STABLE EQUIVARIANT MOTIVIC HOMOTOPY THEORY AND MOTIVIC BOREL COHOMOLOGY

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Contents

1. Introduction

This dissertation deals with rather foundational work in equivariant motivic homotopy theory. Motivic homotopy theory, slightly more revealing also known as $\mathbb{A}^1$-homotopy theory of schemes, emerged over the last decades from a long development of topological methods in algebraic geometry and generalizations of homotopy theory within the field of algebraic topology. Numerous famous conjectures on qualitative invariants of varieties have been a driving force in the development of motivic homotopy theory. In the 1990s one of these conjectures - the Bloch-Kato conjecture, finally proven by Rost and Voevodsky - was the motivation for Voevodsky to investigate $\mathbb{A}^1$-homotopy theory of schemes and led to the groundwork [MV99] in joint work with Morel.

A prerequisite for the basic setup of $\mathbb{A}^1$-homotopy theory was the abstraction of homotopy theoretic methods from point-set topology to a categorical framework in the 1960s, prominently represented by Quillen's model categories. On the algebraic geometry side a generalizing process had already started at that time and was accelerated by the Weil conjectures [Wei49]. It has changed the objects of study from vanishing subsets of some k^n to so-called schemes, allowing more flexibility and increasing the topological information. However, these schemes still do not provide sufficient flexibility to apply Quillen's model category techniques. But the patching methods used to construct schemes were rescaled and developed further to establish the areas of sheaf and topos theory. By 1980 it became clear that these rescaled methods offer convenient embeddings of varieties (or general small categories) into categories of spaces, which are flexible enough to carry the homotopical structure of a model category. It is crucial for the corresponding homotopy theory to control the embedding and to let it reflect a well chosen amount of geometry of the original category. In the beginning of the first part of this dissertation, the focus will be on this issue for the equivariant context. In the Morel-Voevodsky setup in loc. cit. the choice of a Grothendieck topology at this point is influenced by the precursory investigation of descent properties of algebraic K-theory by Brown-Gersten, Nisnevich, Thomason and others. To satisfy descent with respect to a topology is a major obstruction for the representability of a cohomological theory in the homotopy category of spaces and the adjective 'motivic' for the homotopy theory is exactly meant to express a good representational behavior for cohomology theories in the homotopy category. It refers to Grothendieck's impression [Gro85] that despite the diversity of cohomological theories occurring in algebraic geometry they should all express more or less the same information and the reason would be that they have a common motive, they 'factorize' over the 'motive of a variety'. The more precise relation of motivic homotopy theory to motives will not be a topic of this thesis, though we investigate some descent property in the first part and a particular cohomology theory in the second part.

The approach to distinguish between unstable and stable phenomena is also adopted from topological homotopy theory and the hope to gain structural information by attacking the pretendedly easier stable problems isolated from the others. Already in topology the search for an adequate method to isolate stable phenomena has been a long way and created a lot of important mathematics and insights. Stabilization techniques like sequential (symmetric) spectra, S-modules or enriched functors transfer to motivic homotopy theory [Jar00, Hu03, DRØ03b] and clear the way for many sophisticated approaches, including some of the most important spectral sequences. Nevertheless, the transition of results and techniques from topology to algebraic geometry does not proceed as a blind mirroring, but comes almost everywhere at the cost of varying restrictions, for instance restricting results to smooth schemes or to certain characteristics of a base field. This is partially due to the status quo of the foundations of a relatively young discipline and partially in the nature of things.

This dissertation project was essentially triggered by Kriz' talk on a motivic homotopy theory conference in Münster in the summer of 2009. In the conclusion of his talk Kriz asked for an account to equivariant motivic homotopy theory, which the author could have anticipated as an early announcement of the work of Hu, Kriz and Ormsby [HKO] - later published as [HKO11b]. At that time the literature on this aspect only included a few pieces by Voevodsky (cf. [Voe01]), which were not aimed at a thorough study of equivariant homotopy theory, but at the study of non-equivariant foundations, in particular at Steenrod operations for motivic cohomology. But the study of transformation groups inevitably belongs to a detailed analysis of any geometric theory, as it is recognized ever since Klein's 1872 Erlangen program. So there are definitely good reasons to demand a solid framework for equivariant motivic homotopy theory with reasonable properties and it still has to turn out which frameworks do provide which kind properties. This work is meant to be a part of that process.

