$, where f denotes a fibre in $\mathbb{P}(\mathcal{S}^\vee)$. We want to show that $i_*\xi^{n-2}$ and $i_*(\xi^{n-3}f)$ are numerically equivalent to ξ'^{n-1} and $\xi'^{n-2}f'$ respectively. So let $D = a\xi' + bf'$ be a divisor in $\mathbb{P}(\mathcal{S}'^\vee)$. Then we have $i_*\xi^{n-2}.D = i_*(\xi^{n-2}.i^*D)$ by the projection formula and this is equal to $b + a\xi'^n = \xi'^{n-1}.D$. Similarly $i_*\xi^{n-3}f.D = a = \xi'^{n-2}f.D$.

6.3 Example. We now want to provide an example of an extension $0 \to \mathcal{S} \to \mathcal{S}' \to \mathcal{O} \to 0$ where we have curves with positive intersection lying in the support of $\mathbb{P}(\mathcal{S}^\vee)$ that contradict the ampleness of $\mathcal{S}'^\vee$.

Let k be an algebraically closed field and $Y = \mathbb{P}^1_k$. Consider $\mathcal{S} = \mathcal{O}_Y \oplus \mathcal{O}_Y$ and the short exact sequence $0 \to \mathcal{O}_Y(-1) \to \mathcal{S} \to \mathcal{O}_Y(1) \to 0$ (this is just a twist of the defining sequence of the cotangent bundle – cf. [39, Theorem II.8.13]).

Consider the homomorphism $k[x,y] \to k[s,t]$ defined by $x \mapsto s^d, y \mapsto t^d$. This is a finite injective homomorphism and if we attach to both rings the ordinary grading we obtain a finite dominant morphism $\varphi_d : Y' \to Y$ such that $\varphi^*\mathcal{O}_Y(1) = \mathcal{O}_{Y'}(d)$ by virtue of Lemma 4.2. Note that both Y' and Y are isomorphic to $\mathbb{P}^1_k$. Also note that this idea already occurred in the proof of Lemma 4.4

Of course, $H^1(Y, \mathcal{S}) = 0$ so that all extensions of $\mathcal{O}_Y$ by $\mathcal{S}$ split. Dualising this sequence we have a surjection $\mathcal{S}^\vee \to \mathcal{O}_Y(1)$. Pulling back along morphisms φ_d we obtain surjections $\varphi_d^*\mathcal{S}^\vee \to \mathcal{O}_{Y'}(1)$ so that $\frac{\deg \mathcal{O}_{Y'}(1)}{\deg \varphi_d} = \frac{1}{d}$. Therefore, we have curves with positive intersection lying in the support of $\mathbb{P}(\mathcal{S}^\vee)$ which contradict the ampleness of $\mathcal{S}'^\vee$. Consider now $\mathcal{E} = \mathcal{S} \oplus \mathcal{O}_Y(-2)$ (which is of course no longer semistable) then $\dim H^1(Y, \mathcal{E}) = 1$ by Riemann-Roch and we still have surjections $\varphi_d^*\mathcal{E}^\vee \to \mathcal{O}_Y(1)$ as above.

Alternatively, one can pull back the situation to an elliptic curve Y' (a choice of a transcendental element in $K(Y')/k$ defines a finite morphism $Y' \to Y$). Then $\dim H^1(Y', \mathcal{S}) = 2$ and the multiplication by N map can be used as a substitute for φ_d.

Note that both $\mathcal{S}$ and $\mathcal{E}$ (and their pullbacks to Y') are actually twisted syzygy bundles by virtue of Proposition 5.9. We have presenting sequences

$$0 \to \mathcal{S} \to \mathcal{O}_Y(m)^3 \to \mathcal{O}_Y(3m) \to 0 \text{ and } 0 \to \mathcal{E} \to \mathcal{O}_Y(m)^4 \to \mathcal{O}_Y(3m-2) \to 0.$$

Choosing $m = 1$ we have surjective connecting homomorphisms

$$\delta_{\mathcal{S}} : H^0(Y, \mathcal{O}_Y(3)) \to H^1(Y, \mathcal{S}) \text{ and } \delta_{\mathcal{E}} : H^0(Y, \mathcal{O}_Y(1)) \to H^1(Y, \mathcal{E}).$$

Hence, every class is realised via an element that stems from the section ring induced by $\mathcal{O}_Y(1)$. This works similarly if we pull back to Y'.

6.2 Two special cases

We shall now prove that we can indeed choose our curves so that they do not lie in the support of the divisor in the strongly semistable case if Y is elliptic or if the characteristic of the base field k is positive. We will need a considerable amount of machinery before we can also prove this in general in characteristic zero (see Proposition 7.43).

6.4 Proposition. Let $\mathcal{S}$ be a strongly semistable locally free sheaf of $\deg \mathcal{S} \geq 0$ on a smooth projective curve Y over an algebraically closed field k of characteristic $p > 0$. Fix an extension $0 \to \mathcal{S} \to \mathcal{S}' \to \mathcal{O}_Y \to 0$ of locally free sheaves.

Then for every $\varepsilon > 0$ there exists a finite morphism $\varphi : Y' \to Y$ and a line bundle $\mathcal{L}$ on Y' such that we have a surjection $\varphi^* \mathcal{S}'^\vee \to \mathcal{L} \to 0$ where $0 \leq \frac{\deg \mathcal{L}}{\deg \varphi} < \varepsilon$. Moreover, we can choose $\mathcal{L}$ so that it does not factor through $\varphi^* \mathcal{S}^\vee$.

Proof. Choose a finite morphism $\varphi : Y' \to Y$, where Y' is a smooth curve and fix any line bundle $\mathcal{L}'$ of positive degree on Y' such that $0 < \frac{\deg \mathcal{L}'}{\deg \varphi} < \varepsilon$ (for instance, if $1/\deg \varphi < \varepsilon$ take the line bundle corresponding to the Weil divisor P, where $P \in Y'$ is a point). Consider the pullback by φ of the short exact sequence defining the extension and tensor with $\mathcal{L}'$. Taking cohomology we thus have an exact sequence

$$0 \to H^0(Y', \varphi^* \mathcal{S} \otimes \mathcal{L}') \to H^0(Y', \varphi^* \mathcal{S}' \otimes \mathcal{L}') \to H^0(Y', \mathcal{L}') \to H^1(Y', \varphi^* \mathcal{S} \otimes \mathcal{L}').$$

Pulling back the whole situation by a sufficiently high power of the k-linear Frobenius we may assume that $\deg \mathcal{L}' > 2g(Y') - 2$. By Serre duality $H^1(Y', \varphi^* \mathcal{S} \otimes \mathcal{L}') = \operatorname{Ext}^0(\varphi^* \mathcal{S} \otimes \mathcal{L}', \omega_{Y'})^\vee$. Now, since $\mathcal{S}$ is strongly semistable of non-negative degree we have

$$\mu_{\min}(\varphi^* \mathcal{S} \otimes \mathcal{L}') = \mu(\varphi^* \mathcal{S} \otimes \mathcal{L}') = \mu(\varphi^* \mathcal{S}) + \deg \mathcal{L}' > 2g(Y') - 2 = \mu_{\max}(\omega_{Y'}).$$

Hence, the above Ext vanishes by Proposition 2.27 (d). Also $H^0(Y', \mathcal{L}') \neq 0$ since $H^1(Y, \mathcal{L}') = 0$ and $\deg \mathcal{L}' \geq g$.

This implies that we can choose a global section s in $\varphi^* \mathcal{S}' \otimes \mathcal{L}'$ that does not stem from a global section in $\varphi^* \mathcal{S} \otimes \mathcal{L}'$. This defines a morphism $0 \to \mathcal{L}'^\vee \to \varphi^* \mathcal{S}'$ which does not factor through $\varphi^* \mathcal{S}$. Passing to the saturation $\mathcal{L}^\vee$ of $\mathcal{L}'^\vee$ this defines a morphism $\varphi^* \mathcal{S}'^\vee \to \mathcal{L} \to 0$ with $\frac{\deg \mathcal{L}}{\deg \varphi} < \varepsilon$ which a fortiori does not factor through $\varphi^* \mathcal{S}^\vee$. In particular, $\deg \mathcal{L}$ is non-negative for otherwise the curve in $\mathbb{P}(\varphi^* \mathcal{S}'^\vee)$ corresponding to $\mathcal{L}$ would lie in $\mathbb{P}(\varphi^* \mathcal{S}^\vee)$. $\square$

Note that in the proof of Proposition 6.4 we only imposed a numerical condition on $\mathcal{L}'$. Therefore in Situation 3.6 we can for instance choose $\mathcal{L}'$ as roots of $\mathcal{O}_{\operatorname{Proj} R}(1)$.

6.5 Remark. We remark that the same method works if Y is an elliptic curve (and the characteristic is arbitrary). Indeed, let Y be elliptic and consider the finite morphism $N_Y : Y \to Y$ given by multiplication by N which is of degree N^2 (cf. [39, Corollary 4.17] for the case that the base field has characteristic zero. For the general case see e. g. [63, Proposition II.6 just before the Appendix]). Then we can substitute φ by a power of N_Y. Moreover, since $\mathcal{S}$ is semistable of $\deg \mathcal{S} \geq 0$ we have after twisting that $H^1(Y, N_Y^{e\,*} \mathcal{S} \otimes \mathcal{L}) = 0$ in light of [38, Lemma 1.1] and $H^0(Y, \mathcal{L}) \neq 0$ by Riemann-Roch.

Alternatively, disregarding this fact we can also substitute the Frobenius F by N_Y since we only needed the magical property that a pullback along the Frobenius increases the degree while fixing the genus.

We also note that it is already known that $I^{+\mathrm{gr}} = I^*$ in positive characteristic on an elliptic curve (cf. [9, Theorem 4.3]) for primary homogeneous ideals. And the inclusion $I^{+\mathrm{gr}} \subseteq I^{\dagger\mathrm{GR}}$ is trivial.

In principle, we have all the necessary tools to establish the other inclusion in the strongly semistable case in positive characteristic or if Y is an elliptic curve. But since the general proof is essentially the same except for one particular argument we will only present the general version and point out the simplifications in the special cases – see Theorem 7.23 and Remark 7.24.

7 Almost zero for vector bundles on curves

In this section we will develop the notion of almost zero for vector bundles. This will be the key to prove the converse inclusion of Corollary 4.10 in characteristic zero and is also necessary for the reduction step along a strong Harder-Narasimhan filtration.

In their article [66] Roberts, Singh and Srinivas defined a notion of "almost zero" for modules and used this to study the notion of "almost Cohen-Macaulay". Their definition of almost zero is equivalent to our definition whenever both are applicable. Also note that we will recover one of their results in dimension two in the case of an algebraically closed base field with our definition (cf. Proposition 7.29).

7.1 Almost zero and graded dagger closure

First of all, we extend the definition of (graded) dagger closure to modules. Then we define the notion of almost zero for vector bundles, explore the relation to the notion of Roberts, Singh and Srinivas and prove that this characterises graded dagger closure.

7.1 Definition. Let R be an $\mathbb{N}$-graded domain and let $N \subseteq M$ be an inclusion of $\mathbb{Z}$-graded R-modules. Then the *graded dagger closure* of N in M denoted by $N_M^{\dagger\mathrm{GR}}$ is the set of all elements $m \in M$ such that there exists a sequence a_n of elements in $R^{+\mathrm{GR}}$ with $\nu(a_n) \to 0$ as $n \to \infty$ and $m \otimes a_n \in \operatorname{im}(N \otimes R^{+\mathrm{GR}} \to M \otimes R^{+\mathrm{GR}})$. As usual, ν is the valuation induced by the grading (cf. Proposition 1.12).

7.2 Lemma. Let R be an $\mathbb{N}$-graded domain and $I \subseteq R$ an ideal. Then $I_R^{\dagger\mathrm{GR}} = I^{\dagger\mathrm{GR}}$, where the latter denotes the usual graded dagger closure as in Definition 1.13.

Proof. Consider the exact sequence $0 \to I \to R \to R/I \to 0$ and tensor with $R^{+\mathrm{GR}}$ to obtain a right exact sequence

$$I \otimes R^{+\mathrm{GR}} \to R^{+\mathrm{GR}} \to R^{+\mathrm{GR}}/IR^{+\mathrm{GR}} \to 0.$$

Assume $f \in I^{\dagger\mathrm{GR}}$. Then we have a sequence of elements $a_n \in R^{+\mathrm{GR}}$ with $\nu(a_n) \to 0$ such that $a_n f \in IR^{+\mathrm{GR}}$. Since the canonical map $I \otimes R^{+\mathrm{GR}} \to IR^{+\mathrm{GR}}$ is surjective we have preimages inside $I \otimes R^{+\mathrm{GR}}$.

For the other inclusion we have that the $a_n f \in R^{+\mathrm{GR}}$ have preimages in $I \otimes R^{+\mathrm{GR}}$. That is, they map to zero in $R^{+\mathrm{GR}}/IR^{+\mathrm{GR}}$. Hence, $a_n f \in IR^{+\mathrm{GR}}$. □

7.3 Proposition. In the situation of Definition 7.1 consider the short exact sequence $0 \to N \to M \xrightarrow{p} M/N \to 0$ of $\mathbb{Z}$-graded R-modules. Then $m \in N_M^{\dagger \mathrm{GR}}$ if and only if $p(m) \in 0_{M/N}^{\dagger \mathrm{GR}}$.

Proof. After tensoring with $R^{+\mathrm{GR}}$ the sequence is still exact on the right. Therefore, we have

$$M/N \otimes R^{+\mathrm{GR}} \cong M \otimes R^{+\mathrm{GR}} / \operatorname{im}(N \otimes R^{+\mathrm{GR}} \to M \otimes R^{+\mathrm{GR}}).$$

But this means that $\bar{f} \otimes a = 0$ in $M/N \otimes R^{+\mathrm{GR}}$ if and only if $f \otimes a \in \operatorname{im}(N \otimes R^{+\mathrm{GR}} \to M \otimes R^{+\mathrm{GR}})$. If for any $\varepsilon > 0$ the element a can be chosen such that $\nu(a) < \varepsilon$ then, by definition, $f \in N_M^{\dagger \mathrm{GR}}$. $\square$

Proposition 7.3 motivates the following

7.4 Definition. Let R be an $\mathbb{N}$-graded domain and M a graded R-module. We say that $m \in M$ is *almost zero* if for every $\varepsilon > 0$ the element $m \otimes 1 \in M \otimes R^{+\mathrm{GR}}$ is annihilated by an element $a_m \in R^{+\mathrm{GR}}$ with $\nu(a_m) < \varepsilon$ (equivalently $m \in 0_M^{\dagger \mathrm{GR}}$).

7.5 Proposition. Let R be a normal standard graded k-domain of dimension $\dim R \geq 2$ and $Y = \operatorname{Proj} R$, where k is an algebraically closed field. Let I be a homogeneous R_+-primary ideal generated by homogeneous elements $f_1, \ldots, f_n$.

If a homogeneous element $f_0 \in R_{d_0}$ is contained in $I^{\dagger \mathrm{GR}}$ then

$$\delta(f_0) \otimes 1 \in \bigoplus_{l \geq 0} H^1(Y, \operatorname{Syz}(f_1, \ldots, f_n)(l)) \otimes R^{+\mathrm{GR}}$$

is almost zero, where $\delta : H^0(Y, \mathcal{O}_Y(d_0)) \to H^1(Y, \operatorname{Syz}(f_1, \ldots, f_n)(d_0))$ is the connecting homomorphism induced by the cohomology of the twisted presenting sequence of $\operatorname{Syz}(f_1, \ldots, f_n)$.

Proof. The presenting sequence of the syzygy bundle associated to $(f_1, \ldots, f_n)$ is

$$0 \longrightarrow \operatorname{Syz}(f_1, \ldots, f_n) \longrightarrow \bigoplus_{i=1}^n \mathcal{O}_Y(-d_i) \xrightarrow{f_1, \ldots, f_n} \mathcal{O}_Y \longrightarrow 0.$$

Fix an element f_0 of degree d_0. As R is normal and of dimension ≥ 2 one can identify $H^0(Y, \mathcal{O}_Y(d_0))$ with R_{d_0} (cf. Proposition 3.4). Twisting by $\mathcal{O}_Y(d_0)$ and taking cohomology therefore yields the exact sequence

$$0 \longrightarrow H^0(Y, \operatorname{Syz}(f_1, \ldots, f_n)(d_0)) \longrightarrow \bigoplus_{i=1}^n H^0(Y, \mathcal{O}_Y(d_0 - d_i)) \xrightarrow{f_1, \ldots, f_n}$$

$$R_{d_0} \xrightarrow{\delta} H^1(Y, \operatorname{Syz}(f_1, \ldots, f_n)(d_0)) \longrightarrow \bigoplus_{i=1}^n H^1(Y, \mathcal{O}_Y(d_0 - d_i))$$

(see also [16, Section 2] for a more elaborate discussion of this idea with applications to tight closure). From this we can extract the exact sequence

$$0 \longrightarrow (f_1, \ldots, f_n)_{d_0} \longrightarrow R_{d_0} \longrightarrow H^1(Y, \operatorname{Syz}(f_1, \ldots, f_n)(d_0)).$$

Taking the direct sum over all $d_0 \geq 0$ we obtain an exact sequence of graded R-modules and if a (homogeneous) element f_0 is in the graded dagger closure of I then $\delta(f_0)$ is almost zero in $\bigoplus_{l \geq 0} H^1(Y, \mathrm{Syz}(f_1, \ldots, f_n)(l)) \otimes R^{+\mathrm{GR}}$. Note that by [39, Theorem 5.2 (b)] the direct sum is actually finite (this alternatively also follows since $(f_1, \ldots, f_n)$ is R_+-primary). $\square$

7.6 Remark. If $\operatorname{char} k > 0$ then $R^{+\mathrm{GR}}$ is in fact a big balanced Cohen-Macaulay algebra for R (see [48, Main Theorem 5.15]). In particular, if R is regular then $R^{+\mathrm{GR}}$ is flat over R by [48, 6.7]. Hence, if we denote the syzygy module of $(f_1, \ldots, f_n)$ by M we have, in this case, an isomorphism

$$H^2_{R_+ R^{+\mathrm{GR}}}(M \otimes_R R^{+\mathrm{GR}}) = H^2_{R_+}(M) \otimes_R R^{+\mathrm{GR}}$$

(to see this, look at the Čech-complex of M and tensor with $R^{+\mathrm{GR}}$).

In general, we still have an induced map

$$H^2_{R_+}(M) \otimes_R R^{+\mathrm{GR}} \to H^2_{R_+ R^{+\mathrm{GR}}}(M \otimes_R R^{+\mathrm{GR}}).$$

Indeed, consider the Čech complex of M. Tensoring with $R^{+\mathrm{GR}}$ we obtain the Čech complex for $M \otimes R^{+\mathrm{GR}}$ and since cohomology is a functor we get an induced map $H^i_{R_+}(M) \to H^i_{R_+ R^{+\mathrm{GR}}}(M \otimes R^{+\mathrm{GR}})$. As the latter module has an $R^{+\mathrm{GR}}$-module structure we obtain the desired map.

A thorough study of the proof of [12, Theorem 3.2] with dagger closure in mind strongly[7] suggests that one needs a suitable notion of "almost zero" for cohomology classes of vector bundles for the reduction step along a strong Harder-Narasimhan filtration. More surprisingly, this notion is also extremely useful for studying the strongly semistable case. The key observation here is Proposition 7.13 below which is already implicit in the proof of Proposition 6.4.

We now come to the crucial definition of this section.

7.7 Definition. Let $\mathcal{S}$ be a vector bundle on a smooth projective curve Y over an algebraically closed field k together with a cohomology class $c \in H^1(Y, \mathcal{S})$. We say that c is *almost zero* if for all $\varepsilon > 0$ there exists a finite morphism $\varphi : Y' \to Y$ between smooth projective curves and a line bundle $\mathcal{L}$ on Y' with a global section $s \neq 0$ such that $\deg \mathcal{L} / \deg \varphi < \varepsilon$ and such that

$$s\varphi^*(c) = 0 \in H^1(Y', \mathcal{L} \otimes \varphi^* \mathcal{S}).$$

Here $s\varphi^*(c)$ is induced by the morphism

$$0 \longrightarrow \varphi^* \mathcal{S} \xrightarrow{\cdot s} \varphi^* \mathcal{S} \otimes \mathcal{L}$$

We say that $\mathcal{S}$ is almost zero if every $c \in H^1(Y, \mathcal{S})$ is almost zero. By abuse of notation we will also sometimes say that s annihilates c if $s\varphi^*(c) = 0$.

[7] Here the adverb is valid in any characteristic.

7.8 Remark. (a) If $\mathcal{S}$ is almost zero then there exists for every $\varepsilon > 0$ a line bundle $\mathcal{L}$ with a global section $s \neq 0$ such that $\deg \mathcal{L} / \deg \varphi < \varepsilon$ so that s annihilates every $c \in H^1(Y, \mathcal{S})$. This follows since $H^1(Y, S)$ is finitely generated as a k-vector space. We note, however, that this neither implies that $H^1(Y', \varphi^* \mathcal{S}) = 0$ nor that $H^1(Y', \varphi^* \mathcal{S}) \to H^1(Y', \varphi^* \mathcal{S} \otimes \mathcal{L})$ is the zero map.

In order to provide an example for this phenomenon, consider the Fermat cubic $Y = \operatorname{Proj} k[x, y, z]/(x^3 + y^3 + z^3)$, where k is an algebraically closed field of characteristic $\neq 2, 3$. The map $k[x, y, z] \to k[u, v, w]$ induced by $x \mapsto u^2, y \mapsto v^2, z \mapsto w^2$ yields a finite surjective morphism $\varphi : Y' = \operatorname{Proj} k[u, v, w]/(u^6 + v^6 + w^6) \to Y$ of degree 4.

Note that Y is elliptic by the genus formula and smooth and integral by [39, Ex. I.5.9]. Likewise, Y' is a smooth curve of genus 10. Consider the ideal $(x, y) \subseteq k[x, y, z]/(x^3 + y^3 + z^3)$ and note that z^2 is not contained in (x, y) for if we kill x, y we obtain $k[z]/(z^3)$ and z^2 is not zero in this ring.

We start with the Koszul resolution of (x, y) (note that x, y form a regular system of parameters so that this is actually a free resolution (cf. [21, Corollary 1.6.14])):

$$0 \longrightarrow R(-2) \xrightarrow{(y,-x)} R(-1)^2 \xrightarrow{x,y} R \longrightarrow R/(x, y) \longrightarrow 0.$$

This yields the short exact sequence $0 \to \mathcal{O}_Y(-2) \to \mathcal{O}_Y(-1)^2 \to \mathcal{O}_Y \to 0$ on Y. We therefore have $\operatorname{Syz}(x, y)(2) = \mathcal{O}_Y$ and the extension bundle $\mathcal{S}' = \operatorname{Syz}(z^2, x, y)(2)$ corresponding to $\delta(z^2) = \frac{z^2}{xy}$.

Pulling back to Y' the class $\frac{z^2}{xy}$ is mapped to $\frac{w^4}{u^2 v^2} \in H^1(Y', \mathcal{O}_{Y'})$. The class $\frac{w^4}{u^2 v^2}$ is annihilated by multiplication with $u^2 \in H^1(Y', \mathcal{O}_{Y'}(2))$ (of course, $\mathcal{O}_{Y'}(2) = \varphi^* \mathcal{O}_Y(1)$). On the other hand, the class $\frac{w^4}{u^3 v} \in H^1(Y', \mathcal{O}_{Y'})$ is not annihilated by multiplication with u^2.

As $\dim H^1(Y, \mathcal{O}_Y) = 1$ we see that every element of $H^1(Y, \mathcal{O}_Y)$ is annihilated by the map $H^1(Y, \mathcal{O}_Y) \to H^1(Y', \mathcal{O}_{Y'}(2))$ but $H^1(Y', \varphi^* \mathcal{O}_Y)$ is nonzero. Moreover, the induced map $H^1(Y', \varphi^* \mathcal{O}_Y) \to H^1(Y', \mathcal{O}_{Y'}(2))$ is not the zero map since the class $\frac{w^4}{u^3 v}$ is not annihilated.

(b) We remark that the short exact sequence $0 \to \varphi^* \mathcal{S} \to \varphi^* \mathcal{S}' \to \mathcal{O}_{Y'} \to 0$ obtained by the pullback of the short exact sequence defined by c corresponds to the extension of $\mathcal{O}_{Y'}$ by $\varphi^* \mathcal{S}$ which is defined by $\varphi^*(c)$. Indeed, the extension is given by the image of 1 in $\operatorname{Ext}^1(\varphi^* \mathcal{S}, \varphi^* \mathcal{S})$ and we have a commutative diagram

$$\begin{tikzcd} \operatorname{Ext}^0(\mathcal{S}, \mathcal{S}) \arrow[r] \arrow[d, "\varphi^*"] & \operatorname{Ext}^1(\mathcal{O}_Y, \mathcal{S}) \arrow[d, "\varphi^*"] \\ \operatorname{Ext}^0(\varphi^* \mathcal{S}, \varphi^* \mathcal{S}) \arrow[r] & \operatorname{Ext}^1(\mathcal{O}_{Y'}, \varphi^* \mathcal{S}), \end{tikzcd}$$

where the left vertical arrow maps 1 to 1.

7.9 Remark. We want to emphasize that the notion of almost zero for vector bundles and the notion of almost zero for elements of graded R-modules are two separate notions.

In fact, for a given nonzero finitely generated $\mathbb{Z}$-graded R-module, where R is an $\mathbb{N}$-graded domain finitely generated over a field R_0, not every element of M is almost zero as we will see in Proposition 9.37.

We now want to prove that the notion of "almost zero" in the sense of [66, Question 3.3] implies our notion of "almost zero" over an algebraically closed field of characteristic zero. First, we recall this definition in a form adapted to our situation.

7.10 Definition. Let R be an $\mathbb{N}$-graded domain which is finitely generated over a field R_0 of characteristic zero, write $X = \operatorname{Proj} R$ and assume that $\mathcal{O}_X(1)$ is invertible. An element $c \in H^1(X, \mathcal{O}_X(m))$ is ***almost zero*** if for every $\varepsilon > 0$ there exists a finite $\mathbb{Q}$-graded extension $R \subseteq S$ (preserving degrees) such that the image of c under the induced map $H^1(X, \mathcal{O}_X(m)) \to H^1(Y, \varphi^*\mathcal{O}_X(m))$, where $\varphi : Y = \operatorname{Proj} S \to X$ (here the grading of S is multiplied by an integer so that S is $\mathbb{N}$-graded), is annihilated by an element of S of degree $< \varepsilon$.

We say that $\mathcal{O}_X(m)$ is almost zero if every $c \in H^1(X, \mathcal{O}_X(m))$ is almost zero.

We remark that Roberts, Singh and Srinivas do neither require $\mathcal{O}_X(1)$ to be invertible nor R to be normal but all of their applications reduce to the invertible case on the ring level. Likewise, we can assume in addition R to be normal by passing to a finite extension domain.

7.11 Proposition. Let R be a normal two-dimensional $\mathbb{N}$-graded domain which is finitely generated over an algebraically closed field R_0 of characteristic zero such that $\mathcal{O}_X(1)$ is invertible, where $X = \operatorname{Proj} R$. If $c \in H^1(X, \mathcal{O}_X(m))$ is almost zero in the sense of Definition 7.10 then it is also almost zero in the sense of Definition 7.7.

Proof. Assume that $c \in H^1(X, \mathcal{O}_X(m))$ is almost zero. This means that for every $n \in \mathbb{N}$ there is a finite extension $R \subseteq S$ such that the image of c under the map $H^1(X, \mathcal{O}_X(m)) \to H^1(Y, \varphi^*\mathcal{O}_X(m))$ is annihilated by some nonzero homogeneous element $s \in S$ of $\deg s \leq \frac{1}{n}$, where $Y = \operatorname{Proj} S$ and $\varphi : Y \to X$ is induced by the inclusion (here we regrade S before taking the Proj). Since R is normal and of dimension 2 we have $\Gamma(X, \mathcal{O}_X(n)) = R_n$ by Proposition 3.4. Adjoining suitable roots to R and then normalising again we may assume that R is normal and that $\operatorname{Proj} R$ is covered by standard open sets coming from elements of degree 1. Furthermore, we replace S by RS. That is, the subring of $R^{+\mathrm{GR}}$ generated by the elements of S and R.

As in the proof of Lemma 1.19 we may assume that $\deg s = \frac{1}{n}$. Moreover, applying Lemma 4.2 we may assume that S is generated by elements of degree $\frac{1}{n}$. Furthermore, $\varphi^*\mathcal{O}_X(1) = \mathcal{O}_Y(n)$. Consider the normalisation S' of S and denote the induced morphism by $\psi : \operatorname{Proj} S' = Y' \to Y$. The element s then induces a morphism $\mathcal{O}_{Y'} \to \psi^*\mathcal{O}_Y(1)$ and tensoring with $\psi^*\varphi^*\mathcal{O}_X(m) = \psi^*\mathcal{O}_Y(nm)$, and taking cohomology yields a map that annihilates c. Furthermore, $\frac{\deg \psi^*\mathcal{O}_Y(1)}{\deg \varphi\psi} = \frac{\mathcal{O}_Y(1)}{\deg \varphi} = \frac{1}{n} \deg \mathcal{O}_X(1)$. □

See Remark 7.27 as to what extend the converse holds.

7.12 Remark. Let R be an $\mathbb{N}$-graded domain of dimension 2 which is finitely generated over a field R_0 of characteristic zero and write $Y = \operatorname{Proj} R$. Roberts, Singh and Srinivas showed in [66, Corollary 3.5] that $\mathcal{O}_Y(n)$, $n \geq 0$, is almost zero in the sense of Definition 7.10. We will also prove this with different methods in Proposition 7.29. This implies the converse inclusion of Corollary 4.11 in characteristic zero in Situation 3.6 in light of Theorem 3.12.[8]

7.13 Proposition. Let $\mathcal{S}$ be a locally free sheaf on a smooth projective curve Y over an algebraically closed field k. Fix a cohomology class $c \in H^1(Y, \mathcal{S})$ which defines an extension $\mathcal{S}'$ of $\mathcal{O}_Y$ by $\mathcal{S}$. Then the following are equivalent:

(i) The cohomology class c is almost zero.

(ii) For every $\varepsilon > 0$ there is a finite k-morphism $\varphi : Y' \to Y$ of smooth projective curves, a line bundle $\mathcal{L}$ on Y' such that $\deg \mathcal{L} / \deg \varphi < \varepsilon$ and such that $\varphi^* \mathcal{S}'^\vee \to \mathcal{L} \to 0$ is exact and the surjection does not factor through $\varphi^* \mathcal{S}^\vee$.