At the moment there are basically two approaches to equivariant motivic homotopy theory. One that uses the classifying topos of group object in Morel and Voevodsky's category of motivic spaces [Ser01, CJ11a] and another one which constructs a category of equivariant motivic spaces more from a more basic level, starting with a group scheme G and a choice of a topology on a category of schemes with a G-action [HKO11b, Voe01]. The present work takes up the latter approach with the suggestion of a new Grothendieck topology. This topology is a slight variation of the topology in [HKO11b] and the resulting homotopy theories are therefore strongly related, but also have surprisingly essential differences. The topology suggested here allows a characterization of equivariant weak equivalences via their fixed-points while the HKO-topology does not. Conversely, the HKO-topology allows descent for equivariant algebraic K-theory [KØ10] as opposed to our topology. This combination is rather unfortunate since it obstructs

Asok's program [Aso, 2.4.8], involving equivariant motivic homotopy theory, to decide about the role of the Russell cubic $R = \{x + x^2y + z^3 + t^2 = 0\}$ as a counterexample to the cancellation problem in affine algebraic geometry. The cancellation problem asks for the correctness of the implication

$$X \times \mathbb{A}^1 \cong \mathbb{A}^n \Rightarrow X \cong \mathbb{A}^{n-1}$$

for affine varieties X. The Russell cubic is a famous candidate for being a counterexample to the above implication. It is known that R is non-isomorphic to $\mathbb{A}^3$ [ML96] and an open question whether the cylinder $R \times \mathbb{A}^1$ is $\mathbb{A}^4$, which would necessarily imply $\mathbb{A}^1$-contractibility of R. At the moment it is even unknown whether R is $\mathbb{A}^1$-contractible, but as the cubic arose in context of the so-called linearization problem, it has a well studied $\mathbb{G}_m$ action, given by

$$(\lambda, (x, y, z, t)) \mapsto (\lambda^6 x, \lambda^{-6} y, \lambda^2 z, \lambda^3 t),$$

and there is a some hope to reduce the $\mathbb{A}^1$-contractibility question to the fixed-points of R under the restriction to some μ_l action. The Russell cubic has a non-trivial equivariant K_0 [Bel01]. As Asok points out, an equivariant motivic homotopy theory with 'fixed-point wise weak equivalences' and allowing representability of equivariant algebraic K-theory, would force R^{μ_p} to be non-$\mathbb{A}^1$-contractible for all primes p. Despite the K-theory descent statement in Proposition 3.5.4 one can still hope to find similar non-triviality for some representable cohomological theory in our setup.

In the middle part of this thesis, the corresponding stable equivariant motivic homotopy theory is developed to some extent, allowing a later application. According to topological stable equivariant homotopy theory, we distinguish between naive and genuine equivariant motivic spectra, inverting the integer indexed motivic spheres $S^{p,q}$ and all representation spheres S^V respectively. It should be mentioned that a restricting assumption on the transformation group is needed to guarantee the invertability of all representations for our approach. The chosen stabilization technique is that of [Jar00, Hov01], following [HKO11b] by smashing with the regular representation sphere. This way, we have to care about the linear reductivity of the transformation group. As suggested in [CJ11a], this restriction can probably be removed by applying the stabilization technique of [DRØ03b] to our setting.

While the unstable motivic fixed-point functors show a bit more flexibility than their topological counterparts, being left and right adjoint, the genuine stable fixed-point functors share the same problems: for any subgroup H of G there exist two reasonable versions of them, one adjoint to a trivial H-action and one being a direct prolongation of the space level functor. We show that both families of fixed-point functors are able to detect equivariant stable equivalences.

The last part of this thesis is concerned with an application of the findings of the previous sections. We study a motivic Borel cohomology Adams spectral sequence, analogous to topological spectral sequences constructed in [Gre88b, MRS01] but following Greenlees' account. In particular, we follow Greenlees' idea to fundamentally rely on co-Borel cohomology during the construction, which is how the freeness assumption finds its way into Theorem 6.1.3 below: Borel and co-Borel cohomology coincide on spaces with a free action. In general, Adams spectral sequences for a cohomology theory E are applied to recover homotopical information from E-cohomological one and can be seen as a machinery working sort of inverse to the Hurewicz map

$$[X, Y] \xrightarrow{E^{*,*}} \mathrm{Hom}_{E^{**}E}\left(E^{*,*}Y, E^{*,*}X\right), f \mapsto E^{*,*}f.$$

Clearly, the potential of recovering homotopical information depends on the visual acuity [Gre88a] of E, as, for example, the terminal spectrum is not able to give any homotopical information. From this point of view, and also in consideration of the genuine stable homotopy theory developed in the middle part, one is tempted to use a more flexible Bredon cohomology theory instead of motivic Borel cohomology $F(EG_+, H\mathbb{Z}/p_{fixed})$, which is essentially the push forward of the non-equivariant Eilenberg-MacLane spectrum. On the other hand, computations in motivic homotopy theory are still quite rare and one should be content to be able to build on Voevodsky's computations of the motivic cohomology $H^{*,*}(B_{gm}\mathbb{Z}/p)$ of the geometric classifying spaces and the motivic Steenrod algebra $H\mathbb{Z}/p^{**}H\mathbb{Z}/p$, which go into the construction of the spectral sequence this way.

We are finally able to prove the following theorem.

Theorem 6.1.3. *Let $X, Y \in \mathcal{SH}(k, \mathbb{Z}/l)$, where k is a field of characteristic zero containing a primitive l-th root of unity. Furthermore, let Y be*

- *$\mathbb{Z}/l$-free*
- *bounded below, and such that*
- *$H^{p,q}(Y, \mathbb{Z}/l)$ is finite for all p, q.*

Then there exists a spectral sequence with

$$E_2^{s,(t+s,u)} = \mathrm{Ext}_{b^{**}b}^{s,(t+s,u)}(b^{*,*}Y, b^{*,*}X) \Rightarrow [X, Y]^G.$$

Moreover, it is shown that the spectral sequence is conditionally convergent to $[X, Y/\operatorname{holim}_s Y_s]^G$ by Boardman's trick [Boa99], which collapses the obstruction group.

2. Preliminaries

2.1. **Group Actions.** As we want to establish an equivariant homotopy theory of schemes it would be natural to consider actions of arbitrary group objects in schemes. However, there is no convenient homotopy theory for actions of arbitrary group schemes yet and also for the theory developed in the following we find reasons to restrict ourselves to finite abstract groups. We will consider group actions in various contexts, e.g. in categories of schemes and presheaves. Thus we should take care of meanings and relations of these actions, which we do here.

Definition 2.1.1. Let $\mathcal{C}$ be a category with terminal object $*$. A group object in a category $\mathcal{C}$ consists of an object $G \in \mathcal{C}$ together with morphisms

$$m : G \times G \to G, i : G \to G, e : * \to G,$$

such that the diagrams

$$\begin{tikzcd} G \times G \times G \arrow[r, "m\times \mathrm{id}"] \arrow[d, "\mathrm{id}\times m"] & G \times G \arrow[d, "m"] \\ G \times G \arrow[r, "m"'] & X \times G, \end{tikzcd} \tag{2.1}$$

$$\begin{tikzcd} G \arrow[r, "\cong"] \arrow[drr, "\mathrm{id}"'] & * \times G \arrow[r, "e\times \mathrm{id}"] & G \times G \arrow[d, "m"] & G \times * \arrow[l, "\mathrm{id}\times e"'] & G \arrow[l, "\cong"'] \arrow[dll, "\mathrm{id}"] \\ & & G & & \end{tikzcd} \tag{2.2}$$