(iii) For every $\varepsilon > 0$ there is a curve C in $\mathbb{P}(\mathcal{S}'^\vee)$ not contained in $\mathbb{P}(\mathcal{S}^\vee)$ such that $\varphi : C \to Y$ induced by the projection is finite and such that
$$\frac{C.\mathbb{P}(\mathcal{S}^\vee)}{\deg \varphi} < \varepsilon.$$

Proof. We prove the implication from (i) to (ii). For $\varepsilon > 0$ let $\varphi : Y' \to Y$ be a finite morphism of smooth projective curves and $\mathcal{L}$ a line bundle on Y' such that $\deg \mathcal{L} / \deg \varphi < \varepsilon$ and s a nonzero global section of $\mathcal{L}$ with $s\varphi^*(c) = 0$. We consider the exact sequence $0 \to \varphi^* \mathcal{S} \to \varphi^* \mathcal{S}' \to \mathcal{O}_{Y'} \to 0$, tensor with $\mathcal{L}$ and take cohomology. We thus obtain an exact sequence
$$0 \to H^0(Y', \varphi^* \mathcal{S} \otimes \mathcal{L}) \to H^0(Y', \varphi^* \mathcal{S}' \otimes \mathcal{L}) \to H^0(Y', \mathcal{L}) \xrightarrow{\delta} H^1(Y', \varphi^* \mathcal{S} \otimes \mathcal{L}).$$
Now, since $\delta(s) = s\varphi^*(c) = 0$ we have that s stems from a global section in $\varphi^* \mathcal{S}' \otimes \mathcal{L}$. As in the proof of Proposition 6.4 this defines a morphism $\varphi^* \mathcal{S}'^\vee \to \mathcal{L}' \to 0$ with $\frac{\deg \mathcal{L}'}{\deg \varphi} < \varepsilon$, where $\mathcal{L}'^\vee$ is the saturation of $\mathcal{L}^\vee$ in $\varphi^* \mathcal{S}'$. Moreover, as s is nonzero this surjection does not factor through $\varphi^* \mathcal{S}^\vee$.

For the implication from (ii) to (i) let $\varepsilon > 0$. Assume that $\mathcal{L}$ is a line bundle on a smooth curve $\varphi : Y' \to Y$ finite over Y such that $\deg \mathcal{L} / \deg \varphi < \varepsilon$. Furthermore, assume that $\varphi^* \mathcal{S}'^\vee \to \mathcal{L}$ is a surjection which does not factor through $\varphi^* \mathcal{S}^\vee$. Therefore, we have a nonzero section s in $H^0(Y', \varphi^* \mathcal{S}' \otimes \mathcal{L})$ associated to $\varphi^* \mathcal{S}'^\vee \to \mathcal{L} \to 0$. Since this morphism does not factor through $\varphi^* \mathcal{S}^\vee$ the section s does not stem from $H^0(Y', \varphi^* \mathcal{S} \otimes \mathcal{L})$. Again we have an exact sequence
$$0 \to H^0(Y', \varphi^* \mathcal{S} \otimes \mathcal{L}) \to H^0(Y', \varphi^* \mathcal{S}' \otimes \mathcal{L}) \xrightarrow{\sigma} H^0(Y', \mathcal{L}) \xrightarrow{\delta} H^1(Y', \varphi^* \mathcal{S} \otimes \mathcal{L}),$$

[8] One has to read a bit between the lines of the proof of Proposition 7.11 to actually see that this annihilation by roots of $\mathcal{O}(1)$ characterises graded dagger closure – we will prove this in a more general setting in Theorem 7.23.

where $\sigma(s) \neq 0$ and $\sigma(s) \cdot \varphi^*(c) = \delta(\sigma(s)) = 0$ as desired.

The equivalence of (ii) and (iii) is given by the correspondence described in Proposition 3.8. A surjective morphism $\varphi^*\mathcal{S}'^\vee \to \mathcal{L}$ corresponds to a morphism $Y' \to \mathbb{P}(\mathcal{S}'^\vee)$ over Y and that it does not factor through $\mathcal{S}^\vee$ means precisely that its image is not contained in $\mathbb{P}(\mathcal{S}^\vee)$. For the rest of the claim assume that C is finite over Y. The inclusion $C \to \mathbb{P}(\mathcal{S}'^\vee)$ corresponds to a line bundle $\mathcal{L}$ on C. Pulling back to the normalisation Y' of C yields (ii) since $\deg \mathcal{L} = C.\mathbb{P}(\mathcal{S}^\vee)$. Conversely, assuming (ii) the image of Y' in $\mathbb{P}(\mathcal{S}'^\vee)$ yields C. We refer to the proof of Theorem 3.14 (cf. [7, Theorem 2.3]) for a detailed discussion. $\square$

Our next goal is to prove that the notion of almost zero as in Definition 7.7 characterises dagger closure. We need some preparations which are quite interesting in their own right.

7.14 Lemma. Let A be a noetherian ring containing a field k of characteristic $p \geq 0$, $a \in A^\times$ and $B = A[T]/(T^n - a)$. If $(\operatorname{char} k, n) = 1$ then the relative differentials $\Omega_{B/A}$ vanish.

Proof. To begin with, $\Omega_{B/A}$ is generated by dT so that it is enough to show that $dT = 0$. We have $dT^n = da = 0$ and $dT^n = nT^{n-1}dT$. Hence, $T^{n-1}dT = 0$ and since T^n is a unit, so is T^{n-1}. Thus $dT = 0$ as desired. $\square$

7.15 Lemma. Let Y be a smooth projective curve over an algebraically closed field k. Denote by $\mathcal{T}$ a torsion element of $\operatorname{Pic} Y$. Then there is a finite morphism $\varphi : Y' \to Y$ such that $\varphi^*\mathcal{T} \cong \mathcal{O}_{Y'}$ and such that Y' is smooth.

Proof. See also [39, Ex. IV.2.7]. Denote the order of $\mathcal{T}$ by n and consider the coherent $\mathcal{O}_Y$-algebra $\mathcal{A} = \bigoplus_{i=0}^{n-1} \mathcal{T}^i$, where the multiplication is induced by a fixed isomorphism $\alpha : \mathcal{T}^n \to \mathcal{O}_Y$. First, we show that $\psi : \operatorname{Spec} \mathcal{A} \to Y$ is finite (and hence projective by [31, Corollaire 6.1.11]). The morphism is affine by [39, Ex. II.5.17]. Take an open affine cover U_j of Y such that the $\mathcal{T}|_{U_j}$ are free. It follows that the $\mathcal{T}^i|_{U_j}$ are also free and $\mathcal{O}_{\psi^{-1}(U_j)} = \mathcal{A}(U_j)$ is a free $\mathcal{O}_{U_j}$-module of rank n. Next, we show that $\psi^*\mathcal{T} = \mathcal{O}_{Y'}$. One has $(\psi^*\mathcal{T})_U = \mathcal{A}_U \otimes_{\mathcal{O}_U} \mathcal{T} = \mathcal{A}_U$ and clearly these isomorphisms glue.

Note that if $(\operatorname{char} k, n) = 1$ then $\operatorname{Spec} \mathcal{A} = X$ is êtale over Y. Indeed, locally over $U = U_j$ we have that $\mathcal{T}_U$ is generated by some element t. We therefore may identify $\mathcal{O}_{\psi^{-1}(U)}$ with $\mathcal{O}_U[T]/(T^n - \alpha(t^n))$. Applying [39, Ex. III.10.3] it is enough to observe that $\Omega_{X/Y} = 0$ by Lemma 7.14, that the extension over the stalks is separable since $(\operatorname{char} k, n) = 1$ and that the morphism is flat by [23, Proposition 4.1 (b)]. Pulling back to an irreducible component Y' which dominates Y (in fact, every component dominates Y) we have the desired morphism.

If the characteristic of the base divides n we can write $n = p^e m$ with $p \nmid m$. Then we consider a Frobenius pullback $F^e : Y' \to Y$ and thus have $(F^e)^*\mathcal{T} = \mathcal{T}^{p^e}$. Note that Y' is smooth and that $\mathcal{T}^{p^e}$ is an m-torsion element so that we may apply the previous argument. $\square$

7.16 Remark. Note that Lemma 7.15 implies that in characteristic zero any torsion element of $\operatorname{Pic} Y$ is almost zero, since $\mathcal{O}_Y$ is almost zero by [66, Theorem 3.4] (or by Proposition 7.29 below).

7.17 Corollary. Let Y be a smooth projective curve over an algebraically closed field and let $\mathcal{L}$ be an ample line bundle on Y. Then there is a finite morphism $\varphi : Y' \to Y$ of smooth projective curves such that $\varphi^*\mathcal{L}$ is generated by global sections.

Proof. Applying Lemma 4.4 to $\mathcal{L}$ with $\mathcal{O}(1) = \mathcal{L}$ (and $N = m$) we have that after a finite pullback $\psi : X \to Y$ there is a line bundle $\mathcal{M}$ on X which is generated by global sections such that $\psi^*\mathcal{L}^m = \mathcal{M}^m$. Hence, $\psi^*\mathcal{L} \otimes \mathcal{T} = \mathcal{M}$ where $\mathcal{T}$ is an m-torsion element of the Picard group. Applying Lemma 7.15 we are done. $\square$

7.18 Definition. Let X be a normal projective variety over an algebraically closed field. A line bundle $\mathcal{L}$ is called *semiample* if $\mathcal{L}^m$ is globally generated for some $m > 0$.

7.19 Remark. Note that any ample line bundle and the structure sheaf are semiample. Also note that if $\mathcal{L}^k$ is globally generated then so is $\mathcal{L}^{nk}$ for $n \geq 0$. The crucial fact is that semiampleness is sufficient for the conclusion of Lemma 4.4 insofar as we can find a globally generated kth root of any semiample $\mathcal{L}$ for $k \gg 0$.

Finally, we remark that, unlike for ample line bundles, it is not true that there is some n_0 such that $\mathcal{L}^n$ is globally generated for $n \geq n_0$ if $\mathcal{L}$ is merely semiample. To see this, consider the following

7.20 Example. Let X be a projective variety over an algebraically closed field and $\mathcal{T}$ a torsion element of $\operatorname{Pic} X$ of order n. Then $\mathcal{T}^m$ has global sections if and only if $n \mid m$. Assume that $\mathcal{T}^m$ has a global section for some $m \in \mathbb{N}$. This defines an injective morphism $\mathcal{O}_X \xrightarrow{s} \mathcal{T}^m$. In particular, we get an isomorphism $\mathcal{O}_X^{\otimes n} \xrightarrow{s^n} \mathcal{O}_X$ – and this implies that s also defines an isomorphism. Hence, $n \mid m$.

So $\mathcal{T}$ is semiample and $\mathcal{T}^m$ is globally generated if and only if $n \mid m$. We refer to [56, Example 2.1.2] for a similar but more sophisticated example.

The result of Corollary 7.17 continues to hold in higher dimensions. Stated precisely we have

7.21 Proposition. Let Y be a normal projective variety over an algebraically closed field and let $\mathcal{L}$ be a semiample line bundle on Y. Then there is a finite dominant morphism $\varphi : Y' \to Y$ of normal projective varieties such that $\varphi^*\mathcal{L}$ is generated by global sections. Moreover, if Y is smooth and if $\mathcal{L}$ is ample then Y' may be chosen to be smooth as well.

Proof. The exact same proof of Lemma 7.15 also works in higher dimensions. Hence, one can prove this similarly to Corollary 7.17. If $\mathcal{L}$ is ample we can fix $m \gg 0$ such that $\mathcal{L}^m$ is very ample and invoke Lemma 4.5 with $\mathcal{L}^m = \mathcal{O}_X(1)$ and $d = m$. $\square$

We do not know whether the supplement of Proposition 7.21 also holds if $\mathcal{L}$ is merely semiample.

7.22 Lemma. Let $\varphi : Y' \to Y$ be a finite morphism of smooth projective curves over an algebraically closed field, $\mathcal{S}$ a locally free sheaf on Y and $\mathcal{S}'$ an extension corresponding to $c \in H^1(Y, \mathcal{S})$. If $\mathcal{L}$ is a line bundle on Y and $\mathcal{S}'^\vee \to \mathcal{L} \to 0$ a surjection which does not factor through $\mathcal{S}^\vee$, then $\varphi^*\mathcal{S}'^\vee \to \varphi^*\mathcal{L} \to 0$ does not factor through $\varphi^*\mathcal{S}^\vee$.

Proof. Note that we have an exact sequence $0 \to \mathcal{S} \to \mathcal{S}' \to \mathcal{O}_Y \to 0$ and hence an inclusion $H^0(Y, \mathcal{S}) \subseteq H^0(Y, \mathcal{S}')$. The exact sequence $\mathcal{S}'^\vee \to \mathcal{L} \to 0$ yields after dualising and tensoring with $\mathcal{L}$ a global section $s \in H^0(Y, \mathcal{S}' \otimes \mathcal{L})$ which is not contained in $H^0(Y, \mathcal{S} \otimes \mathcal{L})$. We have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H^0(Y, \mathcal{S} \otimes \mathcal{L}) \arrow[r] \arrow[d] & H^0(Y, \mathcal{S}' \otimes \mathcal{L}) \arrow[r] \arrow[d] & H^0(Y, \mathcal{L}) \arrow[d] \\
0 \arrow[r] & H^0(Y', \varphi^*(\mathcal{S} \otimes \mathcal{L})) \arrow[r] & H^0(Y', \varphi^*(\mathcal{S}' \otimes \mathcal{L})) \arrow[r] & H^0(Y', \varphi^*\mathcal{L})
\end{tikzcd}$$

where the vertical arrows are injective (locally this is just a base change $R^n \to R^n \otimes_R S$ whence the injectivity). Since the rightmost vertical arrow is injective $\varphi^*(s)$ cannot lie in $H^0(Y', \varphi^*(\mathcal{S} \otimes \mathcal{L}))$. And this means that the surjection still does not factor. $\square$

7.23 Theorem. Assume Situation 3.6, fix a homogeneous element f_0 of degree d_0 and write $Y = \operatorname{Proj} R$ and $\mathcal{S} = \operatorname{Syz}(f_1, \ldots, f_n)(d_0)$. Then $c = \delta(f_0) \in H^1(Y, \mathcal{S})$ is almost zero if and only if f_0 is contained in the graded dagger closure of I.

Proof. Assume that $f_0 \in I^{\dagger \mathrm{GR}}$. Therefore, for $m \in \mathbb{N}$ we find a nonzero element a_m of degree $\frac{1}{m}$ such that $a_m f_0 \in IS$, where S is a finite $\mathbb{Q}$-graded extension domain of R. Note that we may assume that the minimal degree of S is $\frac{1}{m}$ (cf. proof of Proposition 4.9). Furthermore, we may assume that S is normal and that $Y' = \operatorname{Proj} S$ is covered by finitely many standard open sets coming from elements of degree 1 after regrading. Moreover, we may assume that elements of degree 1 in R are of degree m in the regraded copy of S. To accomplish this adjoin roots to S so that S is generated in degree $\frac{1}{m}$ then normalise and regrade by multiplying the grading by m so that S is $\mathbb{N}$-graded. Thus we have all the desired properties and $S(1)^\sim$ is invertible.

Pulling back the whole situation to $Y' = \operatorname{Proj} S$ we therefore have that $a_m \delta(f_0) = 0$ since it is contained in the extended ideal (cf. Proposition 3.7 and note that standard graded is not necessary – we only need that the $\mathcal{O}_Y(n)$ are invertible). The multiplication by a_m is induced by a morphism $\mathcal{O}_{Y'} \to \mathcal{O}_{Y'}(1)$. Now one can argue as in the proof of Proposition 7.11. Indeed, tensoring $\mathcal{O}_{Y'} \to \mathcal{O}_{Y'}(1)$ with $\mathcal{S}$ and taking cohomology yields that $H^1(Y', \varphi^*\mathcal{S}) \to H^1(Y', \varphi^*\mathcal{S} \otimes \mathcal{O}_{Y'}(1))$ maps $\delta(f_0)$ to $a_m \delta(f_0) = 0$, where $\varphi : Y' \to Y$ is induced by the inclusion of rings. Moreover, $\frac{\deg \mathcal{O}_{Y'}(1)}{\deg \varphi} = \frac{1}{m} \deg \mathcal{O}_Y(1)$.

Alternatively, if f_0 is contained in the graded dagger closure then the line bundles constructed in the proof of Proposition 4.9 do not factor through $\operatorname{Syz}(f_1, \ldots, f_n)$ since they come from syzygies with $a_0 \neq 0$. Hence, Proposition 7.13 (ii) is satisfied.

As to the other implication, Proposition 7.13 implies that for $\varepsilon > 0$ there is a line bundle $\mathcal{L}$ on some smooth curve $f : Y' \to Y$ finite over Y such that $\frac{\deg \mathcal{L}}{\deg f} < \varepsilon$ and an exact sequence $f^*\mathcal{S}'^\vee \to \mathcal{L} \to 0$ which does not factor through $f^*\mathcal{S}^\vee$.

Consider the ring $S = \bigoplus_{n \geq 0} \Gamma(Y', f^*\mathcal{O}_Y(n))$ – this is a finite $\mathbb{N}$-graded extension of R by Proposition 5.3. Moreover, since $\mathcal{O}_Y(1)$ is generated by global sections (it is even very ample) the line bundle $f^*\mathcal{O}_Y(1)$ is also generated by global sections. Hence, Proposition 5.6 implies that $\operatorname{Proj} S$ is covered by finitely many standard open sets coming from elements $s_1, \ldots, s_l$ of degree 1 in S. This in turn allows us to apply Lemma 4.2 and to adjoin mth roots of the $s_1, \dots, s_l$ to obtain rings S'_m that are still covered in degree 1 after regrading. Varying m we obtain finite morphisms $g_m : Y''_m \to Y'$, where $Y''_m = \operatorname{Proj} S'_m$ so that $\deg \mathcal{O}_{Y''_m}(1)/\deg(fg_m)^*\mathcal{O}_Y(1) = \frac{1}{m}$.

In order to obtain syzygies we fix an $m \gg 0$ and omit the index. We have that $\deg g^*\mathcal{L}^\vee/\deg(fg)$ is rational and independent of g. Hence, choosing m sufficiently large we can find a positive integer $t-1$ such that $\frac{t-1}{m}\deg \mathcal{O}_Y(1) = \deg g^*\mathcal{L}/\deg(fg)$. Possibly choosing a larger m (and then a different t) we may moreover assume that $\frac{t}{m}\deg \mathcal{O}_Y(1) < \varepsilon$. Now consider $g^*\mathcal{L}^\vee$ and twist by $\mathcal{O}_{Y''}(t)$ for a t as above so that

$$0 < \frac{\deg g^*\mathcal{L}^\vee(t)}{\deg(fg)} < \varepsilon.$$

Applying Corollary 7.17 we may assume that $g^*\mathcal{L}^\vee(t)$ is generated by global sections after a finite pullback and pulling back again if necessary we may assume that Y'' is smooth. By Proposition 5.4 we have a finite extension of graded domains $R \subseteq \bigoplus_{n\geq 0}\Gamma(Y'', \mathcal{O}_{Y''}(n)) = T$ and an exact sequence

$$0 \longrightarrow g^*\mathcal{L}^\vee(t) \longrightarrow (fg)^*\mathcal{S}'(t).$$

Recalling the presenting sequence for the syzygy bundle we see that global sections of $g^*\mathcal{L}^\vee(t)$ define syzygies of total degree md_0+t on $\operatorname{Proj} T$. We therefore have syzygies $(a_0, \dots, a_n)$ corresponding to relations $-a_0f_0 = \sum_i a_if_i$, where $\deg a_0 = \frac{t}{m} < \varepsilon$ and we still need to make sure that we find a syzygy with $a_0 \neq 0$. But otherwise $g^*\mathcal{L}^\vee(t)$ would factor through $((fg)^*\operatorname{Syz}(f_1, \dots, f_n)(d_0))(t)$ since it is generated by global sections. Twisting by $\mathcal{O}_{Y''}(-t)$ and applying Lemma 7.22 would imply that $\mathcal{L}$ factored as well contradicting our assumptions. □

7.24 Remark. (a) In positive characteristic we can avoid Corollary 7.17. One can simply pull back by a sufficiently high power of the k-linear Frobenius in order to make $\deg g^*\mathcal{L}^\vee(t)$ larger than $2g-2$ without changing $\deg g^*\mathcal{L}^\vee(t)/\deg(gf)$.

(b) On an elliptic curve this is also easier since any line bundle of degree ≥ 2 is generated by global sections. And this can be accomplished by pulling back along the multiplication by N map for some integer $N \geq 2$.

(c) Also note that both on an elliptic curve and in positive characteristic this yields together with Proposition 6.4 the converse inclusion of Corollary 4.10 if we assume Situation 3.6 and also suppose that $\operatorname{Syz}(f_1, \dots, f_n)$ is strongly semistable.

7.25 Proposition. Let Y be a smooth projective curve over an algebraically closed field. Fix an ample line bundle $\mathcal{O}_Y(1)$ and let $\mathcal{S}$ be a locally free sheaf of rank r such that $\det \mathcal{S} = \mathcal{O}_Y(n)$ for some integer n. If $\mathcal{S}$ is almost zero then one can choose the annihilating line bundles as suitable roots of $\mathcal{O}_Y(1)$.

Proof. By Proposition 5.9, $\mathcal{S}$ is up to twist a syzygy bundle. This means that we have an exact sequence[9] $0 \to \mathcal{S} \to \bigoplus_{i=0}^r \mathcal{O}_Y(d) \to \mathcal{O}_Y(l) \to 0$. Making a finite pullback we

[9] A priori the d's might be different but the proof of Proposition 5.9 actually shows that they are equal.

may assume that $\mathcal{O}_Y(1)$ is generated by global sections. Hence, by Proposition 5.6 its section ring is covered by finitely many standard open sets coming from elements of degree 1. Arguing as in the proof of Theorem 7.23 we see that we can annihilate $\mathcal{S}$ by roots of $\mathcal{O}_Y(1)$. □

7.26 Proposition. Let Y be a smooth projective curve over an algebraically closed field and $\mathcal{S}$ a vector bundle such that $\deg(\det \mathcal{S}) = 0$. Then $\mathcal{S}$ is almost zero and the annihilators can be chosen as roots of any ample line bundle $\mathcal{L}$ on Y.

Proof. If $\det \mathcal{S}$ is torsion then the determinant is trivial after a finite pullback and the pullback of $\mathcal{S}$ is a twisted syzygy bundle with respect to any embedding. Thus in this case the claim follows from Proposition 7.25 above.

So assume that $\det \mathcal{S}$ is not a torsion element and fix an ample line bundle $\mathcal{L}$ on Y. In light of Corollary 7.17 and Lemma 4.4 there is a finite morphism $\varphi : Y' \to Y$ of smooth projective curves such that there is a root $\mathcal{L}'$ on Y' of $\mathcal{L}$ which is generated by global sections and such that $\deg \mathcal{L}'/\deg \varphi < \frac{\varepsilon}{2}$. We note that pullbacks and det commute. Consider $\mathcal{L}'' = \mathcal{L}' \otimes (\det \varphi^*\mathcal{S})^\vee$. After a finite pullback we may assume that there is a $\operatorname{rk}\mathcal{S}$th root $\mathcal{M}$ of $\mathcal{L}''$ which is globally generated.

In particular, there is a global section $s \neq 0$ of $\mathcal{M}$ which induces a morphism $H^1(Y', \varphi^*\mathcal{S}) \to H^1(Y', \varphi^*\mathcal{S} \otimes \mathcal{M})$. Moreover, $\det(\varphi^*\mathcal{S} \otimes \mathcal{M}) = \det \varphi^*\mathcal{S} \otimes \mathcal{M}^{\operatorname{rk}\mathcal{S}} = \mathcal{L}'$. Hence, $\varphi^*\mathcal{S} \otimes \mathcal{M}$ is a twisted syzygy bundle with respect to $\mathcal{L}'$. Since the latter is a root of $\mathcal{L}$ the claim follows from Proposition 7.25 above. □

7.27 Remark. Let $\mathcal{S}$ be a locally free sheaf on a smooth projective curve Y over an algebraically closed field. Then $\mathcal{S}$ is almost zero if and only if it can be annihilated with respect to a suitable fixed embedding. That is, the annihilators can be chosen as roots of a suitable fixed line bundle $\mathcal{L}$ – this follows from Propositions 7.25 and 7.26 above. In particular, our definition of almost zero is equivalent to Definition 7.10 whenever both are applicable.

7.2 Almost zero for line bundles

In this section we will investigate line bundles with respect to the property of being almost zero. In characteristic zero this notion will only depend on the degree of the line bundle for a given curve – see Theorem 7.34. In positive characteristic the situation will be a little bit more subtle.

7.28 Lemma. Let Y be a smooth projective curve over an algebraically closed field and $\mathcal{E}$ a locally free sheaf. Let Y' be an irreducible curve in $\mathbb{P}(\mathcal{E})$ which dominates the base. Write $a\xi^{n-1} + b\xi^{n-2}f$ for its numerical equivalence class, where f is a fibre of $\pi : \mathbb{P}(\mathcal{E}) \to Y$ and ξ a Weil divisor whose linear equivalence class corresponds to $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$. Let φ be the morphism $Y' \to Y$ induced by π. Then $a = \deg(\varphi)$.

Proof. We have the following commutative diagram

$$\begin{tikzcd} & \mathbb{P}(\mathcal{E}) \arrow[d, "\pi"] \\ Y' \arrow[ur, "i"] \arrow[r, "\varphi"] & Y \end{tikzcd}$$

and replacing Y' by its normalisation we may assume that Y' is smooth (but i will no longer be a closed immersion). Intersecting i_*Y' with a fibre we have $a = i_*Y'.f$ which is equal to $i_*(Y'.i^*f)$ by the projection formula. Thus we have $i_*Y'.f = i_*(Y'.i^*\pi^*Q)$, where Q is a point on Y. Considering this as an element of $\mathbb{Z}$ (via the degree map) rather than of the Chow ring we may omit the pushforward. Thus the latter is equal to $Y'.\varphi^*Q$ which is equal to $\deg\varphi$ by [39, Proposition II.6.9]. $\square$

7.29 Proposition. Let Y be a smooth projective curve over an algebraically closed field. If $\mathcal{L}$ is a line bundle of degree ≥ 0 then $\mathcal{L}$ is almost zero.

Proof. Fix $c \in H^1(Y, \mathcal{L})$. This defines an extension $0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{O} \to 0$. Dualising we have that $X = \mathbb{P}(\mathcal{E}^\vee)$ is a normalised ruled surface with e-invariant $e = \deg\mathcal{L} \geq 0$. This is normalised since if we tensor the dualised short exact sequence with a line bundle $\mathcal{M}$ of negative degree then $H^0(Y, \mathcal{E}^\vee \otimes \mathcal{M}) = 0$. It also follows that $\mathcal{E}^\vee$ is not ample, since it surjects onto $\mathcal{L}^\vee$ which has degree ≤ 0. Fix a section Y_0 that is equivalent to $\mathcal{O}_{\mathbb{P}(\mathcal{E}^\vee)}(1)$.

In order to apply Proposition 7.13 we need to find for every $\varepsilon > 0$ a curve C in X such that $\varphi : C \to Y$ is dominant and $0 < Y_0.C/\deg\varphi < \varepsilon$. The curve C then defines a section of $\mathbb{P}(\varphi^*\mathcal{E}^\vee)$ which corresponds to a line bundle $\mathcal{M}$ on C with $\deg\mathcal{M} = Y_0.C$ such that $\varphi^*\mathcal{E}^\vee \to \mathcal{M} \to 0$ is exact (see Corollary 3.9). Also note that since $\deg\varphi^*\mathcal{L}^\vee \leq 0$ we cannot have a surjection $\varphi^*\mathcal{L}^\vee \to \mathcal{M} \to 0$.

By [39, Proposition V.2.20 (a)] an irreducible curve $C' \neq Y_0$ is numerically equivalent to $aY_0 + bf$ where f is the class of a fibre and $a > 0, b \geq ae$. Moreover, C' is ample if in addition $b > ae$ (loc. cit.). So fix a divisor C' numerically equivalent to $aY_0 + bf$ with $a > 0$ and choose $b = ae + 1$. Then $nC' = naY_0 + n(ae+1)f$ is very ample for $n \gg 0$. Use nC' to embed X into some $\mathbb{P}^N$ and apply Bertini's Theorem [39, Theorem II.8.18 and Remark III.7.9.1] to see that there is an irreducible nonsingular curve C on X which is linearly equivalent to nC'. Since C is not contained in a fibre it dominates the base Y and we have an induced morphism $\varphi : C \to Y$, where $\deg\varphi = na$. This yields $\frac{C.Y_0}{\deg\varphi} = \frac{n}{an} = \frac{1}{a}$. Therefore, choosing $a \gg 0$ completes the proof. $\square$

7.30 Remark. Assume Situation 3.6 and assume that $n = 2$. Then $\mathrm{Syz}(f_1, f_2)$ is a line bundle so that Proposition 7.29 yields the converse inclusion of Corollary 4.11 in light of Theorem 7.23 (the case $c = 0$ respectively $F^{e*}(c) = 0$ for some $e \geq 0$ being trivial).

We now give an alternative proof of Proposition 7.29 which employs an argument only valid in characteristic zero since it uses a corollary of [66, Theorem 3.4]. More precisely, the authors exploit that for a normal projective variety X over a field of characteristic zero the morphism $\varphi : X \to A$, where A is the Albanese variety, induces an isomorphism $\varphi^* : H^1(A, \mathcal{O}_A) \to H^1(X, \mathcal{O}_X)$.

We will also give an alternative proof of this result valid in characteristic $p > 0$ using the k-linear Frobenius.

7.31 Lemma. Any ample line bundle $\mathcal{L}$ on a smooth projective curve Y over an algebraically closed field of characteristic zero is almost zero.

Proof. By Corollary 7.17 there is a finite morphism $\varphi : Y' \to Y$ of smooth projective curves such that $\varphi^*\mathcal{L}$ is generated by global sections and it follows that $\mathcal{L}$ is almost zero by [66, Corollary 3.5] (simply take $R = \bigoplus_{i \geq 0} \Gamma(Y', \varphi^*\mathcal{L}^i)$). $\square$

We can fully recover Proposition 7.29 since the degree zero case reduces to the ample case (we will only need characteristic zero to apply Lemma 7.31):

7.32 Lemma. Let Y be a smooth projective curve over an algebraically closed field k of characteristic zero. Then any $\mathcal{L}$ in $\operatorname{Pic}^0 X$ is almost zero.

Proof. Let $\varepsilon > 0$. There exists an ample line bundle $\mathcal{G}$ on $\varphi : Y' \to Y$ such that $H^0(Y', \mathcal{G}) \neq 0$ with $\frac{\deg \mathcal{G}}{\deg \varphi} < \frac{\varepsilon}{2}$. Fix a global section $s \neq 0$ of $\mathcal{G}$. This induces a non-trivial morphism $\varphi^*\mathcal{L} \to \varphi^*\mathcal{L} \otimes \mathcal{G}$. Furthermore, $\varphi^*\mathcal{L} \otimes \mathcal{G}$ is ample and hence almost zero by Lemma 7.31.