, and

$$\begin{tikzcd} G \arrow[r, "\Delta"] \arrow[d] & G \times G \arrow[r, "i\times \mathrm{id}"] & G \times G \arrow[d, "m"] & G \times G \arrow[l, "id\times i"'] & G \arrow[l, "\Delta"'] \arrow[d] \\ * \arrow[rr, "e"] & & G & & * \arrow[ll, "e"'] \end{tikzcd} \tag{2.3}$$

commute.

Example 2.1.2. An abstract group is just the same as a group object in the category of sets. A standard trick - the so-called Eckman-Hilton argument - allows to compute that a group object in the category of groups is exactly an abelian group. A Hopf algebra is a group object in the opposite category of some category of algebras. A more involved example is that a group object in smooth manifolds is already a Lie group. We will mostly consider group schemes, i.e. group objects in schemes. A group object in (pre-) sheaves is the same as a (pre-) sheaf of groups.

Let G be a group object in a category $\mathcal{C}$. An action of G on some object X in the same category $\mathcal{C}$ is given by a morphism $\alpha : G \times X \to X$ in $\mathcal{C}$, such that the diagrams

$$
\begin{tikzcd}
G \times G \times X \arrow[r, "m\times \mathrm{id}"] \arrow[d, "id\times\alpha"'] & G \times X \arrow[d, "\alpha"] \\
G \times X \arrow[r, "\alpha"'] & X
\end{tikzcd}
\tag{2.4}
$$
, and

$$
\begin{tikzcd}
X \arrow[r, "\cong"] \arrow[drr, "\mathrm{id}"'] & * \times X \arrow[r, "e\times \mathrm{id}"] & G \times X \arrow[d, "\alpha"] \\
& & X
\end{tikzcd}
\tag{2.5}
$$

commute.

As mentioned we want to study actions of ordinary abstract groups in categories different from sets which does not exactly match the above definition. Hence we introduce Vistoli's notation of a category which 'has discrete group objects' (cf. [Vis04, Def. 2.20.]).

Let $\mathcal{C}$ be a category with terminal object $*$ and suppose that all small coproducts of $*$ exist. We define a discrete object functor

$$\delta : \mathcal{S}\mathrm{et} \to \mathcal{C}, I \mapsto \coprod_I *$$

which sends a map $\varphi : I \to J$ to the morphism $\delta(I) \to \delta(J)$ induced by the morphism $*_i \to *_{\varphi(i)} \to \coprod_J *$.

Definition 2.1.3. A category $\mathcal{C}$ has discrete group objects if all finite products exist and for all objects $X \in \mathcal{C}$ and all sets I

(1) the coproduct $I \times X := \coprod_I X$ exists, and
(2) the canonical morphism

$$I \times X \to \delta(I) \times X$$

is an isomorphism in $\mathcal{C}$.

Lemma 2.1.4. *Let $\mathcal{C}$ be a category with discrete group objects.*

(1) The functor $\delta : \mathcal{S}\mathrm{et} \to \mathcal{C}$ is left adjoint to $\mathrm{Hom}_{\mathcal{C}}(*, -)$, preserves finite products and hence induces a functor $\mathrm{Grps} \to \mathrm{Grps}(\mathcal{C})$.
(2) Let G be an abstract group. An action of the group object δG on an object X in $\mathcal{C}$ is equivalent to a group homomorphism $G \to \mathrm{Aut}_{\mathcal{C}}(X)$.

Proof.

(1) The adjunction does not need much of the assumptions on $\mathcal{C}$. There is a natural isomorphism

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}}(\delta I, X) = \mathrm{Hom}_{\mathcal{C}}\left(\coprod_I *, X\right) &\cong \prod_I \mathrm{Hom}_{\mathcal{C}}(*, X)\\ &= \mathrm{Hom}_{\mathcal{S}\mathrm{et}}(I, \mathrm{Hom}_{\mathcal{C}}(*, X)).\end{aligned}$$

That δ preserves finite products is [Vis04, Proposition 2.21] and such a functor clearly takes group objects to group objects.