Therefore, we have a finite morphism $\psi : Y'' \to Y'$ together with a non-trivial global section t of some line bundle $\mathcal{H}$ such that $\frac{\deg \mathcal{H}}{\deg \varphi\psi} < \frac{\varepsilon}{2}$ and such that the map $H^1(Y'', \psi^*(\varphi^*\mathcal{L} \otimes \mathcal{G})) \to H^1(Y'', \psi^*(\varphi^*\mathcal{L} \otimes \mathcal{G}) \otimes \mathcal{H})$ induced by $\psi^*(s) \otimes t$ annihilates every element of $H^1(Y', \varphi^*\mathcal{L} \otimes \mathcal{G})$. In particular, every element of $H^1(Y, \mathcal{L})$ is annihilated and $\frac{\deg(\psi^*\mathcal{G}) \otimes \mathcal{H}}{\deg \varphi\psi} < \varepsilon$. $\square$

The following lemma is false in positive characteristic as we will see in Remark 7.35 below.

7.33 Lemma. Let Y be a smooth projective curve over an algebraically closed field of characteristic zero. Let $\mathcal{L}$ be a line bundle of negative degree. Then any nonzero $c \in H^1(Y, \mathcal{L})$ is not almost zero.

Proof. The class c defines a non-trivial extension $0 \to \mathcal{O}_Y \to \mathcal{E} \to \mathcal{L}^\vee \to 0$ and $\mathcal{L}^\vee$ is ample since $\deg \mathcal{L}^\vee > 0$. By [27, Proposition 2.2] every quotient bundle of $\mathcal{E}$ has positive degree, that is, $\mu_{\min}(\mathcal{E}) > 0$. Since the characteristic is zero this implies that $\mathcal{E}$ is ample by Theorem 3.14 and we are done by Proposition 7.13. $\square$

7.34 Theorem. Let Y be a smooth projective curve over an algebraically closed field of characteristic zero. A line bundle $\mathcal{L}$ on Y is almost zero if and only if $\deg \mathcal{L} \geq 0$ or if $Y = \mathbb{P}^1$ and $\mathcal{L} = \mathcal{O}_{\mathbb{P}^1}(-1)$.

Proof. For the only if part we assume that $\deg \mathcal{L} < 0$ and apply Lemma 7.33 to see that $c \in H^1(Y, \mathcal{L})$ is almost zero if and only if $c = 0$. Thus $\mathcal{L}$ is almost zero if and only if $H^1(Y, \mathcal{L}) = 0$. Note that $\mathcal{L}$ has no nonzero global sections since its degree is negative. Applying Riemann-Roch yields that $\deg \mathcal{L} + 1 - g = 0$. This is only possible for $\deg \mathcal{L} = -1$ and $g = 0$. Hence, $Y = \mathbb{P}^1$ and $\mathcal{L} = \mathcal{O}_{\mathbb{P}^1}(-1)$.

The other implication follows from Lemmata 7.31 and 7.32 (or from Proposition 7.29) and from applying Serre duality to $\mathcal{O}_{\mathbb{P}^1}(-1)$ to see that $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)) = 0$. $\square$

7.35 Remark. We discuss the notion of almost zero for line bundles in positive characteristic. So let Y be a smooth projective curve over an algebraically closed field of characteristic $p > 0$ and let $\mathcal{L}$ be a line bundle on Y.

If $\deg \mathcal{L} \geq 0$ choose a line bundle $\mathcal{G}$ of positive degree on some finite cover $\varphi : Y' \to Y$, where Y' is smooth, such that $\frac{\deg \mathcal{G}}{\deg \varphi} < \varepsilon$. Pulling back by a sufficiently high power of the k-linear Frobenius one may assume that $\deg \mathcal{G} \geq 2g(Y') - 1$. Therefore, $H^0(Y', \mathcal{G}) \neq 0$ and $H^1(Y', \varphi^* \mathcal{L} \otimes \mathcal{G}) = 0$. This implies that $\mathcal{L}$ is almost zero.

Assume now that $\deg \mathcal{L} < 0$ and that $c \in H^1(Y, \mathcal{L})$ is nonzero. Consider the (dual) extension $0 \to \mathcal{O}_Y \to \mathcal{E} \to \mathcal{L}^\vee \to 0$ defined by c. As in Lemma 7.33 we have that $\mu_{\min}(\mathcal{E}) > 0$. If $\mathcal{E}$ is strongly semistable then it is ample as well (Theorem 3.14). If $\mathcal{E}$ is not semistable then the quotients of the Harder-Narasimhan filtration are line bundles (since $\operatorname{rk} \mathcal{E} = 2$). In particular, this is a strong Harder-Narasimhan filtration and it follows that $\mu_{\min}(\mathcal{E}) = \bar{\mu}_{\min}(\mathcal{E}) > 0$ – and again $\mathcal{E}$ is ample by Theorem 3.14.

So assume now that $\mathcal{E}$ is semistable but not strongly semistable. In this case we have to consider a sufficiently high Frobenius pull back so that the Harder-Narasimhan filtration of $F^{e*}\mathcal{E}$ is strong. The issue is that it may indeed happen that $F^{e*}(c) = 0$ and in this case $\mathcal{E}$ is not ample and c is almost zero. See [38, Example 3.2] for an explicit case where this happens.

We also note that if $\deg \mathcal{E} > \frac{2}{p}(g-1)$, where g is the genus of Y, then $\mathcal{E}$ is ample by [35, Corollary 7.7] since we have $\mu_{\min}(\mathcal{E}) > 0$. Hence, c is not almost zero in this case.

7.36 Example. Consider $R = k[x, y, z]/(x^3 + y^3 + z^3)$ and assume $\operatorname{char} k \neq 3$, where k is an algebraically closed field. The corresponding curve $Y = \operatorname{Proj} R$ is elliptic by the genus formula and smooth and integral by [39, Ex. I.5.9]. It is a classical example of tight closure theory that $z^2 \in (x, y)^*$ (see e. g. [13, Example 1.9]). It is not contained in the ideal as we have seen in Remark 7.8. We will show that $z^2 \in (x, y)^{\dagger \mathrm{GR}}$ using our geometric interpretation of graded dagger closure.

We start with the Koszul resolution of (x, y) (again cf. Remark 7.8):

$$0 \longrightarrow R(-2) \xrightarrow{(y,-x)} R(-1)^2 \xrightarrow{x,y} R \longrightarrow R/(x, y) \longrightarrow 0.$$

This yields the short exact sequence $0 \to \mathcal{O}_Y(-2) \to \mathcal{O}_Y(-1)^2 \to \mathcal{O}_Y \to 0$ on Y. We therefore have $\operatorname{Syz}(x, y)(2) = \mathcal{O}_Y$ and the extension bundle $\mathcal{S}' = \operatorname{Syz}(z^2, x, y)(2)$ corresponding to $\delta(z^2) = \frac{z^2}{xy}$ yields the unique ruled surface $\mathbb{P}(\mathcal{S}'^\vee)$ over Y that is given by the non-split extension $0 \to \mathcal{O}_Y \to \mathcal{S}'^\vee \to \mathcal{O}_Y \to 0$. For the uniqueness see [39, Theorem 2.15] and note that it does not split since $z^2 \notin (x, y)$ (cf. Proposition 3.7).

There are now several approaches to show that $z^2 \in (x, y)^{\dagger \mathrm{GR}}$. The first one is to work with numerical equivalence and to apply Proposition 7.29. The second and third approach are very much related. They both exploit the fact that Y is its own Jacobian, that is, we use the multiplication by N map N_Y to obtain curves that intersect the forcing divisor arbitrarily small in the sense of Theorem 3.14. One can now either do this in a geometric or in an algebraic setting (i. e. working with the ring R and constructing extensions explicitly). We note that (up to a linear change of variables) this example also occurs in [66, Example 2.4], where they used a similar technique. But their ring extensions do not stem from any multiplication map since their extensions multiply the degree by 3 which is not a square.

The algebraic approach has the advantage that one actually sees the ring extensions quite explicitly. But it has the distinct disadvantage that one has to work with the multiplication maps of the elliptic curve. (One could pass to a Weierstrass equation (see [39, Example IV.4.6.2]) and then determine a Weil-divisor corresponding to $\mathcal{O}_Y(1)$ to apply the multiplication maps (see [70, III.2.3])).

Let $N = 2$ and consider the morphism $N_Y : Y \to Y$ which has degree $N^2 = 4$. Pulling back $\mathcal{O}_Y(1)$ to this new copy of Y we have $N_Y^* \mathcal{O}_Y(1) = \mathcal{O}_Y(N^2)$. Taking the section ring R_1 induced by $\mathcal{O}_Y(1)$ on this new copy of Y we obtain a ring extension $R = R_0 \subseteq R_1$, where an element of degree 1 in R maps to an element of degree 4. Hence, after regrading, R_1 is generated by elements of degree $1/4$. Iterating this process we obtain graded ring extensions $R_n \subseteq R_{n+1}$, where every R_n is isomorphic to R and the generators of R_n are in degree $1/4^n$.

We now want to see how $\delta(z^2) = c \in H^1(Y, \mathcal{O}_Y)$ is annihilated in cohomology. So let $Y_n = \operatorname{Proj} R_n$ and pull back the whole situation to Y_n. We have $(N_Y^n)^*(\operatorname{Syz}(x, y)(2)) = \mathcal{O}_{Y_n}$. As $\mathcal{O}_{Y_n}(1)$ is ample and Y_n is elliptic $H^1(Y_n, \mathcal{O}_{Y_n}(1)) = 0$ (cf. [39, Example IV.1.3.4]). In particular, Riemann-Roch yields that $\mathcal{O}_{Y_n}(1)$ has nonzero global sections. Fix any such nonzero global section s and consider the induced morphism $\mathcal{O}_{Y_n} \to \mathcal{O}_{Y_n}(1)$. Tensoring with $(N_Y^n)^* \operatorname{Syz}(x, y)(2)$ (which happens to be trivial) and taking cohomology we obtain that s annihilates $\delta(z^2)$. And since $\mathcal{O}_{Y_n}(N^{2n}) = (N_Y^n)^* \mathcal{O}_Y(1)$ we have that $\frac{\deg \mathcal{O}_{Y_n}(1)}{\deg N_Y^n} = \frac{\deg \mathcal{O}_Y(1)}{N^{2n}} = \frac{\deg \mathcal{O}_Y(1)}{4^n}$ which goes to zero as n tends to infinity.

7.3 Almost zero for vector bundles

In this section we turn our attention to locally free sheaves of arbitrary rank.

7.37 Lemma (Persistence)**.** Let $\mathcal{S}, \mathcal{T}$ be locally free sheaves on a smooth projective curve Y over an algebraically closed field with a morphism $\mathcal{S} \to \mathcal{T}$. If $c \in H^1(Y, \mathcal{S})$ is almost zero then its image in $H^1(Y, \mathcal{T})$ is almost zero as well.

Proof. Assume that $0 \neq s \in H^0(Y', \mathcal{L})$ annihilates c on $\varphi : Y' \to Y$ such that $\frac{\deg \mathcal{L}}{\deg \varphi} < \varepsilon$. Then we have a commutative square

$$\begin{tikzcd}
H^1(Y', \varphi^* \mathcal{S} \otimes \mathcal{L}) \arrow[r] & H^1(Y', \varphi^* \mathcal{T} \otimes \mathcal{L}) \\
H^1(Y', \varphi^* \mathcal{S}) \arrow[r] \arrow[u, "\cdot s"] & H^1(Y', \varphi^* \mathcal{T}) \arrow[u, "\cdot s"]
\end{tikzcd}$$

which proves the assertion. ☐

7.38 Lemma. Let $\mathcal{S}$ be a locally free sheaf on a smooth projective curve Y over an algebraically closed field and $c \in H^1(Y, \mathcal{S})$. Then c is almost zero if and only if $\varphi^*(c) \in H^1(X, \varphi^*\mathcal{S})$ is almost zero for every finite morphism $\varphi : X \to Y$ of smooth projective curves.

Proof. The implication from right to left is trivial. Conversely, assume that c is almost zero. That is, there is a finite morphism $\psi : Y' \to Y$ of smooth projective curves and a line bundle $\mathcal{L}$ on Y' with a nonzero global section s such that $s\psi^*(c) = 0$ and such that $\deg \mathcal{L}/\deg \psi < \varepsilon$.

Consider the normalisation Z of an irreducible component of the reduced fibre product $X \times_Y Y'$ and note that we have a surjection $\eta : Z \to Y$. In particular, η is a finite morphism. And pulling back $\mathcal{L}$ to Z we have that $\varphi^*(c)$ is annihilated by the pullback of s.

$$\begin{tikzcd}
Z \arrow[dr] & & \\
& X \times_Y Y' \arrow[r, "p_1"] \arrow[d, "p_2"'] & X \arrow[d, "\varphi"] \\
& Y' \arrow[r, "\psi"] & Y
\end{tikzcd}$$

□

7.39 Definition. Let Y be a smooth projective curve over an algebraically closed field and $\mathcal{S}$ a locally free sheaf on Y. We say that $\mathcal{S}$ is *universally almost zero* if for every finite morphism $\varphi : Y' \to Y$ of smooth projective curves $\varphi^*\mathcal{S}$ is almost zero.

7.40 Remark. Any line bundle of non-negative degree on a smooth projective curve Y over an algebraically closed field is universally almost zero. If $g(Y) \geq 1$ and the characteristic of the base is zero then a line bundle is almost zero if and only if it is universally so. We will see later (Corollary 8.4 and Remark 8.5) that $\mathcal{O}_{\mathbb{P}^1}(-1)$ on $\mathbb{P}^1$ is essentially the only exception in characteristic zero: A vector bundle on a a smooth projective curve of genus $g \geq 1$ over an algebraically closed field of characteristic zero is almost zero if and only if it is universally almost zero.

We do not know if this is true in positive characteristic if $g \geq 2$. It is however true for elliptic curves (see Remark 8.5).

7.41 Lemma. Let Y be a smooth projective curve over an algebraically closed field. Let $\mathcal{S}$ be a locally free sheaf and $\mathcal{L}$ a line bundle of positive degree. If $\mathcal{S}$ is universally almost zero then so is $\mathcal{S} \otimes \mathcal{L}$.

Proof. Making a finite pullback we may assume that $\mathcal{L}$ is generated by global sections by virtue of Corollary 7.17. Assume that $\mathcal{S}$ is almost zero. Note that the map $H^1(Y, \mathcal{S}) \to H^1(Y, \mathcal{S} \otimes \mathcal{L})$ induced by $s \in H^0(Y, \mathcal{L})$, $s \neq 0$, is surjective. Indeed, $(\mathcal{S} \otimes \mathcal{L})/\mathcal{S}$ is a torsion sheaf on a curve. Hence, its first cohomology vanishes. Now let $\mathcal{G}$ be a line bundle on some finite curve $\varphi : Y' \to Y$, where Y' is smooth, with a non-trivial global section t such that $\frac{\deg \mathcal{G}}{\deg \varphi} < \varepsilon$ and such that $H^1(Y, \mathcal{S})$ is annihilated by t. We obtain the following commutative diagram, where the composition of the top horizontal arrows annihilates every $c \in H^1(Y, \mathcal{S})$.

$$\begin{tikzcd}
H^1(Y,\mathcal{S}) \arrow[r, "\varphi^*"] \arrow[d, "\cdot s"] & H^1(Y',\varphi^*\mathcal{S}) \arrow[r, "\cdot t"] \arrow[d, "\cdot\varphi^*(s)"] & H^1(Y',\varphi^*\mathcal{S}\otimes\mathcal{G}) \arrow[d, "\cdot\varphi^*(s)"] \\
H^1(Y,\mathcal{S}\otimes\mathcal{L}) \arrow[r, "\varphi^*"] & H^1(Y',\varphi^*(\mathcal{S}\otimes\mathcal{L})) \arrow[r, "\cdot t"] & H^1(Y',\varphi^*(\mathcal{S}\otimes\mathcal{L})\otimes\mathcal{G})
\end{tikzcd}$$

Since the map $H^1(Y, \mathcal{S}) \to H^1(Y, \mathcal{S} \otimes \mathcal{L})$ is surjective it follows that the bottom right arrow annihilates every $c \in H^1(Y, \mathcal{S} \otimes \mathcal{L})$ showing that $\mathcal{S} \otimes \mathcal{L}$ is almost zero. $\square$

Note that if we assume $\mathcal{L}$ to have a global section then the assertion of the lemma continues to hold if we replace "universally almost zero" by "almost zero".

7.42 Lemma. Let Y be a smooth projective curve over an algebraically closed field. Let $0 \to \mathcal{S}' \to \mathcal{S} \to \mathcal{S}'' \to 0$ be a short exact sequence of locally free sheaves. If $\mathcal{S}'$ is universally almost zero and $\mathcal{S}''$ is almost zero then $\mathcal{S}$ is almost zero. Moreover, if $\mathcal{S}'$ and $\mathcal{S}''$ are universally almost zero then $\mathcal{S}$ is universally almost zero.

Proof. Fix $c \in H^1(Y, \mathcal{S})$. Consider the image of c in $H^1(Y, \mathcal{S}'')$ and annihilate this by a non-trivial global section s of some line bundle $\mathcal{L}$ over $\varphi : Y' \to Y$ such that $0 < \frac{\deg \mathcal{L}}{\deg \varphi} < \frac{\varepsilon}{2}$. It follows that the image of c in $H^1(Y', \varphi^*\mathcal{S}'' \otimes \mathcal{L})$ is zero. Hence we find a preimage c' in $H^1(Y', \varphi^*\mathcal{S}' \otimes \mathcal{L})$. By Lemma 7.41, $\varphi^*\mathcal{S}' \otimes \mathcal{L}$ is almost zero. Consequently, we find a line bundle $\mathcal{G}$ on some finite curve $\psi : Y'' \to Y'$, where Y'' is smooth, with a global section such that $\frac{\deg \mathcal{G}}{\deg \varphi\psi} < \frac{\varepsilon}{2}$ and such that c' is annihilated by a non-trivial global section t of $\mathcal{G}$. It follows that the image of c is zero in $H^1(Y'', \psi^*(\varphi^*\mathcal{S} \otimes \mathcal{L}) \otimes \mathcal{G})$. We illustrate the situation with the following commutative diagram where we have omitted the pullbacks.

$$\begin{tikzcd}
H^1(Y,\mathcal{S}') \arrow[r] \arrow[d, "\cdot s"] & H^1(Y,\mathcal{S}) \arrow[r] \arrow[d, "\cdot s"] & H^1(Y,\mathcal{S}'') \arrow[d, "\cdot s"] \\
H^1(Y',\mathcal{S}'\otimes\mathcal{L}) \arrow[r] \arrow[d, "\cdot t"] & H^1(Y',\mathcal{S}\otimes\mathcal{L}) \arrow[r] \arrow[d, "\cdot t"] & H^1(Y',\mathcal{S}''\otimes\mathcal{L}) \arrow[d, "\cdot t"] \\
H^1(Y'',\mathcal{S}'\otimes\mathcal{L}\otimes\mathcal{G}) \arrow[r] & H^1(Y'',\mathcal{S}\otimes\mathcal{L}\otimes\mathcal{G}) \arrow[r] & H^1(Y'',\mathcal{S}''\otimes\mathcal{L}\otimes\mathcal{G})
\end{tikzcd}$$

The supplement follows via a similar argument. $\square$

We now have enough tools at our disposal to extend Proposition 6.4 to characteristic zero for an arbitrary smooth projective base curve. The proof will be by induction on the rank using the fact that a vector bundle is a successive extension of vector bundles of lower ranks. Since strong semistability is not preserved when passing to quotients we will relax the assumptions to $\bar{\mu}_{\min} \geq 0$.

7.43 Proposition. Let Y be a smooth projective curve over an algebraically closed field k. Let $\mathcal{S}$ denote a vector bundle over Y with $\bar{\mu}_{\min}(\mathcal{S}) \geq 0$. Then $\mathcal{S}$ is universally almost zero.

Proof. Note that $\deg \mathcal{S} \geq 0$. We do induction on the rank n of $\mathcal{S}$. For $n = 1$ the result follows by Proposition 7.29. So assume that $n > 1$. By Theorem 3.14, $\mathcal{S}^\vee$ is not ample. Let $\varepsilon > 0$, also by Theorem 3.14 we find a finite morphism of smooth curves $\varphi : Y' \to Y$ together with a line bundle $\mathcal{L}$ on Y' such that $\frac{\deg \mathcal{L}}{\deg \varphi} < \frac{\varepsilon}{3}$ and an exact sequence $\varphi^*\mathcal{S}^\vee \to \mathcal{L} \to 0$. Dually this yields an exact sequence $0 \to \mathcal{L}^\vee \to \varphi^*\mathcal{S} \to \mathcal{G} \to 0$ for some locally free sheaf $\mathcal{G}$ on Y'. Note that $\bar{\mu}_{\min}(\mathcal{G}) \geq 0$. If $\deg \mathcal{L}^\vee \geq 0$ then $\mathcal{L}^\vee$ is universally almost zero and applying the induction hypothesis to $\mathcal{G}$ and then Lemma 7.42 to the exact sequence we are done.

So assume that $\deg \mathcal{L}^\vee < 0$. Choose $\mathcal{M} \in \operatorname{Pic} Y'$ such that $0 < \frac{\deg \mathcal{L}^\vee \otimes \mathcal{M}}{\deg \varphi} < \frac{\varepsilon}{3}$ (e. g. $\mathcal{M} = \mathcal{L}^2$ will do). In particular, $\mathcal{M}$ is ample since $\deg \mathcal{M} > 0$. By Corollary 7.17 we may therefore assume that $\mathcal{M}$ is generated by global sections. Tensoring with $\mathcal{M}$ we obtain an exact sequence

$$0 \to \mathcal{L}^\vee \otimes \mathcal{M} \to \varphi^*\mathcal{S} \otimes \mathcal{M} \to \mathcal{G} \otimes \mathcal{M} \to 0.$$

Therefore as in the first case $\varphi^*\mathcal{S} \otimes \mathcal{M}$ is universally almost zero. Since $\mathcal{M}$ has a nonzero global section s we have an induced morphism $\varphi^*\mathcal{S} \to \varphi^*\mathcal{S} \otimes \mathcal{M}$. Annihilating $H^1(Y', \varphi^*\mathcal{S} \otimes \mathcal{M})$ by a non-trivial global section t of some line bundle $\mathcal{N}$ on a smooth curve $\psi : Y'' \to Y'$ finite over Y' such that $\deg \mathcal{N} / \deg \psi < \frac{\varepsilon}{3}$ yields that $\varphi^*\mathcal{S}$ is almost zero. Indeed, $\psi^*(s) \otimes t \in H^0(Y'', \psi^*\mathcal{M} \otimes \mathcal{N})$ annihilates every $c \in H^1(Y', \varphi^*\mathcal{S})$ and $\deg \psi^*\mathcal{M} \otimes \mathcal{N} / \deg \psi < \varepsilon$.

By the same token, $\mathcal{S}$ is universally almost zero. □

7.44 Remark. The converse of Lemma 7.42 is false. To see this consider the sheaf of differentials on $Y = \mathbb{P}^1_k$ which is isomorphic to $\mathcal{O}_Y(-2)$, where k is an algebraically closed field. Twisting its presenting sequence by $\mathcal{O}_Y(1)$ we have an exact sequence

$$0 \to \mathcal{O}_Y(-1) \to \mathcal{O}_Y \oplus \mathcal{O}_Y \to \mathcal{O}_Y(1) \to 0 \quad \text{(cf. [39, Theorem II.8.13])}.$$

Now pull back along a morphism φ_d ($d > 1$) as in Example 6.3 so that $\mathcal{O}_Y(n) = \mathcal{O}_{Y'}(dn)$. Then both $\mathcal{O}_{Y'} \oplus \mathcal{O}_{Y'}$ and $\mathcal{O}_{Y'}(d)$ are strongly semistable of degree ≥ 0, hence universally almost zero. But $\mathcal{O}_{Y'}(-d)$ is strongly semistable of negative degree. Hence, it is not almost zero if $\operatorname{char} k = 0$ since $d > 1$ (cf. Theorem 7.34). If the characteristic of the base is positive this counterexample continues to hold. Indeed, denote by $\mathcal{E}$ the extension defined by the class c. Then $0 \to \mathcal{O}_{Y'}(-d) \to \mathcal{E} \to \mathcal{O}_{Y'} \to 0$ is a strong Harder-Narasimhan filtration of $\mathcal{E}$ and $\mathcal{O}_{Y'}(-d)$ is not almost zero by Remark 7.35.[10]

7.45 Corollary. Let Y be a smooth projective curve over an algebraically closed field k and $\mathcal{S}$ a strongly semistable vector bundle. Then $c \in H^1(Y, \mathcal{S})$ is not almost zero if

[10] It is more generally true that if X is a polarised variety where the Frobenius does not destabilise semistable bundles (e. g. abelian varieties) then the Frobenius is injective on $H^1(X, \mathcal{S})$ for any semistable locally free sheaf $\mathcal{S}$ of negative degree.

and only if $\deg \mathcal{S} < 0$ and $c \neq 0$ (in positive characteristic $F^{e*}(c) \neq 0$ for all Frobenius powers F^e).

Proof. Assume that c is not almost zero. Thus Proposition 7.43 implies that $\deg \mathcal{S} < 0$ and we must clearly have that $F^{e*}(c) \neq 0$ for all Frobenius powers F^e.

For the other direction assume that $\deg \mathcal{S} < 0$ and that $F^{e*}(c) \neq 0$ for all e. Since $\bar{\mu}_{\min}(\mathcal{S}^\vee) > 0$, we have that $\mathcal{S}^\vee$ is ample by Theorem 3.14. Denote by $\mathcal{S}'$ the extension of $\mathcal{O}_Y$ by $\mathcal{S}$ defined by c. Since c defines a non-trivial extension, it follows from [27, Proposition 2.2] that every quotient of $\mathcal{S}'^\vee$ has positive degree. As the extension does stay non-trivial for all Frobenius pullbacks we have $\bar{\mu}_{\min}(\mathcal{S}'^\vee) > 0$. Hence, again by Theorem 3.14 we have that $\mathcal{S}'^\vee$ is ample and therefore c is not almost zero by Proposition 7.13. □

Now we can finally prove

7.46 Theorem. Let R be a normal standard graded two-dimensional integral k-algebra, where k is an algebraically closed field of arbitrary characteristic. Moreover, let I be an R_+-primary homogeneous ideal with homogeneous generators $f_1, \ldots, f_n$ and assume that $\operatorname{Syz}(f_1, \ldots, f_n)$ on $\operatorname{Proj} R$ is strongly semistable. Then we have

$$(f_1, \ldots, f_n)^{\dagger \mathrm{GR}} = (f_1, \ldots, f_n)^\star.$$

Proof. The inclusion from left to right holds by Corollary 4.10. For the other inclusion assume that $f_0 \in (f_1, \ldots, f_n)^\star$, where we may assume that f_0 is homogeneous of degree d_0 in light of Corollary 1.24. It is enough to show that $\operatorname{Syz}(f_1, \ldots, f_n)(d_0)$ is almost zero. By Theorem 3.12 we have that $\mu(\operatorname{Syz}(f_1, \ldots, f_n)(d_0)) \geq 0$ or that the pullback along some Frobenius power F^e annihilates c[11]. In the first case the result follows from Proposition 7.43 and in the second case c is clearly almost zero. □

We will remove the condition of (strong) semistability in Section 8. There we will also relax the conditions on the ring.

7.4 Some numerical considerations

In this section we compute upper and lower bounds for the ε occurring in the definition of graded dagger closure and almost zero in the case where a given element does not lie in graded dagger closure respectively where a cohomology class is not almost zero.

7.47 Definition. Let Y be a smooth projective curve over an algebraically closed field, $\mathcal{S}$ a locally free sheaf on Y and let c be a cohomology class in $H^1(Y, \mathcal{S})$. The infimum over the $\varepsilon > 0$ such that there is a finite morphism of smooth projective curves $\varphi : Y' \to Y$ and a line bundle $\mathcal{L}$ on Y' with a nonzero global section s such that $sc = 0$ in $H^1(\varphi^* \mathcal{S} \otimes \mathcal{L})$ and $\deg \mathcal{L} / \deg \varphi < \varepsilon$ is called the *almost-zero-ε*.

[11] We remind the reader of our convention that $F = id$ if $\operatorname{char} k = 0$.

Of course, $\varepsilon = 0$ if and only if c is almost zero. Hence, this notion is only interesting if c is not almost zero. And in this case it is an interesting question to find bounds for the almost-zero-ε. It is also interesting to ask how this infimum is related to the infimum over the ε in the definition of graded dagger closure whenever a given homogeneous element f_0 is not contained in a homogeneous R_+-primary ideal. – As it turns out these to infima only differ by a constant factor.

7.48 Proposition. Let Y be a smooth projective curve over an algebraically closed field and let $\mathcal{S}$ be a locally free sheaf on Y. If $c \in H^1(Y, \mathcal{S})$ is not almost zero then $-\bar{\mu}_{\max}(\mathcal{S}')$ is a lower bound for the almost-zero-ε, where $\mathcal{S}'$ is the extension of $\mathcal{O}_Y$ by $\mathcal{S}$ induced by c.

Proof. Let $\varphi : Y' \to Y$ be a finite morphism of smooth projective curves and $\mathcal{L}$ a line bundle on Y' with a nonzero global section s such that $sc \in H^1(\varphi^*\mathcal{S} \otimes \mathcal{L})$ is zero. Looking at the cohomology of the short exact sequence

$$0 \to \varphi^*\mathcal{S} \otimes \mathcal{L} \to \varphi^*\mathcal{S}' \otimes \mathcal{L} \to \mathcal{L} \to 0$$

we obtain a nonzero global section $H^0(Y', \varphi^*\mathcal{S}' \otimes \mathcal{L})$ which corresponds to a morphism $\mathcal{L}^\vee \to \varphi^*\mathcal{S}'$ (cf. proof of the implication from (i) to (ii) in Proposition 7.13). Hence, by Proposition 2.27 (d) we obtain $\deg \mathcal{L}^\vee \leq \mu_{\max}(\varphi^*\mathcal{S}')$. This is equivalent to $-\mu_{\max}(\varphi^*\mathcal{S}') \leq \deg \mathcal{L}$. Since $\deg \varphi \bar{\mu}_{\max}(\mathcal{S}') \geq \mu_{\max}(\varphi^*\mathcal{S}')$ we obtain the desired inequality. $\square$

7.49 Proposition. Let Y be a smooth projective curve over an algebraically closed field k and let $\mathcal{S}$ be a locally free sheaf on Y. Then $|\bar{\mu}_{\min}(\mathcal{S})|$ is an upper bound for the almost-zero-ε for every cohomology class $c \in H^1(Y, \mathcal{S})$.

Proof. Write $|\bar{\mu}_{\min}(\mathcal{S})| = \frac{a}{b}$. We may assume that both a and b are non-negative. If $\bar{\mu}_{\min}(\mathcal{S}) \geq 0$ the assertion is immediate from Proposition 7.43. So we may assume that $0 < |\bar{\mu}_{\min}(\mathcal{S})| = -\bar{\mu}_{\min}(\mathcal{S})$. Fix a finite morphism $\varphi : Y' \to Y$ of smooth projective curves of degree b (this boils down to finding a field extension $L/K(Y)$ of degree b). Fix any line bundle $\mathcal{L}$ on Y' of degree a. In particular, $\mathcal{L}$ is ample so that we may assume that $\mathcal{L}$ has a global section after another finite pull back (cf. Proposition 7.17). Then we have

$$\bar{\mu}_{\min}(\varphi^*\mathcal{S} \otimes \mathcal{L}) = \bar{\mu}_{\min}(\varphi^*\mathcal{S}) + \deg \mathcal{L} = \deg \varphi \bar{\mu}_{\min}(\mathcal{S}) + \deg \mathcal{L} = 0.$$

Hence, $\deg \mathcal{L} / \deg \varphi = |\bar{\mu}_{\min}(\mathcal{S})|$ and since $H^0(Y, \mathcal{L}) \neq 0$ we obtain an induced morphism $\varphi^*\mathcal{S} \to \varphi^*\mathcal{S} \otimes \mathcal{L}$ and the latter bundle is (universally) almost zero by Proposition 7.43. $\square$

7.50 Example. (a) As promised in Example 4.12 we finally compute the minimal homogeneous degree of an element u that multiplies x^4y^3 into $(x^5, xy^4, y^5)R^{+\mathrm{GR}}$ for $R = k[x, y]$. A computation using Čech cohomology on $Y = \operatorname{Proj} R$ for the open covering given by $D_+(x)$ and $D_+(y)$ shows that $\delta(f) = \frac{1}{xy} \in H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2))$. Hence, we must have that $u \cdot 1 \in (x, y)$. Thus, $\deg u \geq 1$. And in fact, $u = x$ will do so that the minimal degree is indeed 1.

(b) Somewhat more generally we consider $Y = \mathbb{P}^1_k$ for an algebraically closed field k, $\mathcal{S} = \operatorname{Syz}(x^a, y^b) = \mathcal{O}_Y(-a-b)$ and $\frac{1}{x^a y^b} \in H^1(Y, \mathcal{S})$. The extension bundle $\mathcal{S}' = \operatorname{Syz}(1, x^a, y^b)$ splits as $\mathcal{O}_Y(-a) \oplus \mathcal{O}_Y(-b)$. Then the almost-zero-$\varepsilon$ is $\min\{a, b\}$ and the bounds we obtain via Propositions 7.48 and 7.49 are $\min\{a, b\}$ and $a + b$.

(c) We note that Proposition 7.49 only yields an upper bound in general. Consider the Fermat cubic $Y = \operatorname{Proj} k[x, y, z]/(x^3 + y^3 + z^3)$, where k is an algebraically closed field of characteristic $\neq 3$. Take $\mathcal{S} = \operatorname{Syz}(x, y) = \mathcal{O}_Y(-2)$ (cf. Example 7.36). A computation using Čech cohomology for the open covering given by $D_+(x)$ and $D_+(y)$ shows that $H^1(Y, \mathcal{S})$ is generated by the classes $\frac{1}{xy}, \frac{z}{xy^2}, \frac{z^2}{xy^3}, \frac{z}{x^2y}, \frac{z^2}{x^3y}, \frac{z^2}{x^2y^2}$. The first three classes are annihilated by multiplication with $x : \mathcal{O}_Y \to \mathcal{O}_Y(1)$, the fourth and fifth are annihilated by multiplication with y and the last one is annihilated by multiplication with z. Hence, the minimal ε is in this case ≤ 3 but the upper bound obtained via the slope is 6.

7.51 Definition. Let R be an $\mathbb{N}$-graded domain, I a homogeneous ideal and $f \in R$ homogeneous. We call the infimum over the ε such that there is $a \in R^{+\mathrm{GR}}$ with $af \in IR^{+\mathrm{GR}}$ and $\nu(a) < \varepsilon$ the *dagger-ε* (for (f, I)).

Of course, $f \in I^{\dagger\mathrm{GR}}$ if and only if the infimum is zero and in this case it is a minimum if and only if $f \in I^{+\mathrm{gr}}$.

7.52 Proposition. Assume Situation 3.6, assume that the dagger-ε for some fixed homogeneous element $f_0 \in R$ of degree d_0 is α and write $Y = \operatorname{Proj} R$, $\mathcal{O}_Y(1) = R(1)^\sim$. Then an upper bound for the almost-zero-ε of $\delta(f_0) \in H^1(Y, \operatorname{Syz}(f_1, \ldots, f_n)(d_0))$ is given by $\alpha \cdot \deg \mathcal{O}_Y(1)$. Moreover, if f_0 is multiplied into $IR^{+\mathrm{GR}}$ by an element of degree $\frac{a}{b}$ then $\delta(f_0)$ is annihilated by a global section of a line bundle $\mathcal{L}$ on some smooth curve $\varphi : Y' \to Y$ finite over Y such that $\deg \mathcal{L} / \deg \varphi = \frac{a}{b} \deg \mathcal{O}_Y(1)$.

Proof. The second assertion implies the first one. So assume that $f_0 \in R$, homogeneous of degree d_0, is multiplied into $IR^{+\mathrm{GR}}$ by some homogeneous element u of degree $\frac{a}{b}$ (note that we may assume that u is homogeneous in light of Lemma 1.17). That is, there is a finite $\mathbb{Q}_{\geq 0}$-graded extension domain S of R such that we have a relation $uf = \sum u_i f_i$ with $u_i, u \in S$. Applying Lemma 4.8, normalising and regrading we may assume that $S(1)^\sim$ is globally generated and that $\varphi^* \mathcal{O}_Y(1) = \mathcal{O}_{\operatorname{Proj} S}(b)$, where $\varphi : \operatorname{Proj} S = Y' \to Y$ denotes the morphism induced by the inclusion. As S is normal Y' is smooth. Write $\mathcal{S} = \operatorname{Syz}(f_1, \ldots, f_n)(d_0)$.

Since $uf \in IS$ we have that the map $H^1(Y', \varphi^* \mathcal{S}) \to H^1(Y', \mathcal{S} \otimes \mathcal{O}_{Y'}(a))$ maps $\delta(f_0)$ to $u\delta(f_0) = 0$. Furthermore, $\deg \mathcal{O}_{Y'}(a) = \deg \varphi \deg \mathcal{O}_Y(1) \frac{a}{b}$. This proves the claim. □

7.53 Remark. (a) Under the assumptions of the proposition if R has a presentation $R = k[x, y, z]/(f)$ for some irreducible polynomial f then $\deg \mathcal{O}_Y(1)$ is just $\deg f$.

(b) Note that once we have a definition of almost zero in arbitrary dimension (cf. Definition 10.20 and take $\mathcal{O}_Y(1)$ as polarisation) this proof carries over to the more general setting.

7.54 Proposition. Assume Situation 3.6, write $Y = \operatorname{Proj} R$, $\mathcal{O}_Y(1) = R(1)^\sim$ and let $f_0 \in R_{d_0}$. If $\delta(f_0)$ in $H^1(Y, \operatorname{Syz}(f_1, \ldots, f_n)(d_0))$ is annihilated by a nonzero global section s of some line bundle $\mathcal{L}$ on a smooth projective curve $\varphi : Y' \to Y$ finite over Y then $\deg \mathcal{L}/(\deg \varphi \deg \mathcal{O}_Y(1))$ is an upper bound for the dagger-ε of f_0.

Proof. Write $\mathcal{S} = \operatorname{Syz}(f_1, \ldots, f_n)(d_0)$. Assume that $0 \neq s \in H^0(Y', \mathcal{L})$ annihilates $\delta(f_0)$ on Y'. Since we can construct arbitrarily small roots of $\mathcal{O}_Y(1)$ on finite smooth projective covers we may assume that there is a root $\mathcal{O}_{Y'}(1)$ of $\mathcal{O}_Y(1)$ on Y' such that $0 < \deg \mathcal{L}^\vee \otimes \mathcal{O}_{Y'}(t)/\deg \varphi < \eta$ for given $\eta > 0$ and suitable $t \in \mathbb{N}$ (cf. the second part of the proof of Theorem 7.23). Applying Proposition 7.17 we may assume that $\mathcal{L}^\vee \otimes \mathcal{O}_{Y'}(t)$ is globally generated. The choice of an injective morphism $\mathcal{O}_{Y'} \to \mathcal{O}_{Y'}(t), 1 \mapsto a$ which factors through $\mathcal{O}_{Y'} \to \mathcal{L}, 1 \mapsto s$ is nothing else but the choice of a suitable global section $\mathcal{L}^\vee \otimes \mathcal{O}_{Y'}(t)$. Hence, we obtain $H^1(Y'\varphi^*\mathcal{S}) \to H^1(Y', \varphi^*\mathcal{S} \otimes \mathcal{O}_{Y'}(t))$ and since it factors through $H^1(Y', \varphi^*\mathcal{S} \otimes \mathcal{L})$ it maps $\varphi^*(c)$ to zero.

Now we apply Proposition 5.4 to obtain a finite $\mathbb{Q}_{\geq 0}$-graded extension domain S of R induced by $\mathcal{O}_{Y'}(1)$ to see that $a \in S$ is of degree $\deg \mathcal{L}/(\deg \varphi \deg \mathcal{O}_Y(1))$. □

7.55 Corollary. Assume Situation 3.6, write $Y = \operatorname{Proj} R$, $\mathcal{O}_Y(1) = R(1)^\sim$ and let $f_0 \in R_{d_0}$. Then the almost-zero-ε of $\delta(f_0)$ coincides with the dagger-ε of f_0 up to the factor $\deg \mathcal{O}_Y(1)$.

Proof. This follows from Theorem 7.23 in the case that one of the quantities is zero. And in the other cases this follows from Propositions 7.52 and 7.54. □

8 Reductions

8.1 Geometric reductions

In this section we will often consider some locally free sheaf $\mathcal{E}$ over a smooth projective curve Y over an algebraically closed field together with a cohomology class $c \in H^1(Y, \mathcal{E})$. This class defines an extension of $\mathcal{O}_Y$ by $\mathcal{E}$ and we will frequently denote this new bundle by $\mathcal{E}'$.

Assume Situation 3.6, write $Y = \operatorname{Proj} R$, $\mathcal{S} = \operatorname{Syz}(f_1, \ldots, f_n)(d_0)$ and consider an element $f_0 \in R_{d_0}$. We now want to reduce the issue whether $(f_1, \ldots, f_n)^\star = (f_1, \ldots, f_n)^{\dagger\mathrm{GR}}$ to the strongly semistable case using a strong Harder-Narasimhan filtration. We will need to look at the cohomology class c defined by the image of f_0 via the connecting homomorphism $H^0(Y, \mathcal{O}_Y(d_0)) \to H^1(Y, \mathcal{S})$.

In [12] this reduction along a strong Harder-Narasimhan filtration is carried out for solid closure and we will follow the arguments there suitably adopted to our situation. Let $\mathcal{S}$ be a vector bundle on a smooth projective curve over an algebraically closed field k and $\mathcal{S}_1 \subset \ldots \subset \mathcal{S}_t = F^{e*}\mathcal{S}$ a strong Harder-Narasimhan filtration. We will need to look at the maximal i such that $\mu(\mathcal{S}_i/\mathcal{S}_{i-1}) \geq 0$. If $\mu(\mathcal{S}_j/\mathcal{S}_{j-1}) < 0$ for all $j = 1, \ldots, t$ then set $i = 0$ and $\mathcal{S}_0 = \mathcal{S}_{-1} = 0$ and if $\mu(\mathcal{S}_j/\mathcal{S}_{j-1}) \geq 0$ for all $j = 1, \ldots, t$ then set $i = t+1$ and $\mathcal{S}_{t+1} = \mathcal{S}_t = F^{e*}\mathcal{S}$. We recall that if the characteristic of the field is zero then the Frobenius is replaced by the identity.

We start with an example

8.1 Example. (a) Let Y be a smooth projective curve over an algebraically closed field k of characteristic zero, let $\mathcal{S}$ be a locally free sheaf on Y of rank 2 and $0 \neq c \in H^1(Y, \mathcal{S})$. We want to investigate when c is almost zero.

If $\mathcal{S}$ is semistable then c is almost zero if and only if $\deg \mathcal{S} \geq 0$. This follows by Corollary 7.45. So assume that $\mathcal{S}$ is not semistable and let

$$0 \to \mathcal{L} \to \mathcal{S} \to \mathcal{M} \to 0$$

be its Harder-Narasimhan filtration. If $\deg \mathcal{M} = \mu_{\min}(\mathcal{S}) \geq 0$ then any $c \in H^1(Y, \mathcal{S})$ is almost zero by Proposition 7.43. In the case $\deg \mathcal{M} < 0$ the situation depends on the cohomology class c. Indeed, if the image of c in $H^1(Y, \mathcal{M})$ is nonzero then c is not almost zero by Lemma 7.37 and by Proposition 7.33. If c is zero in $H^1(Y, \mathcal{M})$ then there is a preimage $\tilde{c} \in H^1(Y, \mathcal{L})$ of c. We claim that in this case c is almost zero if and only if $\tilde{c}$ is so. By virtue of Theorem 7.34, $\tilde{c}$ is almost zero if and only if $\mu_{\max}(\mathcal{S}) = \deg \mathcal{L} \geq 0$. And if $\tilde{c}$ is almost zero then so is c by Lemma 7.37. Assume that c is almost zero. That is, for $\varepsilon > 0$ there is a finite pullback to some smooth projective curve $\varphi : Y' \to Y$ and a line bundle $\mathcal{G}$ on Y' with a global section s such that $sc = 0$ in $H^1(Y' \varphi^* \mathcal{S} \otimes \mathcal{G})$ and $\deg \mathcal{G} / \deg \varphi < \varepsilon$. Since for $\varepsilon > 0$ small enough $\deg \mathcal{M} \otimes \mathcal{G} < 0$ the morphism $H^1(Y', \varphi^* \mathcal{L} \otimes \mathcal{G}) \to H^1(Y', \varphi^* \mathcal{S} \otimes \mathcal{G})$ is injective. Hence, $s\tilde{c} = 0$.

We note that these cases can actually occur. Consider $\mathcal{S} = \mathcal{L} \oplus \mathcal{M}$ with $\deg \mathcal{L} > \deg \mathcal{M}$ and $\deg \mathcal{M} < 0$ and fix a nonzero $c \in H^1(Y, \mathcal{S})$. Then $H^1(Y, \mathcal{S}) = H^1(Y, \mathcal{L}) \oplus H^1(Y, \mathcal{M})$ and the Harder-Narasimhan filtration of $\mathcal{S}$ is $0 \to \mathcal{L} \to \mathcal{S} \to \mathcal{M} \to 0$. We get the first case for $c = (0, c_2)$ and the second case for $c = (c_1, 0)$. And $(c_1, 0)$ is almost zero if and only if $\deg \mathcal{L} \geq 0$ (still assuming that $c \neq 0$).

We will see in Theorem 8.3 below how the question whether $c \in H^1(Y, \mathcal{S})$ is almost zero is related to a strong Harder-Narasimhan filtration for an arbitrary vector bundle $\mathcal{S}$.

(b) Let $R = k[x, y, z]/(x^4 + y^4 - z^4)$, where k is an algebraically closed field of characteristic zero[12]. Consider $\mathcal{S} = \operatorname{Syz}(x^4, y^4, xy^3)$ on $Y = \operatorname{Proj} R$ which splits as $\mathcal{O}_Y(-7) \oplus \mathcal{O}_Y(-5)$. Indeed, it is generated by the R-linearly independent syzygies $(-y^3, 0, x^3)$ in degree 7 and $(0, x, -y)$ in degree 5.

Note that Y is a smooth curve by [39, Ex. I.5.9], the genus g of Y is 3 and that $\mathcal{O}_Y(1)$ has degree 4. Let $d_0 = 6$, so that $\mathcal{S}(d_0) = \mathcal{O}_Y(-1) \oplus \mathcal{O}_Y(1)$. We have $H^1(Y, \mathcal{O}_Y(2)) = 0$ since $\deg \mathcal{O}_Y(2) = 8 \geq 2g - 1$. Recalling the defining sequence of $\mathcal{S}(6)$ we thus obtain that the map $\delta : H^0(Y, \mathcal{O}_Y(6)) \to H^1(Y, \mathcal{S}(6))$ is surjective. Moreover, $H^1(Y, \mathcal{S}(6)) = H^1(Y, \mathcal{O}_Y(-1)) \oplus H^1(Y, \mathcal{O}_Y(1))$ and Riemann-Roch yields that both $H^1(Y, \mathcal{O}_Y(-1))$ and $H^1(Y, \mathcal{O}_Y(1))$ do not vanish. With this and with $d_0 = 5$ and $d_0 = 7$ we get a more explicit example of the cases of (a).

8.2 Theorem. Let Y be a smooth projective curve over an algebraically closed field k. Let $\mathcal{S}$ be a locally free sheaf on Y and $c \in H^1(Y, \mathcal{S})$ with corresponding torsor

[12] Since k is algebraically closed R is isomorphic to $k[x, y, z]/(x^4 + y^4 + z^4)$.

$T \to Y$. Let $\mathcal{S}_1 \subset \ldots \subset \mathcal{S}_t = F^{e*}\mathcal{S}$ be a strong Harder-Narasimhan filtration on Y for a suitable e. Choose i such that $\mathcal{S}_i/\mathcal{S}_{i-1}$ has degree ≥ 0 and such that $\mathcal{S}_{i+1}/\mathcal{S}_i$ has degree < 0. Let $0 \to \mathcal{S}_i \to F^{e*}\mathcal{S} \to F^{e*}\mathcal{S}/\mathcal{S}_i = \mathcal{Q} \to 0$. Then the following are equivalent:

(i) T is not an affine scheme.

(ii) Some Frobenius power of the image of $F^{e*}(c)$ in $H^1(Y, \mathcal{Q})$ is zero.

Proof. See [12, Theorem 2.3]. □

We will now establish the same numerical criterion for graded dagger closure.

8.3 Theorem. Let Y be a smooth projective curve over an algebraically closed field k. Let $\mathcal{S}$ be a locally free sheaf on Y and $c \in H^1(Y, \mathcal{S})$. Let $\mathcal{S}_1 \subset \ldots \subset \mathcal{S}_t = F^{e*}\mathcal{S}$ be a strong Harder-Narasimhan filtration on Y. Choose i such that $\mathcal{S}_i/\mathcal{S}_{i-1}$ has degree ≥ 0 and such that $\mathcal{S}_{i+1}/\mathcal{S}_i$ has degree < 0. Let $0 \to \mathcal{S}_i \to F^{e*}\mathcal{S} \to F^{e*}\mathcal{S}/\mathcal{S}_i = \mathcal{Q} \to 0$. Then the following are equivalent:

(i) The class c is almost zero.

(ii) Some Frobenius power of the image of $F^{e*}(c)$ in $H^1(Y, \mathcal{Q})$ is zero.

Proof. Assume that the image of $F^{e*}(c)$ in $H^1(Y, \mathcal{Q})$ is nonzero for all e. Fix an e and let c' be the image of $F^{e*}(c)$ in $H^1(Y, \mathcal{Q})$. Then it follows as in the proof of Corollary 7.45 that $\mathcal{Q}'^{\vee}$ is an ample vector bundle. Hence, Proposition 7.13 yields that c' is not almost zero. Consequently, by Lemma 7.37, c is not almost zero.

Suppose now that (ii) holds. We may assume that $F^{e*}(c)$ is 0 in $H^1(Y, \mathcal{Q})$. Thus $F^{e*}(c)$ stems from a cohomology class c_i in $H^1(Y, \mathcal{S}_i)$. Now the result follows from Propositions 7.43 and 7.37. Indeed, $\mathcal{S}_1 \subset \ldots \subset \mathcal{S}_i$ is a strong Harder-Narasimhan filtration of $\mathcal{S}_i$. Therefore, $\bar{\mu}_{\min}(\mathcal{S}_i) \geq 0$.

Alternatively, consider the short exact sequence $0 \to \mathcal{S}_{i-1} \to \mathcal{S}_i \to \mathcal{S}_i/\mathcal{S}_{i-1} =: \mathcal{Q}_i \to 0$ and note that $\mathcal{S}_i/\mathcal{S}_{i-1}$ is strongly semistable of degree ≥ 0. Then look at the class c'_i in $H^1(Y, \mathcal{S}_i/\mathcal{S}_{i-1})$. By Proposition 7.43, c'_i is almost zero. Therefore, after a pullback by a finite morphism $\varphi_i : Y' \to Y$ of smooth projective curves we find a line bundle $\mathcal{L}_i$ on Y' with $\frac{\deg \mathcal{L}_i}{\deg \varphi_i} < \varepsilon$ and a nonzero global section s of $\mathcal{L}_i$ such that $s\varphi_i^*(c_i)$ is zero in $H^1(Y', \varphi_i^*\mathcal{Q}_i \otimes \mathcal{L}_i)$. Thus $s\varphi_i^*(c_i)$ stems from a class c'_{i-1} in $H^1(Y', \varphi_i^*\mathcal{S}_{i-1} \otimes \mathcal{L}_i)$. Now consider the short exact sequence $0 \to \mathcal{S}_{i-2} \otimes \mathcal{L}_i \to \mathcal{S}_{i-1} \otimes \mathcal{L}_i \to \mathcal{Q}_{i-1} \to 0$ and repeat the above argument. Going on like this we find that c is almost zero. □

8.4 Corollary. Let Y be a smooth projective curve of genus $g \geq 1$ over an algebraically closed field of characteristic zero and $\mathcal{S}$ a locally free sheaf on Y. Let $\mathcal{S}_1 \subset \ldots \subset \mathcal{S}_t$ be the Harder-Narasimhan filtration of $\mathcal{S}$ on Y. Choose i such that $\mathcal{S}_i/\mathcal{S}_{i-1}$ has degree ≥ 0 and that $\mathcal{S}_{i+1}/\mathcal{S}_i$ has degree < 0. Let $0 \to \mathcal{S}_i \to \mathcal{S} \to \mathcal{S}/\mathcal{S}_i = \mathcal{Q} \to 0$. Then the following are equivalent:

(i) $\mu_{\min}(\mathcal{S}) \geq 0$.

(ii) $\mathcal{S}$ is almost zero.

(iii) $\mathcal{S}$ is universally almost zero (i. e. $\varphi^*\mathcal{S}$ is almost zero for all finite morphisms $\varphi : Y' \to Y$ of smooth projective curves).

(iv) $\mathcal{Q} = 0$.

(v) $H^1(Y, \mathcal{Q}) = 0$.

Proof. The equivalence of (v) and (ii) is immediate from Theorem 8.3 since the map $H^1(Y, \mathcal{S}) \to H^1(Y, \mathcal{Q})$ is surjective. Likewise, the equivalence of (i) and (iv) is immediate from the definition of $\mathcal{Q}$. Assume (v) and assume that $\mathcal{Q} \neq 0$. Note that we have $H^0(Y, \mathcal{Q}) = 0$ since $\mu_{\max}(\mathcal{Q}) < 0$ (apply Proposition 2.27 (d)). We must have that $\deg \mathcal{Q} = \chi(\mathcal{Q}) - \operatorname{rk} \mathcal{Q}\, \chi(\mathcal{O}_Y) = \operatorname{rk} \mathcal{Q}(g-1) \geq 0$ – a contradiction. Thus $\mathcal{Q} = 0$. The implication from (i) to (iii) follows from Proposition 7.43. Finally, assume (iii). Then taking for φ the identity shows that (ii) holds. □

8.5 Remark. (a) We need to exclude $\mathbb{P}^1$ in Corollary 8.4 since $\mathcal{O}_{\mathbb{P}^1}(-1)$ is almost zero of negative degree (and hence any vector bundle $\mathcal{S} = \bigoplus_i \mathcal{O}(d_i)$ on $\mathbb{P}^1$ where the minimal d_i is equal to -1). The equivalence of (i) and (ii) still holds. But (iii), (iv) and (v) are no longer satisfied. Indeed, this is clear for (iii), (iv) and pulling back along a morphism φ_d $(d > 1)$ as in Example 6.3 contradicts (v).

(b) As an aside, we note that there are non-trivial bundles with vanishing cohomology (see [3, Theorem 1]) for curves of genus $g \geq 1$.

(c) The statement of the Corollary remains valid in arbitrary characteristic if Y is elliptic. This follows from a theorem of Oda (see e. g. [9, Theorem 2.2] for a more general version) which asserts that the map $F^* : H^1(Y, \mathcal{S}) \to H^1(Y', F^*\mathcal{S})$ is injective, where F is the (relative) Frobenius and $\mathcal{S}$ a locally free sheaf whose indecomposable components are of negative degree. Note that the condition on $\mathcal{S}$ is equivalent to $\mu_{\min}(\mathcal{S}^\vee) > 0$ by [38, Theorem 1.3].

We do not know whether the corollary holds true in general in positive characteristic. There are of course cases, where the Frobenius is not injective in cohomology for vector bundles of degree < 0 (see e. g. [38, Example 3.2]). But for the corollary to be false one would need a map $H^1(Y, \mathcal{Q}) \to H^1(Y', F^{e*}\mathcal{Q})$ such that the Frobenius is identically zero and $\bar{\mu}_{\max}(\mathcal{Q}) < 0$.

8.6 Remark. Let Y be a a smooth projective curve over an algebraically closed field k and let $0 \to \mathcal{S} \to \mathcal{S}' \to \mathcal{O}_Y \to 0$ be an exact sequence of locally free sheaves. Then there are cases where $\mathcal{S}'^\vee$ is not ample and all the curves that contradict the ampleness in the sense of Seshadri's Theorem 3.16 are contained in the support of $\mathbb{P}(\mathcal{S}^\vee)$. Indeed, by Theorems 8.2 and 8.3 we find curves contradicting the ampleness that do not lie in the support of $\mathbb{P}(\mathcal{S}^\vee)$ if and only if the complement $\mathbb{P}(\mathcal{S}'^\vee) \setminus \mathbb{P}(\mathcal{S}^\vee)$ is not affine. But $\mathcal{S}'^\vee$ may not be ample and have affine complement $\mathbb{P}(\mathcal{S}'^\vee) \setminus \mathbb{P}(\mathcal{S}^\vee)$.

Specifically, let $Y = \mathbb{P}^1_k = \operatorname{Proj} k[x, y]$, $\mathcal{S} = \operatorname{Syz}(x^4, y^4, x^4)(2) = \mathcal{O}_Y \oplus \mathcal{O}_Y(-2)$ and $c = \delta(xy) \in H^1(Y, \mathcal{S})$ so that $\mathcal{S}' = \operatorname{Syz}(x^4, y^4, x^4, xy)$ (cf. [13, Example 7.3]). The forcing divisor is then not ample since $\mathcal{S}'^\vee$ surjects onto $\mathcal{S}^\vee$, hence has the non-ample quotient $\mathcal{O}_Y$. But $\mathbb{P}(\mathcal{S}'^\vee) \setminus \mathbb{P}(\mathcal{S}^\vee)$ is affine since $xy \notin (x^4, y^4) = (x^4, y^4)^\star$ as $k[x, y]$ is regular of dimension 2.

8.7 Theorem. Let k be an algebraically closed field. Let R be a standard graded two-dimensional normal domain of finite type over k. Then for every homogeneous R_+-primary ideal I we have

$$I^\star = I^{\dagger \mathrm{GR}}.$$

Proof. In light of Corollary 1.18 and Corollary 1.24 we may restrict our attention to homogeneous elements. So let $I = (f_1, \ldots f_n)$ and let f_0 be homogeneous with corresponding cohomology class $\delta(f_0) = c$ and torsor T. Combining Theorem 8.2 and Theorem 8.3 we have that the torsor T is not affine if and only if the corresponding cohomology class c is almost zero. The non-affineness of T is equivalent to containment in solid closure by [5, Proposition 3.9]. And c is almost zero if and only if the element f_0 is contained in graded dagger closure by Theorem 7.23. □

8.2 Algebraic reductions

In this section we prove Theorem 8.7 without the conditions standard graded, normal, R_+-primary and k algebraically closed. In doing this, we will frequently need to pass from R to a finite graded extension domain S. It is then clear from the definition of graded dagger closure that if an element f of R is in $(IS)^{\dagger \mathrm{GR}}$ it is also contained in $I^{\dagger \mathrm{GR}}$. But this is a priori not clear at all for solid closure. Therefore, we need some preparations.

8.8 Proposition. Let $\varphi : R \to S$ be a ring homomorphism of noetherian rings, let $I \subseteq R$ be an ideal and $f \in R$. Suppose that $\varphi(f) \in (IS)^\star$. Suppose furthermore that one of the following conditions is satisfied:

(a) S is faithfully flat over R.

(b) For every maximal ideal M of R and every minimal prime $\mathfrak{p}$ of $\widehat{R_M}$ there is a prime ideal Q of S lying over M and a prime ideal $\mathfrak{q}$ of $\widehat{S_Q}$ lying over $\mathfrak{p}$ such that the height of the extended ideal M in $\widehat{S_Q}/\mathfrak{q}$ is $\geq \dim \widehat{R_M}/\mathfrak{p}$.

Then $f \in I^\star$.

Proof. See [46, Theorem 5.9]. □

8.9 Proposition. Assume that $R \to S$ is a finite extension of noetherian domains, $I \subseteq R$ is an ideal and let $f \in R$. Then $f \in (IS)^\star$ implies $f \in I^\star$.

Proof. We will verify that the conditions of Proposition 8.8 (b) are satisfied. We basically reproduce the discussion of [49, Remarks 1.7.6] here.

Localising R at some maximal ideal M we obtain a finite inclusion $R_M \to S_M$. Hence, we may assume that (R, M) is local. We have $\widehat{S_M} = S_M \otimes_R \widehat{R_M}$ by [23, Theorem 7.2 (a)] and since completion is flat a finite inclusion $\widehat{R_M} \subseteq \widehat{S_M}$. Note that by [23, Corollary 7.6], $\widehat{S_M}$ is the direct product of the completions of S with respect to the finitely many maximal ideals lying over MS.

Suppose now that $\mathfrak{p}$ is a minimal prime of $\widehat{R_M}$, then there is a minimal prime $\mathfrak{q}$ in $\widehat{S_M}$ lying over $\mathfrak{p}$. And we have an injective homomorphism $\widehat{R_M}/\mathfrak{p} \to \widehat{S_M}/\mathfrak{q}$. Now, $\widehat{S_M}/\mathfrak{q}$ is a quotient of one of the factors of $\widehat{S_M}$. Hence, we may identify it with the local ring $\widehat{S_Q}/\mathfrak{q}$ for some maximal ideal Q of S lying over M. The desired inequality now follows, since it holds by going up in $\widehat{R_M} \subseteq \widehat{S_M}$ and therefore also holds after killing minimal primes. □

Before presenting the first reduction result we need two somewhat technical lemmata. We shall also need the notion of a *paraclass* in the next lemma. Let R be a d-dimensional $\mathbb{N}$-graded domain finitely generated over a field R_0. Let $(x_1, \ldots, x_d)$ be homogeneous parameters for R. This yields an element $1/(x_1 \cdots x_d) \in H^d_{R_+}(R)$. Any such element is called a *canonical element* or a *paraclass*. Since R contains a field such a class is nonzero (see [21, Theorem 9.2.1 and Remark 9.2.4 (b)]). Moreover, if A is a forcing algebra and $d = 2$ then $H^2_{R_+}(A) = 0$ if and only if some (equivalently every) paraclass coming from R vanishes (see [4, Proposition 1.9]). This is *not* true if $d \geq 3$ in equal characteristic zero, and indeed, this is the issue which parasolid closure addresses. We refer to [4, Section 1] for an elaborate discussion of paraclasses and the connection to (para)solid closure and also to [21, Sections 9.2 and 9.3] for further discussion of paraclasses.

Finally, we recall that the vanishing of a paraclass $c = 1/(x_1 \cdots x_d)$ in $H^d_{R_+}(A)$ is equivalent to $(x_1 \cdots x_d)^t \in (x_1^{t+1}, \ldots, x_d^{t+1})$ in A for some $t \in \mathbb{N}$ (see [21, Remark 9.2.4 (b) and the discussion at the beginning of Section 9.3]).

8.10 Lemma. Let R be a two-dimensional domain of finite type over a field $R_0 = k$, $I \subseteq R$ a homogeneous ideal and f a homogeneous element of R. Let A be the forcing algebra for (f, I) and assume that $H^2_{R_+}(A) = 0$. Then there exists a homogeneous R_+-primary ideal J containing I with forcing algebra A' for (f, J) such that $H^2_{R_+}(A') = 0$.

Proof. Let $I = (f_1, \ldots, f_n)$ and $f \in R$. Write

$$R[T_1, \ldots, T_n]/(\sum_{i=1}^{n} f_i T_i - f)$$

for the forcing algebra A and assume that $H^2_{R_+}(A) = 0$. In particular, paraclasses vanish, that is, we have a relation

$$(xy)^t = a_1 x^{t+1} + a_2 y^{t+1} + P(\sum_{i=1}^{n} f_i T_i - f) \text{ in } R[T_1, \ldots, T_n],$$

where x, y are homogeneous parameters for R_+ and $a_1, a_2, P \in R[T_1, \ldots, T_n]$.

Consider $J = I + (x^{t+1}, y^{t+1})$ – this is obviously R_+-primary and contains I. A forcing algebra for (f, J) is given by

$$A' = R[T_1, \ldots, T_n, U_1, U_2]/(\sum_{i=1}^{n} f_i T_i + x^{t+1} U_1 + y^{t+1} U_2 - f).$$

In $R[T_1, \ldots, T_n, U_1, U_2]$ we obtain the equation

$$(xy)^t = (a_1 - PU_1)x^{t+1} + (a_2 - PU_1)y^{t+1} + P(\sum_{i=1}^{n} f_i T_i + x^{t+1}U_1 + y^{t+1}U_2 - f).$$

This means that the paraclass $1/(xy)$ vanishes in $H^d_{R_+}(A')$ and since R has dimension two this implies that $H^2_{R_+}(A') = 0$. $\square$

8.11 Lemma. Let R be a $\mathbb{N}$-graded domain of dimension two that is finitely generated over a field R_0. Let $I \subseteq R$ be a homogeneous ideal and $f \in R$. Then $f \in I^\star$ if and only if $f \in (IR_{R_+})^\star$.

Proof. The only if part is clear by the persistence of solid closure (cf. [46, Theorem 5.6]). If $\operatorname{ht} I$ is 0 or 2 then the assertion is also clear. For $\operatorname{ht} I = 0$ implies $I = 0$ and $0^\star = \operatorname{Rad} 0 = 0$. If $\operatorname{ht} I = 2$ then R_+ is the only maximal ideal containing I (this follows as the minimal primes over I are homogeneous – see [21, Lemma 1.5.6 (a)]). Since we only have to consider the completions at maximal ideals containing I the claim follows.

So we may assume that $\operatorname{ht} I = 1$. By Proposition 8.9 we may pass to finite graded ring extensions. Adjoining roots of generators of R we may assume that R is standard graded (hence $\mathcal{O}_{\operatorname{Proj} R}(1)$ is locally free and generated by global sections) so that we may work with extensions that are section rings. Furthermore, by passing to the section ring corresponding to $\varphi^*\mathcal{O}_{\operatorname{Proj} R}(1)$, where φ denotes the normalisation morphism, we may assume by Proposition 5.7 (d) that R has an isolated normal singularity in R_+. Since $\varphi^*\mathcal{O}_{\operatorname{Proj} R}(1)$ is still generated by global sections we have by Proposition 5.6 that $\operatorname{Proj} R$ is covered by standard open sets coming from elements of degree 1.

Applying Corollary 1.24 we may assume f to be homogeneous. We have that $f \in (IR_P)^\star = IR_P$ for every minimal prime P over I by persistence and since ideals in regular rings of dimension ≤ 2 are solidly closed. Fix a minimal prime P over I. We then have $uf \in I$ for some $u \in R$ that is not contained in P. And we may assume u to be homogeneous since I, P and f are homogeneous.

Let $P_1, \ldots, P_n$ be the minimal primes over I. We then have homogeneous elements $u_i \in R \setminus P_i$ such that $u_i f \in I$. Moreover, we have elements $s_1, \ldots, s_m$ of degree 1 that cover $\operatorname{Proj} R$. This implies ${s_j}^{-\deg f} f \in (IR_{P_i})_0$ for all i and suitable j. Furthermore, the $D_+(u_i s_j) \cap V_+(I)$ cover $V_+(I) = \{P_1, \ldots, P_n\}$. Looking at the cone mapping we see that the $D(u_i s_j) \cap V(I)$ cover $V(I) \setminus R_+$. Hence, $f \in IR_M = (IR_M)^\star$ for any maximal ideal $M \neq R_+$. Since, by assumption, $f \in (IR_{R_+})^\star$ it follows that $f \in (IR_M)^\star$ for every maximal ideal M of R. Hence, $f \in I^\star$. $\square$

8.12 Theorem. Let k be an algebraically closed field. Let R be an $\mathbb{N}$-graded two-dimensional domain of finite type over $R_0 = k$. Then for every homogeneous ideal I we have

$$I^\star = I^{\dagger\mathrm{GR}}.$$

Proof. We first reduce to the primary case. So suppose in addition that R is normal and standard graded. Let $I = (f_1, \ldots, f_n)$ be a homogeneous ideal. Suppose $f \in I^\star$

for some $f \in R$. For every $l \in \mathbb{N}$ we have $f \in (I + R_{\geq l})^\star$ and we may assume f to be homogeneous of degree m due to Corollary 1.24. Since these ideals are R_+-primary we have $f \in (I + R_{\geq l})^{\dagger \mathrm{GR}}$ by Theorem 8.7. Therefore, by Lemma 1.19 we have for $r \in \mathbb{N}$ a nonzero element a_r of degree $\frac{1}{r}$ (which may depend on l but the degree does not) in some finite $\mathbb{Q}$-graded extension domain S of R such that $a_r f = \sum_i s_i f_i + \sum t_j g_j$ with $s_i, t_j \in S$ and $g_j \in R_{\geq l}$. We may assume that everything is homogeneous, hence for $l > m+1$ we get $t_j = 0$ and therefore $a_r f \in IS$.

Suppose now that $f \in I^{\dagger \mathrm{GR}}$ and assume that $f \notin I^\star$. By Lemma 8.11 this happens if and only if $H^2_{R_+}(A) = 0$, where A is the forcing algebra for (f, I) (note that since R is normal and excellent $R' = \widehat{R_{R_+}}$ is integral, and $H^d_{\widehat{R_+}}(R' \otimes_R A) = H^d_{R_+}(A)$ by flat base change [19, Theorem 4.3.2], since $H^d_m(A) = H^d_m(R) \otimes A$ and because $H^d_{\widehat{m}}(\widehat{R}) = H^d_m(R)$). By Lemma 8.10 it follows that there exists an R_+-primary homogeneous ideal $I \subseteq J$ such that $f \notin J^\star$. But this is a contradiction since we must have $f \in J^{\dagger \mathrm{GR}} = J^\star$ by Theorem 8.7.

Assume now that R is a two-dimensional $\mathbb{N}$-graded domain of finite type over an algebraically closed field $k = R_0$. Write $R = k[x_1, \ldots, x_r]/P$ with $\deg x_i = e_i$ and adjoin e_ith roots of the x_i (cf. Lemma 4.8). Call the normalisation of this ring S. We therefore have a finite injective mapping $R \to S$ such that the $D_+(s), s \in S_1$, cover $\operatorname{Proj} S$. Note that Theorem 8.7 still holds under this weaker hypothesis.

By [46, Theorem 5.6] we have that $f \in I^\star$ implies $f \in (IS)^\star$. But by Theorem 8.7 the containment $f \in (IS)^\star$ yields that $f \in (IS)^{\dagger \mathrm{GR}}$ and then $f \in I^{\dagger \mathrm{GR}}$. For the converse suppose that $f \in I^{\dagger \mathrm{GR}} \subseteq (IS)^{\dagger \mathrm{GR}}$. Hence, we have $f \in (IS)^\star$ by Theorem 8.7. Since $R \subseteq S$ is finite Proposition 8.9 implies $f \in I^\star$. □

8.13 Theorem. Let R denote an $\mathbb{N}$-graded two-dimensional domain of finite type over a field R_0 and I a homogeneous ideal of R. Then $I^{\dagger \mathrm{GR}} = I^\star$.

Proof. We may assume that R is normal. Furthermore, R is geometrically integral. Indeed, this is the case if and only if $Q(R) \cap k = R_0$, where k denotes an algebraic closure of R_0 by virtue of [58, Corollary 3.2.14 (c)]. And elements of $Q(R) \cap k$ are integral over R_0 and hence contained in R since R is normal. Moreover, as any such nonzero element is a unit it is necessarily contained in R_0.

Therefore, we may identify $R_k = R \otimes_{R_0} k$ with $R[\alpha \mid \alpha \in \overline{Q(R)}$ is algebraic over $R_0]$, where $\overline{Q(R)}$ is an algebraic closure of the field of fractions of R. Let $f \in I^{\dagger \mathrm{GR}}$. It follows that $f \in (IR_k)^{\dagger \mathrm{GR}} = (IR_k)^\star$ by Theorem 8.12. Since R_k is faithfully flat over R we have by Proposition 8.8 (a) that $f \in I^\star$.

For the converse assume that $f \in I^\star$. The persistence of solid closure [46, Theorem 5.6] implies that $f \in (IR_k)^\star$. And the latter is equal to $(IR_k)^{\dagger \mathrm{GR}}$ again by Theorem 8.12. Therefore, we immediately have $f \in I^{\dagger \mathrm{GR}}$. □

9 Dagger closure

9.1 Basic properties

In this section we propose a definition of dagger closure for arbitrary domains. Our definition is in principle applicable if the ring does not contain a field but we will not prove any results in this case.

9.1 Definition. Let R be a domain and I an ideal. Then we define the *dagger closure* $I^\dagger$ of I as the set of elements $f \in R$ such that for every $\varepsilon > 0$ and every valuation ν of rank at most one on R^+ there exists $a \in R^+$ with $\nu(a) < \varepsilon$ and $af \in IR^+$.

We recall that a valuation of rank one is a non-trivial valuation whose value group is contained in $\mathbb{R}$. A valuation of rank zero is precisely the trivial valuation (see [76, vol. 2, VI §10 Theorem 15]) and we include this only to force that $0^\dagger = 0$ if R is a field of positive characteristic. Indeed, if for instance $R = \mathbb{F}_p$ then there is no non-trivial valuation on R. See also [42, Remark 1.4]. If R is a complete local noetherian domain and R contains a field of positive characteristic or if the ideal is primary to the maximal ideal then this definition agrees with the one of Hochster and Huneke given earlier (Definition 1.6) – see Corollary 9.9 below. Henceforth, the term "dagger closure" refers to Definition 9.1 unless explicitly stated otherwise.

Definition 9.1 is very much inspired by Heitmann's definition of his full rank one closure (see [42, Definition after Remark 1.3]) as cited in the introduction which we shall recall here for the convenience of the reader:

9.2 Definition. Let R be a domain and I an ideal in R. Then the *full rank one closure* of I is given by the set of elements $f \in R$ such that for every valuation ν on R^+ of rank at most one, every prime number p, every positive integer n and every $\varepsilon > 0$ there exists $a \in R^+$ with $\nu(a) < \varepsilon$ such that $af \in (I, p^n)R^+$.

Note that Definition 9.1 and Heitmann's definition coincide in positive characteristic. Obviously, Heitmann's definition of full rank one closure is not applicable in equal characteristic zero. In this case he proposed a different definition which reduces to the mixed characteristic case by looking at a finitely generated $\mathbb{Z}$-algebra containing all relevant data (see [42, Definition after Lemma 1.6]).

As mentioned before we do not have any results in mixed characteristic. Nevertheless, we propose the following alternative definition in the local case which is the same as dagger closure if the ring contains a field and coincides with Heitmann's rank one closure in mixed characteristic.

9.3 Definition. Let (R, m, k) be a local domain, I an ideal and $\operatorname{char} k = p \geq 0$. Then we define the *local dagger closure* $I^\dagger$ of I as the set of elements $f \in R$ such that for every $\varepsilon > 0$, every $n \in \mathbb{N}$ and every valuation ν of rank at most one on R^+ there exists $a \in R^+$ with $\nu(a) < \varepsilon$ and $af \in (I, p^n)R^+$.

Of course, if R is a local domain of mixed characteristic we have that any prime $q \neq \operatorname{char} k$ is invertible in R. Hence, we only need to look at $p = \operatorname{char} k$ in the full rank one closure so that the two definitions agree.

It should be possible to extend these definitions (and most of the results) to rings which are not domains by reduction to the domain case (i. e. via killing minimal primes) but we did not pursue this (see also [42]). We also made no systematic effort of extending Heitmann's results concerning persistence to dagger closure in equal characteristic zero.

9.4 Remark. In the definition of dagger closure it is enough to only consider valuations ν such that $R^+ \subseteq R_\nu$, where R_ν is the valuation ring associated to ν. Indeed, $R^+ \subseteq R_\nu$ means that all elements of R^+ have non-negative valuation. If there is an element $x \in R^+$ such that $\nu(x) < 0$ and $0 \neq I \subseteq R$ is an ideal then $I^\dagger = R$ with respect to ν. To see this let $0 \neq f \in I$. Then $x^n f \cdot 1 \in I$ and for n sufficiently large $\nu(x^n f) < 0$.

If $I = 0$ then $I^\dagger = 0$ in any case.

9.5 Lemma. Let (R, m, k) be a complete local noetherian domain containing a field. Then there is a $\mathbb{Q}$-valued valuation ν on R which is non-negative on R and positive on m.

Proof. By complete Noether normalisation (see [21, Theorem A.22]), R is isomorphic to a finite extension of a power series ring $P = k[[x_1, \ldots, x_d]]$. Attach a valuation to P such that $\nu(x_i) = 1$ for all i and such that $v(k)$ is trivial. Extending this to R yields a valuation with the desired properties. $\square$

9.6 Lemma. Let $R \subseteq S$ be an extension of integral domains and I an ideal. Then $I^\dagger \subseteq (IS)^\dagger \cap R$.

Proof. See [42, Lemma 1.6]. Note that $R^+ \subseteq S^+$. Therefore, any rank one valuation on S^+ restricts to a valuation on R^+ of rank at most one. The rank one case is clear. So assume that ν is a valuation on S^+ which is trivial on R^+ and let $f \in I^\dagger$. Then any element $0 \neq a \in (I : f)$ will do since $\nu(a) = 0$ in this case. Furthermore, $(I : f) = 0$ if and only if $I = 0$ and in this case $I^\dagger = 0$. $\square$

9.7 Proposition. Let R be a domain of characteristic $p > 0$ essentially of finite type over an excellent local ring. Then dagger closure as in Definition 9.1 coincides with tight closure.

Proof. Dagger closure always contains tight closure (see the argument in the proof of Theorem 1.9) so we only have to show one inclusion. Passing to the normalisation we may assume that R is normal. By virtue of [49, Theorem 1.4.11] R has a completely stable test element. Hence, by [49, Theorem 1.4.7 (g)] an element f is contained in the tight closure of an ideal I if and only if f is contained in the tight closure of the extended ideal $I\widehat{R_m}$ for the completion of R at every maximal ideal m. Furthermore, Lemma 9.6 yields that $I^\dagger \subseteq (I\widehat{R_m})^\dagger$. Therefore, we may assume that R is a complete local domain (since R is normal and excellent, and these two properties are stable under localisation, its completions at maximal ideals are again domains).

Let now $I \subseteq R$ be an ideal, $f \in R$ and assume that $f \in I^\dagger$. In particular, this implies that f is multiplied into IR^+ by elements of small order with respect to an extension of a valuation as in Lemma 9.5 above. But by Theorem 1.9 this yields that $f \in I^*$. □

This entails

9.8 Corollary. Let R be a domain of characteristic $p > 0$ essentially of finite type over an excellent local ring. Then Heitmann's full rank one closure coincides with tight closure.

Proof. This is immediate from the previous proposition since dagger closure and full rank one closure coincide in positive characteristic. □

9.9 Corollary. Let (R, m) be a complete local noetherian domain. Then dagger closure is contained in the dagger closure in the sense of Definition 1.6. If R is of positive characteristic or if the ideal is primary to m then the two closures coincide.

Proof. If an element is contained in dagger closure then a fortiori in the dagger closure of Hochster and Huneke. In positive characteristic the result is immediate from Proposition 9.7 and Theorem 1.9.

Assume now that the ideal $I \subseteq R$ is primary. We have already seen that we only need to look at valuations that are non-negative on R (see Remark 9.4). If they are in addition positive on m then they are equivalent by a theorem of Izumi (see [53]). So the only remaining case is that $\nu(m) \geq 0$ and there is an $x \in m$ such that $\nu(x) = 0$. Since I is primary $x^n \in I$ for $n \gg 0$ so with respect to this fixed valuation the dagger closure of I is R. □

Of course, it is interesting to ask how Definition 9.1 compares to the graded version of dagger closure as defined earlier (Definition 1.13). We do have a result in positive characteristic. First of all, we need the following

9.10 Lemma. Let R be an $\mathbb{N}$-graded normal domain finitely generated over a field R_0. Then the valuation by grading extends to a $\mathbb{Q}$-valued valuation on the completion of R_{R_+} which is non-negative on $\widehat{R_{R_+}}$ and positive on $\widehat{R_{R_+}}R_+$.

Proof. Write $R = k[x_1, \ldots, x_n]/P$ for some prime ideal P and assign the x_i a degree such that the mapping is homogeneous of degree zero. In particular, P is homogeneous. We then have $S := \widehat{R_{R_+}} = k[[x_1, \ldots, x_n]]/Pk[[x_1, \ldots, x_n]]$. Note that S is a domain since R is normal. For $f \in S$ we define

$$\nu(f) = \sup\{\tilde{\nu}(\tilde{f}) \,|\, \tilde{f} \in k[[x_1, \ldots, x_n]] \text{ such that } \tilde{f} = f \mod P' := Pk[[x_1, \ldots, x_n]]\},$$

where $\tilde{\nu}$ on $k[[x_1, \ldots, x_n]]$ is the valuation induced by the grading on $k[x_1, \ldots, x_n]$ (cf. Proposition 1.12) – $\tilde{\nu}(f)$ is the degree of the minimal nonzero "homogeneous term" of f. We claim that this yields a valuation with the desired properties.

First of all, we have to check that the supremum is actually a maximum (i. e. $\neq \infty$) whenever $f \neq 0$. So assume that we have $f \in S$ with $\nu(f) = \infty$. In particular, there is a representative $\tilde{f}_l$ of f with $\tilde{\nu}(\tilde{f}_l) \geq l$ so that $\tilde{f}_l \in (x_1, \ldots, x_n)^l + P'$. But by the Krull Intersection Theorem [23, Corollary 5.4], we must have that $\bigcap_{l\geq 1} = (x_1,\ldots,x_n)^l + P' = P'$ so that $f = 0$ (also cf. the introduction of [53]).

Next, we verify that ν satisfies $\nu(fg) = \nu(f)+\nu(g)$ for $f,g \in S$. If $f \in S$ we denote by $\tilde{f}$ a preimage in $k[[x_1,\ldots,x_n]]$ which is maximal with respect to $\tilde{\nu}$.

So let $f,g \in S$ be nonzero elements with maximal preimages $\tilde{f},\tilde{g}$. We claim that $\widetilde{fg} = \tilde{f}\tilde{g}$. Write $\tilde{f}\tilde{g} = \tilde{f}_a\tilde{g}_b + U$, where $\tilde{f}_a$ and $\tilde{g}_b$ are the minimal "homogeneous components" of $\tilde{f}$ and $\tilde{g}$ respectively and U is a powerseries whose "minimal homogeneous" term is bigger than ab. In particular, $\tilde{\nu}(\tilde{f}\tilde{g}) = ab$. Note that $\tilde{f}_a\tilde{g}_b \notin P'$ since P' is prime (if $\tilde{f}_a \in P'$ then $\tilde{f} - \tilde{f}_a$ would be a representative with bigger valuation). Let $p \in P'$. We have to show that $\tilde{\nu}(\tilde{f}\tilde{g}+p) \leq \tilde{\nu}(\tilde{f}\tilde{g})$. This is certainly the case whenever $\tilde{\nu}(p) \neq \tilde{\nu}(\tilde{f}\tilde{g})$. So we only have to consider the case where equality holds and then we need to show that $(p+\tilde{f}\tilde{g})_{ab} \neq 0$. We can write $p = \sum_{i=1}^m p_iU_i$, where $U_i \in k[[x_1,\ldots,x_n]]$ and $p_i \in P$ are homogeneous. Note that the minimal nonzero "homogeneous term" of p is in P. But the minimal homogeneous term of p is p_{ab}. Hence, $p + \tilde{f}\tilde{g}_{ab} \neq 0$.

Clearly, $\nu(f) = \infty$ if and only if $f = 0$.

Next we show that $\nu(f+g) \geq \min\{\nu(f),\nu(g)\}$. Assume that $\nu(g) > \nu(f+g)$ so that we have to show that $\nu(f+g) \geq \nu(f)$. In particular, $f+g \neq 0$. If $g = 0$ the assertion follows, so that we may assume $g \neq 0$. Fix maximal representatives $\tilde{g}$ and $\widetilde{f+g}$ of g and $f+g$. Then $\tilde{f} = \widetilde{f+g} - \tilde{g}$ is a representative of f with $\tilde{\nu}(\tilde{f}) = \nu(f+g)$. If $\tilde{f}$ were not a maximal representative of f then $\widetilde{f+g}$ would not be maximal.

The remaining claims are obvious since they do hold for $\tilde{\nu}$. □

9.11 Proposition. Let R be an $\mathbb{N}$-graded domain finitely generated over a field R_0 of positive characteristic. Then graded dagger closure coincides with dagger closure as in Definition 9.1.

Proof. We prove that graded dagger closure coincides with tight closure in this setting. Then the result follows by Proposition 9.7.

As usual one inclusion is clear. Furthermore, we may assume that R is normal, hence its completion S with respect to R_+ is an integral domain. Now, if an element $f \in R$ is contained in $I^{\dagger\mathrm{GR}}$ for some ideal $I \subseteq R$ then f is multiplied into IS by elements of arbitrarily small order with respect to the valuation ν constructed in Lemma 9.10. Therefore, $f \in (IS)^*$ by Theorem 1.9. Since R has a complete stable test element (fields are excellent local rings – thus [49, Theorem 1.4.11] applies) this implies $f \in I^*$. □

9.12 Corollary. Let R be an $\mathbb{N}$-graded ring finitely generated over a field R_0 of positive characteristic. Then graded dagger closure coincides with tight closure.

Proof. Since both definitions reduce to the domain case by killing minimal primes the result follows from the proof of Proposition 9.11. □

9.13 Proposition. Let R be a noetherian domain. Then dagger closure coincides with integral closure for principal ideals and dagger closure is always contained in integral closure.

Proof. Let $I = (x)$ be a principal ideal in R and let S be the normalisation of R. If $f \in \bar{I}$ then $f \in \overline{IS} = IS$ (this follows from the first part of the proof of [21, Proposition 10.2.3] which does not require the noetherian hypothesis) and hence immediately $f \in I^\dagger$.

For the other inclusion and the second assertion (cf. [42, Proposition 2.6]) assume that I is an ideal and $f \in I^\dagger$ but $f \notin \bar{I}$. The latter means that there is a discrete valuation ring $R \subseteq V$ such that $f \notin IV$ (see [21, Proposition 10.2.4 (b)]). That is, we have $\nu(f) < \nu(IV)$, where ν is the[13] valuation attached to V. This is in particular a rank one valuation on R which extends to R^+. Since f is contained in the dagger closure we find $u \in R^+$ such that $\nu(u) < \nu(IV) - \nu(f)$ and $uf \in IV^+$. But the latter implies $\nu(uf) \geq \nu(IV)$ which is a contradiction. □

Note that this result very much depends on the fact that we consider *all* rank one valuations on R^+. So one probably should allow some flexibility with respect to the valuations.

9.2 Dagger closure in regular domains

We now want to prove that ideals in a regular noetherian domain containing a field are dagger-closed (meaning $I^\dagger = I$ for any ideal I). We need some lemmata.

9.14 Lemma. Let (R, m) be a local noetherian domain with $\operatorname{depth} R \geq 2$. Then $H^0(D(m), \mathcal{O}_{D(m)}) = R$.

Proof. This is a local version of Proposition 3.4. We remark that since R is local any permutation of a regular sequence is again a regular sequence even if R is not a domain (see [23, Corollary 17.2]). □

9.15 Lemma. Let (R, m) be a regular local noetherian domain of dimension d, $x_1, \ldots, x_d$ generators of m, I an m-primary ideal and $0 \neq c \in H^d_I(R)$. Then there is an $r \in R$ such that $rc = 1/(x_1 \cdots x_d)^t$, that is, rc is a paraclass.

Proof. We identify $H^d_I(R) = H^d_m(R)$ with $R_x/(\sum_i R_{y_i})$, where $x = x_1 \cdots x_d$, and $y_i = x_1 \cdots x_{i-1} x_{i+1} \cdots x_d$ so that $x_i y_i = x$. Since $(x_1, \ldots, x_d) = m$ we can write

$$c = \sum_i \frac{a_i}{x_1^{t_{1,i}} \cdots x_d^{t_{d,i}}},$$

where $a_i \in R^\times$. Fix any maximal tuple $(t_{1,i_0}, \ldots, t_{d,i_0})$. Then $c \cdot x_1^{t_{1,i_0}-1} \cdots x_d^{t_{d,i_0}-1} a_{i_0}^{-1}$ is a paraclass. □

9.16 Definition. Let R be a noetherian local ring. A *big balanced Cohen-Macaulay algebra* is an R-algebra A such that the images of every system of parameters in R form a regular system in A.

[13] This is of course only unique up to equivalence of valuations.

The existence of such algebras has been proven by Hochster and Huneke if R is an excellent local domain containing a field(see [48]). In fact, they also proved that if, in addition, R is of positive characteristic then R^+ is a big balanced Cohen-Macaulay algebra.

9.17 Lemma. Let R be a local noetherian domain and A a big balanced Cohen-Macaulay algebra for R. Then the structure homomorphism $R \to A$ is injective and the images of elements of R in A are not zero divisors.

Proof. Denote the dimension of R by d and fix a nonzero element $r \in R$. Choose a sequence of parameters $x_1, \ldots, x_{d-1}$ in $R/(r)$. Choosing preimages x'_i we have that $r, x'_1, \ldots, x'_d$ are parameters in R. Since A is balanced r cannot map to zero for otherwise $r, x'_1, \ldots, x'_d$ would not form a regular sequence in A. This proves the injectivity. By the same token no element $r \in R$ is a zero divisor in A. □

9.18 Lemma. Let (R, m) be a local noetherian domain, A a big balanced Cohen-Macaulay algebra for R and ν a rank one valuation on R^+ positive on m and non-negative on R. Fix elements $x_1, \ldots, x_d$ in R and let $r \in (x_1, \ldots, x_d)A$, where $r \in R$. Then $\nu(r) \geq \min_i \nu(x_i)$.

Proof. Write $r = \sum_i \alpha_i x_i$ and consider the noetherian R-algebra $B = R[\alpha_1, \ldots, \alpha_d] \subseteq A$. Since A is a big Cohen-Macaulay algebra we have that R embeds into B/P, where P is a minimal prime of B (B is noetherian so minimal primes only contain zero divisors. Thus Lemma 9.17 yields the claim). Trivially extend the valuation $\nu|_{Q(R)}$ to the transcendental part of $Q(B/P)/Q(R)$ and then extend it to the (now) algebraic part – call this extension μ and note that it is non-negative on B/P.

We thus have a non-negative valuation on B/P which lifts to a map $\nu' : B \to \mathbb{R}$ extending ν. Specifically, one defines $\nu'(b) = \mu(\bar{b})$ for all $b \in B$. One still has $\nu'(ab) = \nu'(a)\nu'(b)$ since $\nu'(ab) = \mu(\overline{ab}) = \mu(\bar{a})\mu(\bar{b}) = \nu'(a)\nu'(b)$. Similarly, one sees that $\nu'(a + b) \geq \min\{\nu(a), \nu(b)\}$. Therefore, we obtain $\nu(a) = \nu'(a) \geq \min_i \nu'(x_i) = \min_i \nu(x_i)$ as desired. □

9.19 Definition. Let R be a domain and M an R-module. We say that $m \in M$ is *almost zero* if for every valuation ν on R^+ of rank at most one and for every $\varepsilon > 0$ the element $m \otimes 1 \in M \otimes R^+$ is annihilated by an element $a \in R^+$ with $\nu(a) < \varepsilon$.

Note that this is a special case of dagger closure for modules which would be defined similarly to Definition 7.1.

9.20 Theorem. Let R be a regular noetherian domain containing a field k and $I \subseteq R$ an ideal. Then $I^\dagger = I$.

Proof. Localising at a maximal ideal $I \subseteq m$ and completing we may assume that (R, m) is a noetherian complete regular local domain by Lemma 9.6 and since the completion of a regular noetherian local ring is again a regular noetherian local domain.

Furthermore, if $\dim R \leq 1$ then the result follows from Proposition 9.13 so we may assume $\dim R = d \geq 2$. Since $I \subseteq J$ implies $I^\dagger \subseteq J^\dagger$ and since I is the intersection of m-primary ideals (see e. g. [50, Ex. 1.1]) we may assume that I is m-primary.

Consider a resolution

$$0 \to R^{n_d} \to R^{n_{d-1}} \to \ldots \to R^{n_1} \to R \to R/I \to 0$$

of R/I. The corresponding sequence on $U = D(I) \subseteq \operatorname{Spec} R$ is

$$0 \to \mathcal{O}_U^{n_d} \to \mathcal{O}_U^{n_{d-1}} \to \ldots \to \mathcal{O}_U^{n_1} \to \mathcal{O}_U \to 0.$$

We can extract the following short exact sequences from the resolution for $j = 2, \ldots, d$

$$0 \to \mathrm{Syz}_1 \to \mathcal{O}_U^{n_1} \to \mathcal{O}_U \to 0, \tag{1}$$

$$0 \to \mathrm{Syz}_j \to \mathcal{O}_U^{n_j} \to \mathrm{Syz}_{j-1} \to 0, \tag{2}$$

where Syz_j is the kernel in the jth term. In particular, $\mathrm{Syz}_d = 0$.

Note that by [29, Proposition 2.2] we have R-module isomorphisms $H^j(D(I), \mathcal{O}_U) = H_m^{j+1}(R)$ for $j \geq 1$ and [21, Theorem 3.5.7] implies $H_m^j(R) = 0$ for $j = 1, \ldots, d-1$. Taking cohomology of the exact sequences in (2) hence yields R-module isomorphisms

$$H^{j-1}(D(I), \mathrm{Syz}_{j-1}) \to H^j(D(I), \mathrm{Syz}_j) \text{ for } j = 2, \ldots, d-2$$

and an inclusion

$$H^{d-2}(D(I), \mathrm{Syz}_{d-2}) \to H^{d-1}(D(I), \mathrm{Syz}_{d-1}) = H_m^d(R^{n_d}).$$

By Lemma 9.14 we have $H^0(D(I), \mathcal{O}_U) = R$. The cohomology sequence of (1) yields the connecting homomorphism $\delta : R \to H^1(D(I), \mathrm{Syz}_1)$ and, as in the proof of Proposition 7.5, if an element f is contained in $I^\dagger$ then $\delta(f)$ is almost zero in $H^1(D(I), \mathrm{Syz}_1)$.

So assume that $f \notin I$ but $f \in I^\dagger$ and denote $\delta(f)$ by c. It is enough to show that the image of c is not almost zero in $H_m^d(R)$ since we have an inclusion $H^1(D(I), \mathrm{Syz}_1) \subseteq H_m^d(R^{n_d})$ and since tensor products and cohomology commute with direct sums. Henceforth, we will identify c with its image in $H_m^d(R)$.

By Lemma 9.15 we find $r \in R$ such that $rc = 1/(x_1 \cdots x_d)$ is a parameter class.[14] And if c is almost zero then a fortiori rc. So we have reduced to showing that paraclasses are not almost zero. Assume to the contrary that $c = 1/(x_1 \cdots x_d)$ is almost zero. That is, for every $\varepsilon > 0$ there is an element $a \in R^+$ such that $ac = 0$ and $\nu(a) < \varepsilon$. Since $ac = 0$ if and only if $a(x_1 \cdots x_d)^t = \sum_i a_i x_i^{t+1}$ for some $t \in \mathbb{N}$ and suitable $a_i \in R^+$ we may assume that all relevant data are contained in a finite ring extension $R \subseteq S$. Note that S is still local by [23, Corollary 7.6] since S is a domain.

By [48, Theorem 8.1] and Lemma 9.17 there is a big balanced Cohen-Macaulay algebra A for S which is induced by a local injective homomorphism. We want to prove that the annihilator of $1/(x_1 \cdots x_d)$ in A is $(x_1, \ldots, x_d)A$.

Indeed, by [4, Lemma 2.8] the annihilator of $1/(x_1 \cdots x_d)$ in R is $(x_1, \ldots, x_d)$. Thus we have an exact sequence $0 \to (x_1, \ldots, x_d) \to R \to R \cdot 1/(x_1 \cdots x_d) \to 0$ and since A is flat over R (by [48, 6.7]) tensoring with A yields that the annihilator of $1/(x_1 \cdots x_d)$ in A is $(x_1, \ldots, x_d)A$.

[14] Here we absorbed the exponent in the notation. In particular, the x_i merely form a system of parameters.

In order to deduce a contradiction fix a rank one valuation ν on R which is non-negative and positive on m and extend it to R^+ (this is then necessarily a rank one valuation on R^+). Consider the restriction of ν to S and apply Lemma 9.18 to see that $\nu(a) \geq \min_i \nu(x_i)$ which is a contradiction for $\varepsilon > 0$ small enough. $\square$

9.21 Remark. (a) Since the assertion of the Proposition can be reduced to the complete local case the result follows in characteristic $p > 0$ from Theorem 1.9 and from the fact that ideals in regular rings are tightly closed.

Furthermore, in positive characteristic R^+ is a big balanced Cohen-Macaulay algebra for R if R is an excellent local domain (see [48, Theorem 1.1] or [50, Theorem 7.1]). But this is wrong in characteristic zero if $\dim R \geq 3$.

(b) The last argument of the proof of Theorem 9.20 is considerably simpler in dimension two. Indeed, if $d = 2$ then (in the notation of 9.20) the normalisation S' of S is Cohen-Macaulay. Hence, S' is flat over R by [23, Corollary 18.17] and we can argue as in the theorem to see that the annihilator of $c = 1/(x_1 x_2)$ in S' is precisely $(x_1, x_2)S'$.

Now fix any valuation ν which is non-negative on R and positive on m (e. g. take the one from Lemma 9.5). Then $a \in (x_1, x_2)S'$ implies $\nu(a) \geq \min_i \nu(x_i)$ which is a contradiction for $\nu(a)$ sufficiently small since all x_i have positive valuation.

(c) We note again that we do need some flexibility concerning the valuation (or one that is sufficiently well-behaved). Since we pass to the completion we need to have that the valuation ν we consider on R extends to a rank one valuation on the completion. This is not true for arbitrary rank one valuations – see [42, Remark 3.6].

9.22 Remark. In a somewhat different direction we note that there are balanced big Cohen-Macaulay algebras that are domains provided that R is an affine domain over an uncountable field of characteristic zero. This follows from Schoutens' construction of big Cohen-Macaulay algebras using ultrafilters (cf. [67] and [68]). One considers a relative family $\prod_{p \in X} R_p^+$, where R_p are certain non-trivial domains of varying prime characteristics p and X is an infinite subset of the set of all primes. Then one can mod out the prime ideal $\mathfrak{p}$ associated to an ultrafilter. Schoutens shows among other things that such ultraproducts are big balanced Cohen-Macaulay algebras for R.

9.23 Corollary. Let R be an $\mathbb{N}$-graded regular ring finitely generated over a field R_0 and $I \subseteq R$ an ideal. Then $I^{\dagger \mathrm{GR}} = I$. In particular, graded dagger closure is not solid closure in dimension bigger than two if R_0 is a field of characteristic zero.

Proof. First of all, by [21, Corollary 2.2.20] R is a direct product of regular domains. Hence, killing a minimal prime we may assume that R is a regular domain. The case $\dim R = 0$ is trivial.

Assume that $\dim R = 1$. We may localise at R_+ and extend ν to a valuation on the absolute integral closure of the localisation. Thus R is a regular local ring of dimension one. Hence, $I = (t^n)$, where t is a generator of the maximal ideal. Clearly $\nu(t) > 0$.

If $af \in (t^n)R^+$ we must have that $\nu(a) + \nu(f) \geq n\nu(t)$. Hence, $\nu(f) \geq n\nu(t)$ since we can choose $\nu(a)$ arbitrarily small. But this implies that $f \in (t^n)$.

If $\dim R \geq 2$ then one can employ essentially the same arguments as in the proof of Theorem 9.20. The reduction argument has only to be carried out with respect to a valuation as in Lemma 9.10.

The last claim follows due to Roberts' example showing that ideals in a regular ring of dimension ≥ 3 need not be solidly closed (cf. [46, Discussions 7.22 and 7.23 or Corollary 7.24]). Specifically, one has $x^2y^2z^2 \in (x^3, y^3, z^3)^\star$ in $\mathbb{Q}[x, y, z]$. $\square$

9.24 Corollary. Let (R, m) be an excellent local domain containing a field k. Then no paraclass $c \in H^d_m(R)$ is almost zero.

Proof. Let $x_1, \ldots, x_d$ be parameters and $c = 1/(x_1 \cdots x_d)$ the corresponding paraclass. We may assume that R is normal. For if c is almost zero then a fortiori c is almost zero in $H^d_{mS}(S)$, where S is the normalisation of R. Since $H^d_m(R) = H^d_m(\widehat{R})$ and R is excellent and normal we may moreover assume that R is a complete local domain.

By virtue of complete Noether normalisation (see [21, Theorem A.22]) the ring $T = k[[x_1, \ldots, x_d]]$ is a regular local ring over which R is finite. In particular, $T^+ = R^+$ and since $1/(x_1 \cdots x_d) \in H^d_m(T)$ it cannot be almost zero by the proof of Theorem 9.20. $\square$

9.25 Lemma. Let R be a one-dimensional regular noetherian domain and I an ideal in R. Then $\bar{I} = I$.

Proof. If R is local then the assertion follows from [21, Proposition 10.2.3]. In general, we have $\bar{I} \subseteq \bigcap_P \bar{I}R_P = \bigcap_P IR_P = I$, where the intersection runs over all primes $P \in \operatorname{Spec} R$ and the latter equality holds since $I^\sim$ is a sheaf on $\operatorname{Spec} R$. $\square$

9.26 Corollary. Let R be an $\mathbb{N}$-graded ring of dimension one which is finitely generated over a field R_0 and let $I \subseteq R$ an ideal. Then $\bar{I} = I^{\dagger\mathrm{GR}}$.

Proof. We may reduce to the domain case. This immediately follows from the definition for graded dagger closure and from [21, Proposition 10.2.2 (c)] for integral closure. Denote the normalisation of R by S. We then have $I^{\dagger\mathrm{GR}} \subseteq (IS)^{\dagger\mathrm{GR}} \cap R = IS \cap R$ by Corollary 9.23. Since the inclusion $IS \cap R \subseteq I^{\dagger\mathrm{GR}}$ is immediate equality holds.

Furthermore, we have $\bar{I} \subseteq \overline{IS} \cap R = IS \cap R$ by Lemma 9.25. If $f \in IS \cap R$ then write $f = \sum_i s_i t_i$, where $t_i \in I$ and $s_i \in S$. As $\bar{I}$ is an ideal we only need to show that $s_i t_i \in \bar{I}$ for each i. So fix i and omit the index. Since $R \subseteq S$ is integral we have an equation $s^m + a_1 s^{m-1} + \ldots + a_{m-1}s + a_m = 0$ with $a_i \in R$. Therefore, $(st)^m + a_1 t (st)^{m-1} + \ldots + a_{m-1}t^{m-1}(st) + a_m t^m = 0$ yields that $f \in \bar{I}$. $\square$

9.27 Corollary. Let R be a one-dimensional noetherian domain containing a field k such that the normalisation is finite over R (e. g. R excellent or, more generally, Japanese[15]). Then $I^\dagger = \bar{I}$ for any ideal I.

[15] See [33, Chapitre 0, Définition 23.1.1] for a definition of Japanese rings.

Proof. See the proof of Corollary 9.26. Also note that for principal ideals this is already stated in Proposition 9.13. □

Next we shall need the notion of a parasolid algebra as introduced in [4]. Since we are not concerned with the mixed characteristic case here we may state a simpler definition.

9.28 Definition. Let R denote a local noetherian ring of dimension d containing a field. An R-algebra A is called *parasolid* if the image of every paraclass $c \in H^d_m(R)$ in $H^d_m(A)$ does not vanish.

An algebra A over noetherian ring R containing a field is called *parasolid* if A_m is parasolid over R_m for every maximal ideal m of R.

9.29 Lemma. Let $R \subseteq S$ be a finite extension of noetherian domains containing a field k and let A be an R-algebra. If $A \otimes_R S$ is parasolid as an S-algebra then A is parasolid.

Proof. We may assume that R is a local complete noetherian ring and that $R \subseteq S$ is quasi-local (cf. [4, Proposition 1.5], [23, Corollary 7.6] and recall that completion is flat). Fix a paraclass $c = 1/(x_1 \cdots x_d)$ in $H^d_m(R)$. Let n be a maximal ideal in S containing $(x_1, \ldots, x_d)$. Localising S at n one has that $x_1, \ldots, x_d$ are parameters for S_n. In particular, the image of c in $H^d_{mS}(S) = H^d_n(S)$ and in $H^d_n(S) \to H^d_{nS_n}(S_n)$ is nonzero. We have a commutative diagram

$$\begin{tikzcd} H^d_{nS_n}(S_n) \arrow[r] & H^d_{nS_n}(A \otimes_R S_n) \\ H^d_m(R) \arrow[u] \arrow[r] & H^d_m(A) \arrow[u] \end{tikzcd}$$

and since $A \otimes_R S$ is parasolid c cannot vanish in $H^d_m(A)$. □

9.30 Corollary. Let R be an excellent domain of dimension d containing a field and $I = (f_1, \ldots, f_n)$ an ideal. If $f \in I^\dagger$ then the forcing algebra for (f, I) is parasolid.

Proof. Since we may pass to finite ring extensions we can assume that R is normal (this is clear for dagger closure and follows from Lemma 9.29 for the parasolid case). By Lemma 9.6 we have that $f \in (IR_m)^\dagger$ for every maximal ideal m of R and we may also pass to the completion (this does not affect parasolidity – see [4, Proposition 1.5]). Hence, we may assume that (R, m) is a complete local domain of dimension d.

Assume to the contrary that the forcing algebra

$$B = R[T_1, \ldots, T_n]/(\sum_{i=1}^{n} f_i T_i - f_0)$$

is not parasolid. That is, there exists a paraclass $1/(x_1 \cdots x_d)$ coming from $H^d_m(R)$ which vanishes in $H^d_m(B)$. This is the case if and only if we have an equation

$$b_1 x_1^{t+1} + \ldots + b_d x_d^{t+1} = (x_1 \cdots x_d)^t,$$

where $b_i \in B$ (see [21, Remark 9.2.4 (b) and the discussion at the beginning of Section 9.3]).

Since $f \in I^\dagger$ we have for every $\varepsilon > 0$ a relation $a_0 f_0 = \sum_i^n a_i f_i$ with $a_i \in R^+$ and $\nu(a_0) < \varepsilon$. Hence, in $Q(R^+)$ we have $f_0 = \sum_i^n \frac{a_i}{a_0} f_i$. These relations induce homomorphisms

$$\varphi_\varepsilon : B \to Q(R^+), \quad T_i \mapsto \frac{a_i}{a_0}.$$

This in turn induces a relation

$$a_0^l \sum_{i=1}^d \varphi_\varepsilon(b_i) x_i^{t+1} = a_0^l (x_1 \cdots x_d)^t$$

in R^+ for sufficiently large l. Since the b_i are polynomials in the T_i the exponent l does not depend on a_0. Thus we find that $1/(x_1 \cdots x_d) \in H_m^d(R)$ is almost zero. This is a contradiction in light of Corollary 9.24. $\square$

9.31 Remark. Since we do not have general persistence results for dagger closure we cannot extend Corollary 9.30 to the universally parasolid case.

The result of Corollary 9.30 also holds for graded dagger closure if we only consider homogeneous ideals and only localise at R_+ by virtue of Lemma 9.10.

This result allows us to recover (and extend) an inclusion we proved earlier using geometric methods.

9.32 Corollary. Let R be an $\mathbb{N}$-graded domain finitely generated over a field $k = R_0$ of dimension d and I a homogeneous ideal of height d (e. g. R_+-primary). Then graded dagger closure is contained in solid closure.

Proof. To begin with, we may assume R to be normal. Then for a given ideal I and an element f in R with forcing algebra A we have that $f \in I^\star$ if and only if $H_{R_+}^d(A) \neq 0$ (as in Theorem 8.12 we may omit completions and since R_+ is the only maximal ideal over I we only need to look at local cohomology with respect to R_+). But by the previous remark $f \in I^{\dagger\mathrm{GR}}$ implies $H_{R_+}^d(A) \neq 0$. $\square$

9.33 Remark. We remark that the condition that $\operatorname{ht} I$ be equal to the dimension of R is necessary for this approach. For if $\operatorname{ht} I < \dim R$ then there is some non-homogeneous maximal ideal m containing I. The valuation on R extends to R_m but it is no longer non-negative. Hence, any element of $H_m^d(A)$ is almost zero with respect to this valuation.

This Corollary also has interesting geometric consequences. We will come back to this in Remark 10.21 (b).

9.34 Corollary. Let R be an excellent domain of dimension d containing a field. Then dagger closure is contained in solid closure.

Proof. We may assume that R is normal. Let I be an ideal and $f \in I^\dagger$. Then by Lemma 9.6 we have that $f \in (IR_m)^\dagger$ for every maximal ideal m containing I. Thus Corollary 9.30 implies that $H^d_m(A) \neq 0$, where A is a forcing algebra for the data R_m, f, I. Hence, $f \in I^\star$. □

9.35 Corollary. Let R be an $\mathbb{N}$-graded domain finitely generated over a field $k = R_0$ and let I be a homogeneous ideal of $\operatorname{ht} I = \dim R$. Then $I^{\dagger\mathrm{GR}} \subseteq \bar{I}$.

Proof. Solid closure is contained in integral closure for noetherian rings by [46, Theorem 5.10]. Thus the result follows from Corollary 9.32. □

As another immediate application we obtain an exclusion bound for graded dagger closure. This was proven for tight closure in [71, Theorem 2.2].

9.36 Corollary. Let R be an $\mathbb{N}$-graded domain of dimension $d \geq 2$ finitely generated over an algebraically closed field $k = R_0$ and $(f_1, \ldots f_n)$ an R_+-primary homogeneous ideal, where the f_i are homogeneous of degrees d_i. Let f_0 be another homogeneous element of degree $d_0 \leq \min_i d_i$. Then $f_0 \in (f_1, \ldots, f_n)^{\dagger\mathrm{GR}}$ is only possible if $f_0 \in (f_1, \ldots, f_n)$

Proof. Passing to a finite ring extension we may assume that R is normal and that $R(1)^\sim$ on $\operatorname{Proj} R$ is globally generated (hence invertible). If f_0 is contained in the graded dagger closure of $(f_1, \ldots, f_n)$ then we must have $f_0 \in (f_1, \ldots, f_n)^\star$ by Corollary 9.32. But this is only possible if $f_0 \in (f_1, \ldots, f_n)$ by virtue of [5, Corollary 4.5]. □

We also note that this result is trivial if $d_0 < \min_i d_i$.

Having established the inclusion $I^{\dagger\mathrm{GR}}$ in special cases we can prove that not every element of a given graded module is almost zero.

9.37 Proposition. Let R be an $\mathbb{N}$-graded domain finitely generated over a field R_0 and $M \neq \{0\}$ a finitely generated $\mathbb{Z}$-graded R-module. Then not every element of M is almost zero in the sense of Definition 7.4.

Proof. If M is generated by a single element then we have a presentation $0 \to I \to R \to M \to 0$ and we may assume that $M = R/I$, where I is homogeneous. If M is free so that $I = 0$ then $M \otimes_R R^{+\mathrm{GR}} = R^{+\mathrm{GR}}$ and clearly this module is not almost zero.

So assume that $I \neq 0$. If the characteristic of R is positive then we have that $I^{\dagger\mathrm{GR}} \subseteq I^* \subseteq \operatorname{Rad}(I)$ and hence R/I is not almost zero by Proposition 7.3. If R is of characteristic zero then by Corollary 9.32 any homogeneous ideal J of height $\dim R$ is contained in $\operatorname{Rad}(J)$. It follows that if $I \subseteq J$ and every element of R/I were almost zero then also every element of R/J.

Assume now that M is minimally generated by homogeneous elements $m_1, \ldots, m_n$, $n \geq 2$ and assume that every element of M is almost zero. Consider the short exact sequence $0 \to N \to M \to M/N \to 0$, where N denotes the submodule of M generated by $m_1, \ldots, m_{n-1}$. Then since every element of M is almost zero so is every element of M/N. But M/N is generated by the class of m_n. This contradiction proves the proposition. □

10 An inclusion result for graded dagger closure

In this section we will extend some of the theory of almost zero for vector bundles on curves (as developed in Section 7) to projective varieties of arbitrary dimension. We will then use this to prove an inclusion result for graded dagger closure in certain section rings of abelian varieties. The inspiration for this section is Brenner's article [8] where he proved similar inclusion bounds for tight closure and Frobenius closure in positive characteristic using the (absolute) Frobenius morphism. The key property that one needs is that the Frobenius can manipulate the degree of a vector bundle for a given polarisation while having the same source and target, thereby fixing all invariants of the variety. Similarly, for the multiplication by N map N_X on an abelian variety X we have that it is a finite surjective morphism whose source and target are X.

There are however some difficulties which we will have to overcome. For instance, Brenner requires that the ring R be Cohen-Macaulay. But the section ring of an abelian variety X is not Cohen-Macaulay if $\dim X \geq 2$ (see Proposition 10.31 below). However, the deviation from being Cohen-Macaulay only comes from the intermediate cohomology of $\mathcal{O}_X$. Hence, tensoring by a suitable root of the polarising line bundle this difficulty may be circumvented.

10.1 Semistability in higher dimensions

As mentioned before there are different concepts of semistability for varieties of dimension ≥ 2. We will exclusively deal with the notion of *slope semistability* due to Mumford and Takemoto.

We refer to [51] for an exhaustive treatment of slope semistability and other concepts of (semi)stability in characteristic zero. Here, we will only collect the definitions and facts that we need in the sequel.

10.1 Definition. A *polarised* projective variety X is a projective variety with a fixed ample invertible sheaf $\mathcal{O}_X(1)$. In this case $\mathcal{O}_X(1)$ is called a *polarisation* of X.

10.2 Proposition. Let $(X, \mathcal{O}_X(1))$ be a polarised projective variety and $\mathcal{E}$ a coherent sheaf of $\operatorname{rk}\mathcal{E} \geq 1$ on X. Then the *Hilbert polynomial* $P(\mathcal{E}, m) = \chi(\mathcal{E} \otimes \mathcal{O}_X(m))$ can be uniquely written in the form

$$P(\mathcal{E}, m) = \sum_{i=0}^{\dim X} \alpha_i(\mathcal{E}) \frac{m^i}{i!},$$

with integral coefficients $\alpha_i(\mathcal{E})$. Furthermore, $\alpha_{\dim X}(\mathcal{E}) > 0$.

Proof. See [51, Lemma 1.2.1]. □

10.3 Definition. Let $(X, \mathcal{O}_X(1))$ be a polarised projective variety of dimension d and let $\mathcal{E}$ be a coherent sheaf of $\operatorname{rk}\mathcal{E} \geq 1$. Then the *degree* of $\mathcal{E}$ is defined by $\deg \mathcal{E} = \alpha_{d-1}(\mathcal{E}) - \operatorname{rk}\mathcal{E} \cdot \alpha_{d-1}(\mathcal{O}_X)$, where $\alpha_i(\mathcal{E})$ are the coefficients of the Hilbert polynomial of $\mathcal{E}$.

10.4 Remark. Let $(X, \mathcal{O}_X(1))$ be a smooth projective polarised d-dimensional variety over an algebraically closed field and $\mathcal{E}$ a coherent sheaf of $\operatorname{rk} \geq 1$ on X. Then we have $\deg \mathcal{E} = \deg(\det \mathcal{E}) = \det \mathcal{E}.\mathcal{O}_X(1)^{d-1}$ (see [51, Remark after Definition 1.2.11]) if we consider this as an element of $\mathbb{Z}$ rather than as a zero cycle in the Chow ring of X.

10.5 Definition. Let $(X, \mathcal{O}_X(1))$ be a polarised projective variety and let $\mathcal{E}$ be a coherent torsion-free sheaf on X with $\operatorname{rk} \mathcal{E} \geq 1$. Then $\mu(\mathcal{E}) = \frac{\deg \mathcal{E}}{\operatorname{rk} \mathcal{E}}$ is called the *slope of* $\mathcal{E}$.

The sheaf $\mathcal{E}$ is called *semistable* if for every coherent subsheaf $\mathcal{F} \subset \mathcal{E}$ with $0 < \operatorname{rk} \mathcal{F} < \operatorname{rk} \mathcal{E}$ the inequality $\mu(\mathcal{F}) \leq \mu(\mathcal{E})$ holds. If strict inequality always holds $\mathcal{E}$ is said to be *stable*.

We note that this definition reduces to Definition 2.15 if X is normal, k algebraically closed and $\dim X = 1$. In particular, semistability does not depend on the choice of polarisation in this case. This is wrong in general in dimension ≥ 2 as we will see in Example 10.9 below. But we still need some preparations.

10.6 Definition. Let X be a projective variety. A coherent sheaf $\mathcal{F}$ is *reflexive* if the natural map $\mathcal{F} \to \mathcal{F}^{\vee\vee}$ is an isomorphism.

In particular, any locally free sheaf is reflexive (this is local on the stalks).

10.7 Lemma. Let X be a non-singular surface. Then every reflexive sheaf is locally free.

Proof. See [40, Corollary 1.4]. □

10.8 Proposition. Let $(X, \mathcal{O}_X(1))$ be a polarised normal projective variety over an algebraically closed field and $\mathcal{E}$ a locally free sheaf on X. Then $\mathcal{E}$ is semistable if and only if for all reflexive subsheaves $\mathcal{F}$ of $\mathcal{E}$ one has $\mu(\mathcal{F}) \leq \mu(\mathcal{E})$.

Proof. We only have to prove the implication from right to left. To begin with, [59, Corollary 1.11.1 and Remark 1.12] imply that for any torsion-free sheaf $\mathcal{F}$ on X one has $\mu(\mathcal{F}) = \mu(\mathcal{F}^{\vee\vee})$. Moreover, $\mathcal{F}^{\vee\vee}$ is reflexive (see [40, Corollary 1.2]). Now assume that $\mathcal{F} \subseteq \mathcal{E}$ is a destabilising subsheaf of $\mathcal{E}$. Then we have the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F} \arrow[r] \arrow[d] & \mathcal{E} \arrow[d] \\ & \mathcal{F}^{\vee\vee} \arrow[r] & \mathcal{E}^{\vee\vee}, \end{tikzcd}$$

where the right vertical arrow is an isomorphism since $\mathcal{E}$ is reflexive. We claim that the bottom horizontal arrow is injective. This is a local question so that we may reduce to a situation where R is a local domain and we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d] & R^n \arrow[d] \\ & M^{\vee\vee} \arrow[r] & R^{n\vee\vee} \end{tikzcd}$$

for some finitely generated R-module M. Assume to the contrary that the bottom map is not injective. Hence, there is a nonzero $\varphi \in M^{\vee\vee}$ which is mapped to zero. As φ is nonzero there is a linear form $f : M \to R$ such that $\varphi(f) \neq 0$.

We claim that there is a nonzero $a \in R$ such that af extends to a linear form $af : R^n \to R$. Consider the following commutative diagram:

$$\begin{tikzcd} R \arrow[r, "i"] & Q(R) \\ M \arrow[u, "f"] \arrow[r] & R^n \arrow[u] \end{tikzcd}$$

The horizontal arrows are the inclusions, the right vertical arrow is an extension of if which we obtain since $Q(R)$ is an injective R-module (use [23, Lemma A3.4]). Multiplying by a common denominator a of the images of the standard basis vectors e_i in $R^n \to Q(R)$ we get an extension $g : R^n \to R$ of $af : M \to R$.

Now $\varphi(g)$ is mapped to $\varphi(g|_M) = \varphi(af) = a\varphi(f)$ which is nonzero. But the image of φ was assumed to be the zero map – this is the desired contradiction. $\square$

10.9 Example. We now come to the promised example that semistability in dimension ≥ 2 may depend on the choice of polarisation. Consider the nonsingular quadric surface $X = V_+(xy - zw)$ in $\mathbb{P}^3_k$, where k is an algebraically closed field.[16] Then X is isomorphic to $\mathbb{P}^1_k \times \mathbb{P}^1_k$ and $\operatorname{Pic} X = \mathbb{Z}^2$ is generated by the lines $\mathcal{O}_X(l) = p_1^*\mathcal{O}_{\mathbb{P}^1}(1)$ and $\mathcal{O}_X(m) = p_2^*\mathcal{O}_{\mathbb{P}^1}(1)$ with the properties $l^2 = 0$, $m^2 = 0$ and $l.m = 1$ (see [39, Examples II.6.6.1, V.1.4.3]). Consider a non-trivial extension $0 \to \mathcal{O}_X(l - 3m) \to \mathcal{E} \to \mathcal{O}_X(3m) \to 0$. There is indeed a non-trivial extension since $\operatorname{Ext}^1(\mathcal{O}_X(3m), \mathcal{O}_X(l - 3m)) = H^1(X, l - 6m)$. The latter is isomorphic to $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) \otimes_k H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-6))$ by the Künneth-formula (see [32, Théorème 6.7.8]) and of dimension 10. The Nakai Criterion (see [39, Theorem V.1.10]) implies that $\mathcal{O}_X(l + 7m)$ and $\mathcal{O}_X(l + 5m)$ are ample

We claim that $\mathcal{E}$ is not semistable with respect to the polarisation $\mathcal{O}_X(l + 7m)$. Indeed, we have

$$\mu(\mathcal{E}) = \frac{\det \mathcal{E}.(l + 7m)}{2} = \frac{l.(l + 7m)}{2} = \frac{3}{2}$$

(here we used [39, Ex. II.5.16 (d)] to compute the determinant). Furthermore, $(l - 3m).(l + 7m) = 4 > \frac{3}{2} = \mu(\mathcal{E})$.

We now want to show that $\mathcal{E}$ is semistable with respect to $l + 5m$. In light of Lemma 10.7 and Proposition 10.8 we may restrict our attention to invertible subsheaves.

So let $\mathcal{O}_X(D)$ be an invertible subsheaf of $\mathcal{E}$ with corresponding divisor D. We either have $\mathcal{O}_X(D) \subseteq \mathcal{O}_X(l - 3m)$ or $\mathcal{O}_X(D) \subseteq \mathcal{O}_X(3m)$ (the map $\mathcal{O}_X(D) \to \mathcal{O}_X(3m)$ is either zero or injective).

In the first case $l - 3m - D$ is effective since we have a non-trivial global section (see [39, Proposition II.7.7]). Hence, $(l - 3m - D).(l + 5m) \geq 0$ and we get $\mu(\mathcal{O}_X(D)) = D.(l + 5m) \leq (l - 3m).(l + 5m) = 2$ and $\mu(\mathcal{E}) = \frac{5}{2}$.

[16] I learned this example from Rosa M. Miró-Roig.

In the second case we have that $3m-D$ is effective. Write $D=\alpha l+\beta m$. Intersecting $3m-D=-\alpha l+(3-\beta)m$ with l and m respectively yields $\alpha\leq 0$ and $\beta\leq 3$. If equality held in both cases the sequence would split which it does not. So we have $\beta<3$ or $\alpha<0$. In the first case the slope of D with respect to $l+5m$ is at most 2 and in the second case it is at most -2 so that $\mathcal{E}$ is semistable (in fact, stable) with respect to $l+5m$.

10.10 Definition. Let $(X,\mathcal{O}_X(1))$ be a polarised projective variety and $\mathcal{E}$ a coherent torsion-free sheaf on X of $\operatorname{rk}\mathcal{E}\geq 1$. We call

$$\mu_{\min}(\mathcal{E})=\min\{\mu(\mathcal{Q})\,|\,\mathcal{E}\to\mathcal{Q}\to 0 \text{ is a torsion-free quotient sheaf of rank } \geq 1\}$$

the *minimal slope* of $\mathcal{E}$.

Similarly,

$$\mu_{\max}(\mathcal{E})=\max\{\mu(\mathcal{F})\,|\,0\to\mathcal{F}\to\mathcal{E} \text{ is a subsheaf of rank } \geq 1\}$$

is called the *maximal slope* of $\mathcal{E}$.

10.11 Lemma. Let $(X,\mathcal{O}_X(1))$ be a polarised projective variety and $\mathcal{E}$, $\mathcal{F}$ coherent torsion-free sheaves of $\operatorname{rk}\geq 1$. If $\mu_{\min}(\mathcal{E})>\mu_{\max}(\mathcal{F})$ then $\operatorname{Hom}(\mathcal{E},\mathcal{F})=0$.

Proof. Let $\varphi:\mathcal{E}\to\mathcal{F}$ be a morphism and denote the coherent image sheaf by $\mathcal{I}$. Assume that $\varphi\neq 0$ and hence $\mathcal{I}\neq 0$. Then we necessarily have $\operatorname{rk}\mathcal{I}\geq 1$ since $\mathcal{F}$ is torsion-free. Therefore, $\mu_{\min}(\mathcal{E})\geq\mu(\mathcal{I})$ and $\mu(\mathcal{I})\leq\mu_{\max}(\mathcal{F})$. That is, $\mu_{\min}(\mathcal{E})\leq\mu_{\max}(\mathcal{F})$. □

10.12 Lemma. Let $(X,\mathcal{O}_X(1))$ be a polarised smooth projective variety of dimension d over an algebraically closed field and $\mathcal{S}$ a locally free sheaf on X. If $\mu_{\min}(\mathcal{S})>\deg\omega_X$ then $H^d(X,\mathcal{S})=0$.

Proof. Serre duality yields that $H^d(X,\mathcal{S})=\operatorname{Ext}^0(\mathcal{S},\omega_X)^\vee$. Since $\mu_{\min}(\mathcal{S})>\deg\omega_X$ the claim follows from Lemma 10.11. □

10.13 Proposition. Let $f:X'\to X$ be a finite dominant separable morphism of projective varieties over an algebraically closed field. Fix an ample invertible sheaf $\mathcal{O}_X(1)$ on X. A locally free sheaf $\mathcal{S}$ on X is semistable if and only if $f^*\mathcal{S}$ is semistable with respect to $f^*\mathcal{O}_X(1)$. Moreover, $\mu(f^*\mathcal{S})=\deg f\cdot\mu(\mathcal{S})$ for a locally free sheaf $\mathcal{S}$ and $\mu_{\max}(\mathcal{S})$ and $\mu_{\min}(\mathcal{S})$ transform in the same way.

Proof. Note that by [39, Ex. III.5.7] (or [37, Proposition I.4.4]) $f^*\mathcal{O}_X(1)$ is ample as well. See [51, Lemma 3.2.2] for a proof. □

10.2 Some properties of abelian varieties

In this section we collect some results on abelian varieties. The basic reference is Mumford's book [63].

10.14 Definition. An abelian variety X is a complete variety over an algebraically closed field k with a group structure $\mu : X \times_k X \to X$ such that μ and the inverse are morphisms.

10.15 Proposition. Let X be an abelian variety. Then the following hold:

(a) The group structure on X is commutative.

(b) X is smooth and projective.

Proof. For (a) see either [63, Corollary II.4.2] or [ibid., discussion after Question II.4.4]. We prove that X is smooth. By [39, Corollary II.8.16] there is an open dense subset $U \subseteq X$ where X is smooth. Choose any non-singular point x and fix $y \in X$. The translation morphism $T = T_{y \cdot x^{-1}} : X \to X$ given by $T(z) = y \cdot x^{-1} z$ is an automorphism of X sending x to y. Hence, X is non-singular in y. For the projectivity see [63, discussion before Application II.6.2]. $\square$

10.16 Proposition. Let X be an abelian variety and N an integer with $\operatorname{char} k \nmid N$. Then the multiplication map $N_X : X \to X$ is a finite surjective separable morphism of degree $N^{2 \dim X}$. Furthermore, for all $N \in \mathbb{Z}$ and $\mathcal{L}$ a line bundle we have $N_X^* \mathcal{L} = \mathcal{L}^{\frac{N^2+N}{2}} \otimes (-1_X)^* \mathcal{L}^{\frac{N^2-N}{2}}$.

Proof. The surjectivity is proved in [63, (iv) after Question II.4.4] and the separability and the claim about the degree are treated in [ibid., Application II.6.3]. In order to prove that N_X is finite note that $N_X = \mu^{N-1} \Delta_{N-1}$, where Δ_{N-1} is the $N-1$-fold diagonal morphism. Since these are both projective ([31, Proposition 5.5.5 (i) and (v)]) and N_X has finite fibres it is finite by [39, Ex. III.11.2]. For the second assertion see [63, Corollary II.6.3]. $\square$

10.17 Definition. A line bundle $\mathcal{L}$ on an abelian variety X is called *symmetric* if $\mathcal{L} = (-1)_X^* \mathcal{L}$.

10.18 Lemma. If $\mathcal{L}$ is an ample line bundle on an abelian variety then $\mathcal{L}' = \mathcal{L} \otimes (-1)_X^* \mathcal{L}$ is ample and symmetric. Moreover, if $\mathcal{M}$ is a symmetric line bundle then $N_X^* \mathcal{M} = \mathcal{M}^{N^2}$.

Proof. Since $(-1)_X$ is an automorphism $(-1)_X^* \mathcal{L}$ is ample. Hence, $\mathcal{L}'$ is ample as well by [39, Ex. II.7.5 (c)]. The last claim is immediate from Proposition 10.16. $\square$

10.19 Proposition. Let X be an abelian variety and $\mathcal{L}$ an ample invertible sheaf on X. Then the following hold:

(a) $\dim H^i(X, \mathcal{O}_X) = \binom{\dim X}{i}$.

(b) There exists an $0 \leq i = i(\mathcal{L}) \leq \dim X$ such that $H^i(X, \mathcal{L}) \neq 0$ and $H^j(X, \mathcal{L}) = 0$ for $j \neq i$. Moreover, $H^l(X, \mathcal{L}^{-1}) \neq 0$ if and only if $l = \dim X - i(\mathcal{L})$.

(c) For any $n \geq 3$, $\mathcal{L}^n$ is very ample.

Proof. All references are to [63]. For (a) see [Corollary III.13.2]. Part (b) is the Vanishing Theorem in [III.16]. The assertion in (c) is proved in [III.17]. $\square$

10.3 Almost zero for vector bundles on projective varieties

Throughout this section we work over a fixed algebraically closed field of arbitrary characteristic.

10.20 Definition. Let $\mathcal{S}$ be a locally free sheaf on a polarised projective variety $(Y, \mathcal{O}_Y(1))$ over an algebraically closed field together with a cohomology class $c \in H^1(Y, \mathcal{S})$. We say that c is *almost zero* if for all $\varepsilon > 0$ there exists a finite dominant morphism $\varphi : Y' \to Y$ between projective varieties and a semiample line bundle $\mathcal{L}$ on Y' with a global section $s \neq 0$ such that $\deg \mathcal{L} / \deg \varphi < \varepsilon$ and such that $s\varphi^*(c) = 0$ in $H^1(Y', \mathcal{L} \otimes \varphi^*\mathcal{S})$. Here $s\varphi^*(c)$ is induced by the morphism

$$0 \longrightarrow \varphi^*\mathcal{S} \xrightarrow{\cdot s} \varphi^*\mathcal{S} \otimes \mathcal{L}$$

and the degree on Y' is with respect to $\varphi^*\mathcal{O}_Y(1)$. We say that $\mathcal{S}$ is almost zero if every $c \in H^1(Y, \mathcal{S})$ is almost zero.

10.21 Remark. (a) The situation of interest from the view point of graded dagger closure is $Y = \operatorname{Proj} R$, where R is an $\mathbb{N}$-graded domain finitely generated over an algebraically closed field R_0. Since we may always pass to finite ring extensions for graded dagger closure we may assume that R is normal and that Y is covered by standard open sets coming from elements of degree 1. Hence, $R(1)^\sim = \mathcal{O}_Y(1)$ is an ample invertible sheaf which is generated by global sections. Therefore, in this case there is a canonical choice for a polarisation.

(b) This definition agrees with Definition 7.7 if $\dim X = 1$. To begin with, we may assume X to be normal. Moreover, [39, Lemma IV.1.2] and the fact that a line bundle is ample if and only if its degree is positive ([ibid., Corollary IV.3.3]) yield that semiample line bundles with a global section are precisely those of non-negative degree with a global section.

Also note that the line bundles in Definition 10.20 are of degree ≥ 0 since they are effective and by the Nakai Criterion (cf. [37, Theorem I.5.1].

10.22 Proposition. Let R be an $\mathbb{N}$-graded normal domain finitely generated over an algebraically closed field $R_0 = k$ of dimension $d \geq 2$. Furthermore, assume that $\operatorname{Proj} R$ is covered by open sets $D_+(g)$, $g \in R_1$, and let $I = (f_1, \ldots, f_n)$ be a homogeneous R_+-primary ideal. Fix a homogeneous element f_0 of degree d_0 and write $Y = \operatorname{Proj} R$ and $\mathcal{S} = \operatorname{Syz}(f_1, \ldots, f_n)(d_0)$. Then f_0 is contained in the graded dagger closure of I if and only if $c = \delta(f_0) \in H^1(Y, \mathcal{S})$ is almost zero and we can choose the annihilating line bundles as roots of $\mathcal{O}_Y(1)$.

Proof. If $f_0 \in I^{\dagger \mathrm{GR}}$ then exactly the same argument as in the first paragraph of the proof of Theorem 7.23 shows that $\delta(f_0) = c$ is almost zero and that the annihilating line bundles can be chosen as roots of $\mathcal{O}_Y(1)$.

Conversely, let $\varepsilon > 0$. Assume that there is a finite dominant morphism $\varphi : X \to Y$ and an ample line bundle $\mathcal{L}$ on X such that $\mathcal{L}^m = \varphi^*\mathcal{O}_Y(1)$ for some m together with a nonzero global section $s \in H^0(X, \mathcal{L})$ such that the map $H^1(X, \varphi^*\mathcal{S}) \xrightarrow{\cdot s} H^1(X, \varphi^*\mathcal{S} \otimes$

$\mathcal{L}$) annihilates c, where $\deg \mathcal{L}/\deg \varphi < \varepsilon$. By virtue of Proposition 7.21 we may assume that $\mathcal{L}$ is generated by global sections and passing to the normalisation we may assume X to be normal.

Consider the section ring $S = \bigoplus_{n \geq 0} \Gamma(X, \mathcal{L}^n)$, where we fix a minimal m such that $\mathcal{L}^m$ is isomorphic to $\varphi^* \mathcal{O}_Y(1)$ and identify these line bundles along a fixed isomorphism. This is then a finite normal graded extension domain of R in light of Propositions 5.4 and 5.7. Since we have $m \deg \mathcal{L} = \deg \varphi^* \mathcal{O}_X(1)$, elements in $H^0(X, \mathcal{L})$ correspond to homogeneous elements of degree $\frac{1}{m}$ in S. Moreover, we have $\frac{1}{m} \leq \deg \mathcal{L}/\deg \varphi < \varepsilon$ since $\deg \mathcal{L}^m/\deg \varphi \geq 1$.

Now consider the exact sequence

$$0 \to \varphi^* \mathcal{S} \to \varphi^* \mathcal{S}' \to \mathcal{O}_X \to 0,$$

where $\mathcal{S}'$ is the extension given by c, tensor with $\mathcal{L}$ and take cohomology. This yields that $s \in H^0(X, \mathcal{L})$ has a nonzero preimage $t \in H^0(X, \varphi^* \mathcal{S}' \otimes \mathcal{L})$ which is not contained in $H^0(X, \varphi^* \mathcal{S} \otimes \mathcal{L})$.

The section t therefore corresponds to a relation $a_0 f_0 = \sum_i a_i f_i$ in S and since it is not contained in $\varphi^* \mathcal{S}$ we must have $a_0 \neq 0$ and a_0 has degree $\frac{1}{m} < \varepsilon$. □

10.23 Remark. (a) Of course, we would very much like to drop the assumption that the annihilating line bundles be roots of $\mathcal{O}_Y(1)$. But we do not know whether this is possible. The critical point where the proof of Theorem 7.23 fails to work is that it is not enough that $\deg \mathcal{L}^\vee(t) > 0$ any more. This ensured that the line bundle was ample so that it was globally generated after some finite dominant pullback. In general, we need that $\mathcal{L}^\vee(t)$ is generated by global sections (after some finite dominant pullback) while at the same time bounding $\deg \mathcal{L}^\vee(t)/\deg \varphi$. Of course, there is some power t such that $\mathcal{L}^\vee(t)$ is generated by global sections but we cannot control t while bounding the quotient – not even on a curve.

The Nakai Criterion (see e. g. [37, Theorem I.5.1]) tells us that $\mathcal{L}^\vee(t)$ is ample if and only if it has positive intersection with every subvariety of X. For example, on a surface we have that the (top) self-intersection is positive if $\deg \mathcal{L}^\vee(t) > 0$ but we would still have to check the positivity along every (integral) curve $C \subseteq X$. Of course for $t \gg 0$, $\mathcal{L}^\vee(t)$ is ample since $\mathcal{O}_X(1)$ is ample.

Thus the ampleness property is most likely too restrictive in higher dimensions. Ideally one would like to have a property on the annihilating line bundle $\mathcal{L}$ such that a "small" twist of $\mathcal{L}^\vee$ by a suitable root of $\mathcal{O}_Y(1)$ is semiample.

(b) Assume that R is an $\mathbb{N}$-graded normal domain over an algebraically closed field R_0, $\dim R = d \geq 2$ and that $R(1)^\sim$ on $\operatorname{Proj} R$ is invertible. Then the issue whether $f_0 \in I^\star$ for a homogeneous R_+-primary ideal is equivalent to the issue whether the cohomological dimension (cd for short) of the torsor T associated to $\delta(f_0) \in H^1(\operatorname{Proj} R, \operatorname{Syz}(f_1, \ldots f_n)$ is $d-1$ (see the discussion after Remark 3.5).

As solid closure and graded dagger closure both agree with tight closure in positive characteristic the condition that $\delta(f_0)$ is almost zero with respect to roots of $R(1)^\sim$ is equivalent to $\operatorname{cd} T = d - 1$ if the characteristic of the base is positive. And in characteristic zero being almost zero with respect to roots of $R(1)^{\sim}$ still implies that $\operatorname{cd} T = d-1$ by virtue of Corollary 9.32.

It would be very interesting to have geometric proofs of these facts. In particular, having a geometric proof in characteristic zero might yield an alternative characterisation of almost zero with respect to roots of $R(1)^{\sim}$.

10.24 Proposition. Let $(X, \mathcal{O}_X(1))$ be a polarised projective variety over an algebraically closed field of characteristic $p > 0$ and $t_0 \in \mathbb{Z}$. Then there is a finite dominant morphism $f : X' \to X$ of varieties such that the induced maps $H^i(X, \mathcal{O}_X(t)) \to H^i(X', f^*\mathcal{O}_X(t))$ are almost zero for all $t \geq t_0$ and $1 \leq i < \dim X$.

Proof. To begin with one has $F^*\mathcal{O}_X(1) = \mathcal{O}_X(p)$, where F denotes the relative Frobenius, so we may assume that $\mathcal{O}_X(1)$ is very ample. Moreover, there is a t_1 such that for all $t \geq t_1$ the cohomology groups $H^i(X, \mathcal{O}_X(t))$ vanish for $i > 0$ by virtue of [39, Proposition III.5.3]. By a vanishing theorem of Hochster and Huneke (see [48, Theorem 1.2]) there are finite dominant morphisms $f_{i,t} : X_{i,t} \to X$ for $1 \leq i < \dim X$ and $t \in \mathbb{Z}$, where $X_{i,t}$ are varieties, such that the induced morphisms $H^i(X, \mathcal{O}_X(t)) \to H^i(X_{i,t}, f_{i,t}^*\mathcal{O}_X(t))$ are zero. Repeatedly applying this theorem for $t_1 \leq t < t_0$ and all $1 \leq i < \dim X$ we deduce that there is a finite dominant morphism with the desired properties. $\square$

10.25 Remark. If in addition X is Cohen-Macaulay then $H^i(X, \mathcal{O}(-t))$ is dual to $H^{\dim X - i}(X, \mathcal{O}(t) \otimes \omega_X^\circ)$, where ω_X° is the dualising sheaf of X (cf. [39, Corollary III.7.7]). And again by [39, Proposition III.5.3] these groups vanish for $i > 0$ and $t \gg 0$. In particular, there is a finite dominant morphism $f : X' \to X$, where X' is a variety, such that all maps $H^i(X, \mathcal{O}_X(t)) \to H^i(X', f^*\mathcal{O}_X(t))$ are zero for $i > 0$ and $t \in \mathbb{Z}$.

10.26 Corollary. Let $(X, \mathcal{O}_X(1))$ be a polarised projective variety over an algebraically closed field k of positive characteristic. Then any semiample line bundle $\mathcal{L}$ on X is almost zero and we can choose the annihilators as suitable roots of $\mathcal{O}_X(1)$.

Proof. Let $\varepsilon > 0$. Applying Proposition 7.21 we may assume that $\mathcal{L}$ is globally generated. Applying Lemma 4.4 and Proposition 7.21 we may assume that there is a finite dominant morphism $\varphi : X' \to X$ of varieties such that there is a globally generated line bundle $\mathcal{M}$ on X' which is an nth root of $\mathcal{O}_X(1)$ such that $\frac{\deg \mathcal{M}}{\deg \varphi} < \varepsilon$ (simply choose n sufficiently large). Since $\mathcal{M}$ is globally generated we have an induced morphism $\varphi^*\mathcal{L} \to \varphi^*\mathcal{L} \otimes \mathcal{M}$ and $\varphi^*\mathcal{L} \otimes \mathcal{M}$ is ample since $\varphi^*\mathcal{L}$ is globally generated (see [39, Ex. II.7.5 (a)]). By Proposition 10.24 there is a pullback along a finite dominant morphism which annihilates $H^1(X', \varphi^*\mathcal{L} \otimes \mathcal{M})$. In particular, $\mathcal{L}$ is almost zero and we can choose the annihilators as suitable roots of $\mathcal{O}_X(1)$. $\square$

10.27 Proposition. Let Y be a polarised projective variety over an algebraically closed field k and $\mathcal{S}$ a locally free sheaf on Y. If $c \in H^1(Y, \mathcal{S})$ is almost zero then $\bar{\mu}_{\max}(\mathcal{S}') \geq 0$, where $\mathcal{S}'$ denotes the extension of $\mathcal{O}_Y$ by $\mathcal{S}$ induced by c.

Proof. This follows exactly as Proposition 7.48 except that one has to use Lemma 10.11 instead of Proposition 2.27 (d). □

10.28 Corollary. Let R be an $\mathbb{N}$-graded domain such that $\mathcal{O}_Y(1) = R(1)^\sim$ on $Y = \operatorname{Proj} R$ is invertible. Furthermore, let I be a homogeneous R_+-primary ideal with homogeneous generators $f_1, \ldots, f_n$. If $f_0 \in (f_1, \ldots, f_n)^{\dagger \mathrm{GR}}$ then $\bar{\mu}_{\max}(\operatorname{Syz}(f_0, \ldots, f_n)) \geq 0$.

Proof. This is immediate from Proposition 10.27. □

10.29 Remark. We do not think that the result of Corollary 10.28 actually provides a useful exclusion bound if $\dim R = d + 1 \geq 3$. The heuristic here is that in higher dimensions the first syzygy bundle has to be "very positive" if one wants to have nontrivial containment relations. It seems that in this case one should look at the dth cohomology of the dth syzygy bundle in a resolution on $\operatorname{Proj} R$.

Consider $R = k[x, y, z]$ and $I = (x^a, y^a, z^a)$, where k is an algebraically closed field of characteristic zero. Since x^a, y^a, z^a form a regular sequence of parameters the Koszul complex is a free resolution of R/I.

Sheafifying this complex on $\operatorname{Proj} R = \mathbb{P}^2_k$ we obtain $\mathcal{S}_1 = \operatorname{Syz}(x^a, y^a, z^a)$ as the first syzygy bundle. The bundle $\mathcal{S}_1$ is semistable of slope $\mu(\mathcal{S}_1) = -\frac{3}{2}a$ and the minimal l such that $R_l \subseteq (x^a, y^a, z^a)^{\dagger \mathrm{GR}} = (x^a, y^a, z^a)$ is $l = 3a - 2$ (of course the polarisation is with respect to $\mathcal{O}_{\mathbb{P}^2_k}(1)$). The top-dimensional syzygy bundle is given by $\mathcal{S}_d = \mathcal{O}_{\mathbb{P}^2_k}(-3a)$ which is of slope $\mu(\mathcal{S}_d) = -3a$.

Using restriction theorems (e. g. [24, Theorem 1.2]) one has that $i^*\mathcal{S}_1$ is semistable on a suitable smooth curve $i : C \to \mathbb{P}^2_k$ and the slope of $i^*\mathcal{S}_1$ is $-\frac{3}{2}a \deg C$. In particular, the minimal l is in this case $\frac{3}{2}a$ (use Proposition 7.43). Of course, on a curve $\mathcal{S}_1$ is the top dimensional syzygy bundle.

10.4 An inclusion result for graded dagger closure in certain section rings of abelian varieties

10.30 Definition. Let X be a projective variety over an algebraically closed field and let $\mathcal{L}$ be an ample line bundle on X. The pair $(X, \mathcal{L})$ is called *arithmetically Cohen-Macaulay* if the section ring $\bigoplus_{n \geq 0} \Gamma(X, \mathcal{L}^n)$ is Cohen-Macaulay.

Note that our definition of arithmetically Cohen-Macaulay is slightly different from the standard one. Our definition looks at the section ring of $\mathcal{L}$ which is, in the case that $\mathcal{L}$ is very ample and X normal, the normalisation of the homogeneous coordinate ring of the embedding (see [39, Ex. II.5.14]).

10.31 Proposition. Let X be an abelian variety and $\mathcal{L}$ an ample invertible sheaf. Then $(X, \mathcal{L})$ is arithmetically Cohen-Macaulay if and only if $\dim X = 1$, that is, X is an elliptic curve.

Proof. Since in dimension 2 normality implies Cohen-Macaulayness the if-part follows from Proposition 5.7. Conversely, the section ring S is Cohen-Macaulay if and only if $H^i_{S_+}(S) = 0$ for $0 \leq i \leq \dim S - 1$ (see [21, Propositions 3.5.4, 3.6.4 and Theorem

3.6.3]) and by [23, Theorem A4.1] we have $H^i(X,\mathcal{O}_X(n)) = H^{i+1}_{S_+}(S)_n$ for $i > 0$. Proposition 10.19 (a) implies that $\mathcal{O}_X$ has non-vanishing intermediate cohomology if $\dim X \geq 2$. Hence, S cannot be Cohen-Macaulay in this case. $\square$

10.32 Lemma. Let X denote an abelian variety of dimension d and $\mathcal{O}_X(1)$ a very ample line bundle on X. Let $\mathcal{S}$ be a locally free sheaf on X. Let

$$\ldots \to \mathcal{G}_3 \to \mathcal{G}_2 \to \mathcal{S} \to 0$$

denote an exact complex of sheaves, where $\mathcal{G}_j$ has type $\bigoplus_{(k,j)} \mathcal{O}_Y(-\alpha_{k,j})$ with $\alpha_{k,j} \neq 0$. Set $\mathcal{S}_1 = \mathcal{S}$ and set $\mathcal{S}_j = \operatorname{im}(\mathcal{G}_{j+1} \to \mathcal{G}_j) = \ker(\mathcal{G}_j \to \mathcal{G}_{j-1})$ for $j \geq 2$. Fix $i \in \{1,\ldots,d\}$. Then there are isomorphisms $H^i(X,\mathcal{S}_1) = H^{i+1}(X,\mathcal{S}_2) = \ldots = H^{d-1}(X,\mathcal{S}_{d-i})$ and an inclusion $H^i(X,\mathcal{S}_1) \to H^d(X,\mathcal{S}_{d+i-1})$.

Proof. Since $\mathcal{O}_X(1)$ is very ample the only non-vanishing cohomology group of $\mathcal{O}_X(n)$ is $H^0(X,\mathcal{O}_X(n))$ for $n > 0$ respectively $H^d(X,\mathcal{O}_X(n))$ for $n < 0$ by Proposition 10.19 (b). Since all the $\alpha_{k,j}$ are nonzero this means that $H^i(X,\mathcal{G}_j) = 0$ for $j \geq 2$ and $i = 1,\ldots,d-1$.

Looking at the cohomology of the short exact sequences $0 \to \mathcal{S}_{j+1} \to \mathcal{G}_{j+1} \to \mathcal{S}_j \to 0$ we can thus extract isomorphisms

$$H^i(X,\mathcal{S}_j) = H^{i+1}(X,\mathcal{S}_{j+1}) \text{ for } i = 1,\ldots,d-2,$$

and inclusions $H^{d-1}(X,\mathcal{S}_j) \to H^d(X,\mathcal{S}_{j+1})$. In particular, we have isomorphisms $H^i(X,\mathcal{S}_1) = H^{i+1}(X,\mathcal{S}_2) = \ldots = H^{d-1}(X,\mathcal{S}_{d-i})$ and an inclusion $H^i(X,\mathcal{S}_1) \to H^d(X,\mathcal{S}_{d+1-i})$. $\square$

10.33 Lemma. Let $(X,\mathcal{O}_X(1))$ be a polarised abelian variety of dimension d and assume that $\mathcal{O}_X(1)$ is symmetric. Then for a locally free sheaf $\mathcal{S}$ on X we have $\mu(N_X^*\mathcal{S}) = N^2\mu(\mathcal{S})$, where both degrees are with respect to $\mathcal{O}_X(1)$.

Proof. The slope of $N_X^*\mathcal{S}$ with respect to $N_X^*\mathcal{O}_X(1) = \mathcal{O}_X(N^2)$ is $N^{2d}\mu(\mathcal{S})$ by Proposition 10.13. The degree with respect to $\mathcal{O}_X(l)$ of a line bundle $\mathcal{L}$ is given by

$$\mathcal{L}.\mathcal{O}_X(l)^{d-1} = l^{d-1}\mathcal{L}.\mathcal{O}_X(1)^{d-1} \text{ (see also Remark 10.4).}$$

Hence, the slope with respect to $\mathcal{O}_X(1)$ is given by $N^{2d}/N^{2d-2}\mu(\mathcal{S}) = N^2\mu(\mathcal{S})$. $\square$

10.34 Situation. Let $R = \bigoplus_{n\geq 0}\Gamma(X,\mathcal{O}_X(n))$ be the section ring of a polarised abelian variety $(X,\mathcal{O}_X(1))$ over an algebraically closed field of arbitrary characteristic and $\dim R = d+1$ (i.e. $\dim X = d$). Assume moreover that $\mathcal{O}_X(1)$ is symmetric. Let I be a homogeneous R_+-primary ideal and let

$$\ldots \to F_2 = \bigoplus_{(k,2)} R(-\alpha_{k,2}) \to F_1 = \bigoplus_{(k,1)} R(-\alpha_{k,1}) \to R \to R/I \to 0$$

be a homogeneous complex of graded R-modules that is exact on $D_+(R_+)$ (e.g. a graded resolution of I). Let

$$\ldots \to \mathcal{G}_2 = \bigoplus_{(k,2)} \mathcal{O}_X(-\alpha_{k,2}) \to \mathcal{G}_1 = \bigoplus_{(k,1)} \mathcal{O}_X(-\alpha_{k,1}) \to \mathcal{O}_X \to 0$$

denote the corresponding exact complex of sheaves on X. Denote by $\mathcal{S}_j = \ker(\mathcal{G}_j \to \mathcal{G}_{j-1})$ the kernel sheaves. Finally, let $\nu = \mu_{\min}(\mathcal{S}_d)/\deg \mathcal{O}_X(1)$.

10.35 Lemma. The $\mathcal{S}_j$ in Situation 10.34 are locally free sheaves.

Proof. We have short exact sequences of sheaves for $j = 1$

$$0 \to \mathcal{S}_1 \to \mathcal{G}_1 \to \mathcal{O}_X \to 0,$$

and for $j = 2, \dots d$

$$0 \to \mathcal{S}_j \to \mathcal{G}_j \to \mathcal{S}_{j-1} \to 0.$$

Hence, the claim follows by inductively applying Lemma 2.8. $\square$

10.36 Theorem. Assume Situation 10.34. Then we have an inclusion $R_{\geq \nu} \subseteq I^{\dagger \mathrm{GR}}$.

Proof. Fix an integer $N \geq 3$ such that $N \nmid \operatorname{char} k$. Pulling back along N_X we may assume $\mathcal{O}_X(1)$ to be very ample in light of Proposition 10.19 (c). If we pull back along a finite dominant separable morphism $f : X' \to X$ then the polarisation on X' will be given by $f^*\mathcal{O}_X(1)$.

By Proposition 10.22 it is enough to show that $\mathcal{S}_1(m)$ is almost zero for $m \geq \nu$ with respect to roots of $\mathcal{O}_X(1)$. So let $\varepsilon > 0$.

In order to be in a situation to apply Lemma 10.32 we pull back the whole situation along N_X^e for $e \gg 0$ and then tensor with $\mathcal{O}_X(1)$. Then we may assume that all $\alpha_{i,j}$ are nonzero and that $\deg \mathcal{O}_X(1)/\deg N_X^e < \varepsilon$. We therefore have an inclusion

$$H^1(X, {N_X^e}^*\mathcal{S}_1(m) \otimes \mathcal{O}_X(1)) \to H^d(X, {N_X^e}^*\mathcal{S}_d(m) \otimes \mathcal{O}_X(1)).$$

Note that $\mu_{\min}({N_X^e}^*\mathcal{S}_d(m) \otimes \mathcal{O}_X(1)) \geq \deg \mathcal{O}_X(1) > 0$. Again pulling back along N_X^c for $c \gg 0$ we have that $\mu_{\min}({N_X^c}^*({N_X^e}^*\mathcal{S}_d(m) \otimes \mathcal{O}_X(1))) \geq \deg \mathcal{O}_X(N^{2c})$. Thus the degree (relative to the original $\mathcal{O}_X(1)$) is $\geq N^{2c} \deg \mathcal{O}_X(1) > \deg \omega_X$ for c sufficiently large. Hence, by Lemma 10.12 we have that the cohomology group $H^d(X, {N_X^e}^*\mathcal{S}_d(m) \otimes \mathcal{O}_X(1))$ vanishes. Consequently, $\mathcal{S}_1(m)$ is almost zero and we have the desired inclusion. $\square$

10.37 Remark. There is actually no advantage in only working with separable morphisms since for a locally free sheaf $\mathcal{E}$ on an abelian variety one has $\mu_{\min}(\mathcal{E}) = \bar{\mu}_{\min}(\mathcal{E})$ (see [60, Theorem 2.1 and Remark 2.2]).

10.38 Remark. (a) We note that for an arbitrary polarised projective variety X over an algebraically closed field of characteristic $p > 0$ the assumption that the section ring be Cohen-Macaulay is also not necessary in order to transfer this problem into a top-dimensional cohomology class. To see this fix a resolution of the syzygy bundle in question. Since we only need to look at finitely many twists we may apply Proposition 10.24 to see that all intermediate cohomology of the occurring twisted structure sheaves is mapped to zero along a finite dominant pullback. Hence, if a cohomology class is mapped to zero during "cohomology hopping" with respect to a fixed resolution then it is almost zero. In particular, the assumption that R be Cohen-Macaulay in [8, Theorems 1 and 2] may be relaxed to $\operatorname{Proj} R$ being Cohen-Macaulay (this assumption is necessary to apply the Serre duality argument – cf. [39, Theorem III.7.6]) and one may also relax the condition that the ring R is standard graded to R being a section ring.

(b) Assuming that ample line bundles are almost zero in characteristic zero (with annihilators as roots of the given line bundle)[17] one can show that the Cohen-Macaulay property is also not necessary for smooth projective surfaces. In this case the intermediate cohomology groups are just the first cohomology groups and Kodaira vanishing takes care of the negative twists. Furthermore, $H^1(X, \mathcal{O}_X)$ is almost zero by [66, Theorem 3.4] (provided that ample line bundles are almost zero this also follows by twisting with suitable roots of an ample line bundle).

In general, in characteristic zero, Kodaira's vanishing theorem (see [39, Remark 7.15]) implies that the intermediate cohomology of $\mathcal{O}_X(n)$ vanishes for $n < 0$ (again assuming that X is a smooth projective variety). But one is still left with the problem of annihilating the higher intermediate cohomology groups for positive twists. This question is also related to the property of *almost Cohen-Macaulay* – see [66, Definition 1.2 and the discussion thereafter]. Almost Cohen-Macaulay does imply the vanishing of these intermediate cohomology groups. But it seems unclear whether it is equivalent to this condition.

10.39 Example. Let $(X, \mathcal{O}_X(1))$ be a polarised abelian variety of dimension g (e. g. the Jacobian of a curve of genus g). Assume that $\mathcal{O}_X(1)$ is symmetric, generated by global sections and denote its section ring by R. Fix homogeneous parameters $x_1, \ldots, x_{g+1}$ in R_1. We want to show that $(x_1, \ldots, x_{g+1})^{\dagger \mathrm{GR}}$ is non-trivial.

Let

$$0 \to \mathcal{O}_X(-g-1) \to \bigoplus_{i=1}^{g+1} \mathcal{O}_X(-g) \to \ldots \to \bigoplus_{i=1}^{g+1} \mathcal{O}_X(-1) \to \mathcal{O}_X \to 0$$

be the sheafified Koszul complex (see [21, Remark 1.6.15]). Of course, the Koszul complex on $\operatorname{Spec} R$ is not exact since $x_1, \ldots, x_{g+1}$ do not form a regular sequence but the complex is exact for every localisation at a homogeneous prime $P \neq R_+$ (cf. [21, Proposition 1.6.7] and [ibid., Theorem 1.6.16] – H_0 vanishes since the ideal is primary). Hence, it is an exact complex on $\operatorname{Proj} R = X$.

Twisting by $\mathcal{O}_X(g+1)$ one has that the intermediate cohomology vanishes (apply Lemma 10.32). We therefore have an isomorphism $H^1(X, \mathcal{S}_1(g+1)) \to H^g(X, \mathcal{O}_X)$ (this is surjective since $H^g(X, \mathcal{O}_X(1)) = 0$). Pulling back along N_X and tensoring with suitable roots $\mathcal{L}$ of $\mathcal{O}_X(1)$ one still has an isomorphism $H^1(X, \mathcal{S}_1(g+1) \otimes \mathcal{L}) \to H^g(X, \mathcal{O}_X \otimes \mathcal{L})$. But the latter cohomology group now vanishes since $\mathcal{L}$ is very ample (apply Proposition 10.19 (b)). Hence, we obtain an inclusion $R_{g+1} \subseteq (x_1, \ldots, x_{g+1})^{\dagger \mathrm{GR}}$. Moreover, since $\dim H^g(X, \mathcal{O}_X) = 1$ there are indeed elements in R_{g+1} that are not contained in the ideal.

Also note that this is a generalisation of Example 7.36.

[17] We do not know whether this is true but we suspect so.

11 Open Questions

In this section we collect open questions that arise in the context of this thesis.

11.1 Question. Does (graded) dagger closure have the colon capturing property? That is, given parameters $x_1, \ldots, x_d$ in a local (or positively graded with R_0 a field) domain R which is a homomorphic image of a noetherian Cohen-Macaulay local (graded) ring, is it true that $((x_1, \ldots, x_n) : x_{n+1})$ is contained in the (graded) dagger closure of $(x_1, \ldots, x_n)$ for $n = 0, \ldots, d-1$?

This is true in positive characteristic since tight closure has this property. It is also true in the graded setting in dimension two since then graded dagger closure coincides with solid closure (this also readily follows since the normalisation is Cohen-Macaulay in this case). Heitmann proved in [42, Theorem 4.4] that if plus closure or full rank one closure in mixed characteristic have the colon capturing property for finite extensions of regular local rings then ideals in regular rings are closed with respect to plus closure or full rank one closure respectively. Furthermore, he proved in [43, Corollary 2.9] that full rank one closure has the colon-capturing property for equidimensional three-dimensional excellent semilocal domains of mixed characteristic.

One possible approach to answer Question 11.1 in the affirmative in equal characteristic zero is to show that (graded) dagger closure always contains tight closure (see the discussion after Question 11.5 below).

11.2 Question. Does the Briançon-Skoda theorem hold for (graded) dagger closure? That is, if R is a noetherian domain and $I \subseteq R$ an ideal generated by n elements is it then true that $\overline{I^{n+w}} \subseteq (I^{w+1})^\dagger$ for all $w \in \mathbb{N}$?

Again this is true in positive characteristic and in the graded setting in dimension two. Furthermore, Heitmann proved this for full rank one closure in mixed characteristic (see [42, Theorem 4.2]).

11.3 Question. Assume that $R \to S$ is a faithfully flat ring homomorphism of noetherian rings and that $I \subseteq R$ is an ideal. Is it true that $I^\dagger = (IS)^\dagger \cap R$?

See [42, Theorem 3.4 and Proposition 3.8], where Heitmann proved this for $S = R^h$ the Henselisation of R, where R is a noetherian local ring, and in the case where $S = R[x]$ for full rank one closure in mixed characteristic. It would be especially interesting to have this result for the completion S of a local ring R and also for base field changes $R \otimes_k K$, when R is a k-algebra.

11.4 Question. What is the behaviour of (graded) dagger closure under arithmetic deformations?

We shall briefly outline the ideas involved and point out some difficulties. To make sense of this question let B be an integral $\mathbb{Z}$-algebra of finite type containing $\mathbb{Z}$. Then we can consider the fibres of the morphism $\operatorname{Spec} B \to \operatorname{Spec} \mathbb{Z}$. The generic fibre $B \otimes_{\mathbb{Z}} \mathbb{Q}$ is a domain of characteristic zero whereas we have rings of varying prime characteristics over the closed fibres $B \otimes_{\mathbb{Z}} \mathbb{F}_p$.

Let R be an $\mathbb{N}$-graded domain finitely generated over a field R_0 of characteristic zero and $I = (f_1, \ldots, f_m) \subseteq R$ an ideal. If $f \in I^{\dagger\mathrm{GR}}$ then we find for $\varepsilon = \frac{1}{n}$ an element α_n in some finite graded extension S_n of R with $\nu(\alpha_n) < \varepsilon$ and such that $\alpha_n f = \sum_i a_i f_i$. Then since S_n is of finite type we can write $S_n = k[x_1, \ldots, x_l]/(g_1, \ldots, g_s)$ for polynomials g_i. Adjoin all coefficients of the g_i, f_j, a_j to $\mathbb{Z}$ - call this ring A. Then one can take $B = A[x_1, \ldots, x_l]/(g_1, \ldots, g_s)$ as a model. The generic fibre $B \otimes_{\mathbb{Z}} \mathbb{Q}$ is a sub-algebra of R and the equation $\alpha_n f = \sum_i a_i f_i$ is defined over this sub-algebra.

These data yield for almost all prime reductions $B \otimes_{\mathbb{Z}} \mathbb{F}_p$ an equation $\alpha_n f = \sum_i a_i f_i$ with $\alpha_n \neq 0$. A given element α_n is defined for almost all fibres. The issue is that we need infinitely many α_n and hence we possibly need to exclude more and more fibres.

We do note that it may indeed happen that this containment does not hold for almost all prime reductions while holding generically. This is due to an example by Brenner and Katzman in graded dimension two – see [18]. In this article they gave an example where generically $f \in I^\star$ but $f \notin I^\star = I^*$ for infinitely many prime reductions.

We also note that for ordinary dagger closure one has the additional problem of dealing with valuations in such families.

Explicitly we pose the following

11.5 Question. Let R be an $\mathbb{N}$-graded domain finitely generated over a field R_0 of characteristic zero, I an ideal and $f \in R$. If $f \in I^{\dagger\mathrm{GR}}$ for almost all prime reductions is it then true that $f \in I^{\dagger\mathrm{GR}}$ in R?

Solid closure as well as parasolid closure have these properties. Tight closure in characteristic zero is essentially defined in this way: an element belongs to the tight closure of an ideal I if and only if this holds for almost all prime reductions (see [49] or [50, Appendix 1]).

The idea for establishing this for parasolid closure is that if generically $f \notin I^\star$ then some paraclass vanishes in a suitable algebra A. This vanishing can be characterised by an equation which descends for almost all prime reductions. The problem in proving this for dagger closure is that we cannot characterise non-containment by equations – only by a lack thereof.

Question 11.5 is especially interesting since a positive answer would imply positive answers to Questions 11.1 and 11.2 for graded dagger closure. Indeed, assume that $f \in I^{\dagger\mathrm{GR}} = I^*$ for almost all primes implies $f \in I^{\dagger\mathrm{GR}}$ over the generic fibre. Then by definition f is contained in the tight closure of I in characteristic zero. Hence, $I^* \subseteq I^{\dagger\mathrm{GR}}$ in characteristic zero. Since colon capturing and the Briançon-Skoda theorem hold for tight closure in characteristic zero both would also hold for graded dagger closure.

11.6 Question. Is dagger closure (as in Definition 9.1) equivalent to graded dagger closure for homogeneous ideals?

As in the other cases our only result (Proposition 9.11) is in positive characteristic. Again this depends very much on the result of Hochster and Huneke and we do not know what the situation is in characteristic zero.

If R is a standard graded domain of positive characteristic p, $\dim R = d$ and I an R_+-primary ideal then one can consider the limit

$$e_{\mathrm{HK}}(I) = \lim_{q \to \infty} \frac{\operatorname{length} R/I^{[p^e]}}{p^{ed}}.$$

This is called the *Hilbert-Kunz multiplicity* of I. Monsky has shown that this limit always exists and is a positive real number – see [62]. The connection to tight closure arises from the fact that if $f \in R$ and R is analytically unramified and formally equidimensional then $e_{\mathrm{HK}}(I) = e_{\mathrm{HK}}(f + I)$ if and only if $f \in I^*$ (see [50, Theorem 5.4]).

It has been shown in [6] that one can attach a Hilbert-Kunz multiplicity to solid closure in Situation 3.6 in characteristic zero. Therefore we may pose the following

11.7 Question. Is there a multiplicity attached to dagger closure in characteristic zero in the same way that Hilbert-Kunz multiplicity is related to tight closure?

We briefly discuss a possible approach to this issue. If R is a standard graded normal integral k-algebra of dimension 2 of positive characteristic then one can compute the Hilbert-Kunz multiplicity $e_{\mathrm{HK}}(I)$ of an ideal I in terms of invariants of a strong Harder-Narasimhan filtration of the corresponding syzygy bundle (see [11] or [74]). Brenner used the corresponding invariants in characteristic zero to give a definition of multiplicity for solid closure as cited above. Furthermore, Trivedi has shown (see [75, Corollary 2.3]) that this multiplicity is the limit of the Hilbert-Kunz multiplicities as $p \to \infty$ over a relative curve.

Therefore, one might hope that $\lim_{p \to \infty} e_{\mathrm{HK}}(I_p)$ is a possible candidate. We note however, that the existence of this limit is very much unclear. Trivedi's proof is built on the expression of e_{HK} in terms of invariants of a strong Harder-Narasimhan filtration of the syzygy bundle. Such a characterisation is not known in higher dimensions. We also note that there have been some efforts to establish Hilbert-Kunz multiplicities in characteristic zero using symmetric powers of syzygy bundles (see [15] – up to now this only works in dimension 2 though).

11.8 Question. What is the precise connection between dagger closure and almost ring theory in the sense of [26]?

There have been some steps in this direction in [66] and in [2].

11.9 Question. Let R be an $\mathbb{N}$-graded k-algebra of finite type. How does (graded) dagger closure for R_+-primary ideals compare in higher dimensions to parasolid closure?

There are two somewhat natural settings to explore this question (at least if one wants to stick to a geometric setting). The first would be to suppose that R has an isolated normal singularity at R_+ so as to be able to work on a smooth projective variety. The more general approach would be to only suppose that R is normal (this is not much of a limitation since one may always pass to the normalisation for dagger closure). Note that because of dagger closure's asymptotic definition one cannot expect to work entirely in a smooth setting if the dimension of the ring is bigger than two. Therefore, one would have to set up a theory for normal varieties.

From the dagger closure side the point of departure should be to study the cohomology class c in $H^1(\operatorname{Proj} R, \operatorname{Syz}(f_1, \ldots, f_n)(m))$ defined by a homogeneous element f of degree m in R as outlined in the proof of Proposition 7.5. One should probably concentrate on setting up a suitable geometric theory of almost zero for these cohomology classes. A first step in this direction is Proposition 10.22. We also note that, at least if the ring is Cohen-Macaulay, one can embed c into a top-dimensional cohomology group, namely into $H^d(\operatorname{Proj} R, \operatorname{Syz}_d)$, where Syz_d is the dth syzygy bundle in a resolution of $(f_1, \ldots, f_n)$ (in positive characteristic this is possible in any case – cf. Proposition 10.24). This is especially interesting for parameters, for then Syz_d is a line bundle if one looks at the Koszul resolution on $\operatorname{Proj} R$.

It is known that $\mathcal{O}_{\operatorname{Proj} R}$ is almost zero by [66, Theorem 3.4] in characteristic zero. But we do not even know whether ample line bundles are almost zero in characteristic zero. Furthermore, the quotients of a strong Harder-Narasimhan filtration will only be torsion-free sheaves in higher dimensions. We also note that the forcing divisor will not be ample in any case of interest if $\dim R \geq 3$ – see [5, Proposition 4.1 (iii)]. So most of the methods employed here fail to work if $\dim \operatorname{Proj} R \geq 2$.

Quite remarkably though one may assume that the syzygy bundle in question is a successive extension of invertible sheaves by virtue of the following

11.10 Theorem. Let X be a smooth quasi-projective scheme separated and of finite type over an algebraically closed field k. Let $\mathcal{E}$ be a locally free sheaf of constant rank on X. Then there is a quasi-projective smooth separated scheme X' of finite type over k satisfying the following conditions:

(a) There is a finite and faithfully flat morphism $f : X' \to X$.

(b) $f^*(\mathcal{E})$ is a successive extension of line bundles. That is, there is a sequence of subbundles $f^*(\mathcal{E}) = \mathcal{F}_0 \supset \mathcal{F}_1 \supset \ldots \supset \mathcal{F}_r = 0$ such that every quotient bundle $\mathcal{F}_i/\mathcal{F}_{i+1}$ is a line bundle on X'.

Proof. See [73, Theorem 2.1]. □

It should also be noted that Question 11.9 is bound to be very difficult since dagger closure is tight closure in positive characteristic in this setting. And tight closure is still poorly understood in higher dimensions in a geometric setting. Moreover, it is not clear what parasolid closure looks like in a geometric setting either. Nevertheless, the dagger closure side might help to understand tight closure better geometrically.

Theorem 11.10 makes the following problem very intriguing.

11.11 Problem. Let X be a smooth projective variety over an algebraically closed field k. Classify line bundles on X with respect to being almost zero.

Related to Question 11.9 but perhaps easier to understand is

11.12 Question. Assume that R is an $\mathbb{N}$-graded k-algebra, finitely generated over the field $R_0 = k$ so that $\operatorname{Proj} R$ is an abelian variety. How does graded dagger closure for R_+-primary ideals compare in higher dimensions to parasolid closure?

This is especially interesting in characteristic zero. The key here is that one has a multiplication map which might work as a substitution for the Frobenius morphism. See also [66, Theorem 3.4], its proof and Section 10 of this thesis. Another key advantage is that one might only have to pass to isomorphic copies of the variety to obtain multipliers of small order. In particular, one would be able to work in a smooth setting at least for one inclusion.

11.13 Question. In Proposition 5.4 and Remark 5.5 it is hinted that one should obtain $R^{+\mathrm{GR}}$ as a direct limit of a suitable direct system of varieties with fixed ample line bundles. Can one make this precise? What happens if one looks at arbitrary ample line bundles? – This should be a ring which contains the graded integral closures of all rings whose Proj yield a given projective variety. What is the relation of this ring with respect to a fixed $R^{+\mathrm{GR}}$? For instance, is it a direct summand?

The existence of this direct limit is not immediate. One should consider as objects pairs $(X, \mathcal{L})$ where X is a variety finite over Y such that $\dim X = \dim Y$ and where $\mathcal{L}$ is a fixed ample invertible sheaf on X. Then one can declare $(X, \mathcal{L}) \leq (X', \mathcal{L}')$ if there exists a projective variety Z finite over Y such that X' is an irreducible component of the reduced fibre product $X \times_Y Z$ which dominates the base and such that the ring extension

$$\bigoplus_{n \geq 0} \Gamma(Y, \mathcal{O}_Y(n)) \subseteq \bigoplus_{n,m,l \geq 0} \Gamma(X', \mathcal{O}_Y(n) \otimes \mathcal{L}^m \otimes \mathcal{L}'^l)/\sim$$

is finite. The morphism is then given by the composition of the closed immersion with the projection on X. The universal property of the fibre product yields that this is a functor from an ordered set[18] to $\mathcal{C}$. An upper bound for $X_1, \ldots, X_n$ is then a suitable irreducible component of the reduced fibre product $X_1 \times_Y \ldots \times_Y X_n$.

But there are some difficulties. It is not clear what this limit should look like since the ring $R^{+\mathrm{GR}}$ is not $\mathbb{N}$-graded (even worse – its grading is not given by a finitely generated group) but the section ring of any variety is. So one should probably work in the category of schemes and examine whether one can extend the Proj-construction to positively $\mathbb{Q}$-graded rings.

11.14 Question. Can one drop the assumption in Proposition 10.22 that the annihilating line bundles be roots of the fixed embedding in dimension ≥ 2?

To make this work one should first of all investigate whether semiampleness is the right generalisation in Definition 10.20. Specifically, nef, big or a combination of these are properties that seem worth exploring. The heuristic idea is that these properties are better behaved than ampleness and that they might be enough to yield the desired results. Of course, assuming nef and big still implies that the intersection with every effective curve is non-negative and that the top-selfintersection is positive which is a priori not easy to control numerically.

A second possible approach might be the following. Assuming that Question 11.13 has a positive answer, it is worth investigating, whether dagger closure contracts from

[18] For this to be a set one should at the very least restrict to isomorphism classes of varieties.

the "big ring". That is, whether for $f \in R$ and an ideal I in R the containment $f \in (IS)^{\dagger \mathrm{GR}}$ for some graded extension domain S of $R^{+\mathrm{GR}}$ implies that $f \in I^{\dagger \mathrm{GR}}$.

11.15 Problem. Find a geometric proof of Corollary 9.32.

Another question is how to actually compute dagger closure (esp. in characteristic zero). There are some favourable situations e. g. if the ring R defines an elliptic curve (see Example 7.36 and also [66, Section 2] for two explicit computations on an elliptic curve and in the case of a hypersurface ring). But in general this is not clear at all and our methods are not constructive.

11.16 Question. Is there a similar theory of test elements for dagger closure? Are there certain circumstances where one can fix a single element a and take as multipliers a_n suitable roots of a?

The second question has an affirmative answer in some cases in mixed characteristic (see [42, Proposition 5.1]) but the proof does not carry over to equal characteristic zero.

The following question came up in Section 8.

11.17 Question. Let Y be a smooth projective curve of genus $g \geq 2$ over an algebraically closed field of positive characteristic. Is there a locally free sheaf $\mathcal{Q}$ with $\bar{\mu}_{\max}(\mathcal{Q}) < 0$ such that the map $H^1(Y, \mathcal{Q}) \to H^1(Y, F^{e*}\mathcal{Q})$ is identically zero for some $e \geq 0$, where F denotes the k-linear Frobenius?

11.18 Question. Let X be a (smooth) projective variety over an algebraically closed field k and D an effective divisor on X. Assume that D is not ample. Under which circumstances can one find curves that contradict the ampleness of D in the sense of Theorem 3.16 that do not lie in the support of D?

A more special version of the question above is

11.19 Question. Let X be a (smooth) projective variety over an algebraically closed field k. Let $\mathcal{E}$ be a locally free sheaf on X such that $H^0(X, \mathcal{E}) \neq 0$ and assume that it is not ample. Under which circumstances can one find curves in $\mathbb{P}(\mathcal{E})$ that contradict the ampleness of $\mathcal{E}$ in the sense of Theorem 3.16 that do not lie in the support of the divisor?

For $\dim X = 1$ and a surjection $\mathcal{E}^{\vee} \to \mathcal{O}_X \to 0$ this is answered by Proposition 7.13.

References

[1] M. Artin, *On the joins of Hensel rings*, Adv. in Math. **7** (1971), 282–296.

[2] M. Asgharzadeh and K. Shimomoto, *Almost Cohen-Macaulay and almost regular algebras via almost flat extensions*, arXiv:1003.0265v1 (2010).

[3] I. Biswas, G. Hein, and N. Hoffmann, *On semistable vector bundles over curves*, C. R. Acad. Sci. Paris **I 346** (2008), 981–984.

[4] H. Brenner, *How to rescue solid closure*, J. Algebra **265** (2003), 579–605.

[5] ______, *Tight closure and projective bundles*, J. Algebra **265** (2003), 45–78.

[6] ______, *A characteristic zero Hilbert-Kunz criterion for solid closure in dimension two*, Math. Research Letters **11** (2004), 563–574.

[7] ______, *Slopes of vector bundles and applications to tight closure problems*, Trans. Amer. Math. Soc. **356** (2004), no. 1, 371–392.

[8] ______, *A linear bound for Frobenius powers and an inclusion bound for tight closure*, Michigan Math. J. (2005), 585–596.

[9] ______, *Tight closure and plus closure for cones over elliptic curves*, Nagoya Math. J. **177** (2005), 31–45.

[10] ______, *Bounds for test exponents*, Compositio Math. **142** (2006), 451–463.

[11] ______, *The rationality of the Hilbert-Kunz multiplicity in graded dimension two*, Math. Ann. **334** (2006), no. 1, 91–110.

[12] ______, *Tight closure and plus closure in dimension two*, Amer. J. Math. **128** (2006), 531–539.

[13] ______, *Tight closure and vector bundles*, Three Lectures on Commutative Algebra (J. Elias S. Zarzuela G. Colomé-Nin, T. Cortodellas Benitez, ed.), University Lecture Series, vol. 42, AMS, 2008, pp. 1–71.

[14] ______, *Forcing algebras, syzygy bundles, and tight closure*, Commutative Algebra: Noetherian and non-Noetherian Perspectives, Springer, 2010, also available via arXiv:0901.3051v1.

[15] H. Brenner and H. Fischbacher-Weitz, *Symmetric powers of vector bundles and Hilbert-Kunz theory*, unpublished preprint.

[16] ______, *Generic bounds for Frobenius closure and tight closure*, to appear Amer. J. Math. (2010), arXiv:0810.4518v3.

[17] H. Brenner and M. Katzman, *On the arithmetic of tight closure*, J. Amer. Math. Soc. **19** (2006), no. 3, 659–672.

[18] H. Brenner and P. Monsky, *Tight closure does not commute with localization*, Ann. Math. **171** (2010), 571–588.

[19] M. P. Brodmann and R. Y. Sharp, *Local cohomology*, Cambridge University Press, 1998.

[20] W. Bruns, *Tight closure*, Bull. Amer. Math. Soc. **33** (1996), 447–458.

[21] W. Bruns and J. Herzog, *Cohen-Macaulay rings, revised edition*, Cambridge University Press, 1998.

[22] L. Bădescu, *Algebraic Surfaces*, Springer, 2001.

[23] D. Eisenbud, *Commutative algebra with a view toward algebraic geometry*, Springer-Verlag, 1995.

[24] H. Flenner, *Restrictions of semistable bundles on projective varieties*, Comment. Math. Helv. **59** (1984), 635–650.

[25] M. Fulger, *The cones of effective cycles on projective bundles over curves*, arXiv:0910.370v1 (2009).

[26] O. Gabber and L. Ramero, *Almost ring theory*, Springer, 2003.

[27] D. Gieseker, *p-ample bundles and their Chern classes*, Nagoya Math. J. **43** (1971), 91–116.

[28] ———, *Stable vector bundles and the Frobenius morphism*, Ann. Sci. Ecole Norm. Sup. (4) **6** (1973), 95–101.

[29] A. Grothendieck, *Local cohomology*, Springer, 1967.

[30] A. Grothendieck and J. Dieudonné, *Eléments de Géométrie algébrique I*, vol. 4, Inst. Hautes Études Sci. Publ. Math., 1960.

[31] ———, *Eléments de Géométrie algébrique II*, vol. 8, Inst. Hautes Études Sci. Publ. Math., 1961.

[32] ———, *Eléments de Géométrie algébrique III*, vol. 11, 17, Inst. Hautes Études Sci. Publ. Math., 1961.

[33] ———, *Eléments de Géométrie algébrique IV*, vol. 20,24,28,32, Inst. Hautes Études Sci. Publ. Math., 1964-1967.

[34] G. Harder and M. S. Narasimhan, *On the cohomology groups of moduli spaces of vector bundles on curves*, Math. Ann. **212** (1975), 215–248.

[35] R. Hartshorne, *Ample vector bundles*, Publ. Math. I.H.E.S. **29** (1966), 63–94.

[36] ———, *Cohomological dimension of algebraic varieties*, Ann. of Math. **88** (1968), 403–450.

[37] ———, *Ample subvarieties of algebraic varieties*, Springer, Berlin Heidelberg New York, 1970.

[38] ———, *Ample vector bundles on curves*, Nagoya Math. J. **43** (1971), 73–89.

[39] ———, *Algebraic Geometry*, Springer, New York, 1977.

[40] ———, *Stable reflexive sheaves*, Math. Ann. **254** (1980), 121–176.

[41] R. C. Heitmann, *Plus closure in mixed characteristic*, J. Algebra **193** (1997), no. 2, 688–708.

[42] ———, *Extensions of plus closure*, J. Algebra **238** (2001), 801–826.

[43] ———, *The direct summand conjecture in dimension three*, Ann. Math. (2) **156** (2002), no. 2, 695–712.

[44] M. Hochster, Tight closure in zero characteristic, ch. Appendix 1 to : C. Huneke, Tight Closure and Its Applications.

[45] ———, *Contracted ideals from integral extensions of regular rings*, Nagoya Math. J. **51** (1973), 25–43.

[46] ———, *Solid closure*, Contemp. Math. **159** (1994), 103–172.

[47] M. Hochster and C. Huneke, *Tight closure and elements of small order in integral extensions*, J. of Pure and Applied Algebra **71** (1991), 233–247.

[48] ———, *Infinite integral extensions and big Cohen-Macaulay algebras*, Ann. Math. **135** (1992), 53–89.

[49] ———, *Tight closure in equal characteristic zero*, Preprint, available at `www.math.lsa.umich.edu/~hochster`, 1999.

[50] C. Huneke, *Tight closure and its applications*, CBMS Lecture Notes in Mathematics, vol. 88, AMS, Providence, 1996.

[51] D. Huybrechts and M. Lehn, *The Geometry of Moduli Spaces of Sheaves*, Viehweg, 1997.

[52] E. Hyry and K. E. Smith, *Core versus graded core and global sections of line bundles*, Trans. Amer. Math. Soc. **356** (2004), 3143–3166.

[53] S. Izumi, *A Measure of Integrity of Local Analytic Algebras*, Publ. Res. Inst. Math. Sci. **21** (1985), 719–735.

[54] H. Lange and C. Pauly, *On Frobenius-destabilized rank-2 vector bundles over curves*, Comm. Math. Helv. **83** (2008), 179–209.

[55] A. Langer, *Semistable sheaves in positive characteristic*, Ann. Math. **159** (2004), 251–276.

[56] R. Lazarsfeld, *Positivity in algebraic geometry*, Springer, 2004.

[57] J. Le Potier, *Lectures on vector bundles*, Cambridge Studies in Advanced Math., vol. 54, Cambridge University Press, 1997.

[58] Q. Liu, *Algebraic geometry and arithmetic curves*, Oxford University Press, 2007.

[59] M. Maruyama, *The theorem of Grauert-Mülich-Spindler*, Math. Ann. **255** (1981), 317–333.

[60] V. B. Mehta and A. Ramanathan, *Homogeneous bundles in characteristic p*, Algebraic Geometry - open problems, Lect. Notes Math., vol. 997, 1982, pp. 315–320.

[61] Y. Miyaoka, *The Chern class and Kodaira dimension of a minimal variety*, Algebraic Geometry, Sendai 1985, Adv. Stud. Pure Math., vol. 10, 1987, pp. 449–476.

[62] P. Monsky, *The Hilbert-Kunz function*, Math. Ann. **263** (1983), 43–49.

[63] D. Mumford, *Abelian varieties*, Oxford University Press, 1974.

[64] D. P. Patil and U. Storch, *Introduction to Algebraic Geometry and Commutative Algebra*, World Scientific Publishing, 2010.

[65] P. Roberts, *A computation of local cohomology*, Contemp. Math. **159** (1994), 351–356.

[66] P. Roberts, A. Singh, and V. Srinivas, *Annihilators of local cohomology in characteristic zero*, Illinois J. Math. **51** (2007), 237–254.

[67] H. Schoutens, *Non-standard tight closure for affine $\mathbb{C}$-algebras*, Manus. Math. **111** (2003), 379–412.

[68] ———, *Canonical big Cohen-Macaulay algebras and rational singluarities*, Illinois J. Math. **48** (2004), no. 1, 131–150.

[69] J. P. Serre, *Local algebra*, Springer, 2000.

[70] J. H. Silverman, *The arithmetic of elliptic curves*, Springer, Berlin, 1986.

[71] K. E. Smith, *Tight closure in graded rings*, J. Math. Kyoto Univ. **37** (1997), no. 1, 35–53.

[72] ———, *Vanishing, singularities and effective bounds via prime characteristic local algebra*, available at `http://www.math.lsa.umich.edu/~kesmith/`, 1997.

[73] H. Sumihiro, *A theorem on splitting of algebraic vector bundles and its applications*, Hiroshima Math. J. **12** (1982), 435–452.

[74] V. Trivedi, *Semistability and Hilbert-Kunz multiplicity for curves*, J. Algebra **284** (2005), no. 2, 627–644.

[75] ———, *Hilbert-Kunz multiplicity and reduction mod p*, Nagoya Math. J. **185** (2007), 123–141.

[76] O. Zariski and P. Samuel, *Commutative algebra*, Springer Verlag, 1960.