(2) This is [Vis04, Proposition 2.22].

□

Lemma 2.1.5. *The categories* Sch_k *of schemes over a base field* k *and* sPre *of simplicial presheaves on some category have discrete group objects.*

Proof. For sPre this is an implication of the completeness and cartesian closedness of presheaf categories. For the category Sch_k the terminal object is given by $\mathrm{Spec}(k)$ and finite products exist by [Har97, Theorem II.3.3]. It remains to show that for any k-scheme U the canonical morphism is an isomorphism

$$\nu : \coprod_I U \xrightarrow{\cong} \left(\coprod_I \mathrm{Spec}(k)\right) \times U.$$

By the Yoneda lemma it is equivalent to show that the induced morphism of Zariski sheaves on the big Zariski site Sch_k is an isomorphism. Following Example 2.2.8 we take a regular local ring R and observe that any morphism

$$\mathrm{Spec}(R) \to \left(\coprod_I \mathrm{Spec}(k)\right) \times U$$

is given by a morphism $\mathrm{Spec}(R) \to U$ and a choice of an element $i \in I$. This data defines a morphism $\mathrm{Spec}(R) \to \coprod_I U$, which lifts the given morphism. Moreover, two morphisms $f, g : \mathrm{Spec}(R) \to \coprod_I U$ which coincide after composition with ν do still coincide after composition with the two projection maps. But this means that f and g give the same morphism $\mathrm{Spec}(R) \to U$, mapping into the same component of the coproduct, so that f is equal to g. □

Remark 2.1.6. In the following we will not consider the whole category of schemes over a fixed base field, but only the full subcategory $\mathcal{S}\mathrm{m}/k$ of smooth finite type separated k-schemes. Because of the finiteness condition - which is important for some noetherian induction arguments to make the topologies well behaved - this category does not have all the required coproducts, but it follows that it still has *finite* discrete group objects.

After the above discussion it seems to be natural to identify

$$\mathbb{Z}/n = \delta(\mathbb{Z}/n) = \coprod_{\mathbb{Z}/n} \mathrm{Spec}(k) \text{ in } \mathcal{S}\mathrm{m}/k$$

and also to identify

$$\mathbb{Z}/n = \delta(\mathbb{Z}/n) = \coprod_{\mathbb{Z}/n} \Delta^0 \text{ in sPre}.$$

The Yoneda Lemma implies that for a scheme $X \in \mathcal{S}\mathrm{m}/k$ the automorphism group $\mathrm{Aut}_{\mathcal{S}\mathrm{m}/k}(X)$ is isomorphic to the automorphism group $\mathrm{Aut}_{\mathrm{sPre}}(X)$ of the represented simplicial presheaf and thus by Lemma 2.1.4 both identifications lead to the same set of group actions in that case.

Another idea to regard $\mathbb{Z}/n$ as a group scheme in a natural way might be to consider the kernel μ_n of the n-th power endomorphism of the multiplicative group, i.e.

$$1 \to \mu_n \to \mathbb{G}_m \xrightarrow{\cdot^n} \mathbb{G}_m \to 1.$$

The equivalence of this approach depends on the base scheme:

Lemma 2.1.7.

(1) *Let k be a field with $\frac{1}{n} \in k$ and containing a primitive n-th root of unity. Then there are isomorphisms*

$$\mathbb{Z}/n \cong (\mu_n)^*$$

of groups objects in $\mathcal{S}\mathrm{m}/k$, where $(\mu_n)^*$ denotes the Cartier dual of the group scheme $\mu_n = \mathrm{Spec}(k[T]/(T^n - 1))$.

(2) *Let $\mathrm{char}(k) > 0$ divide n, then $\mathbb{Z}/n$ and μ_n are non-isomorphic.*

Proof.

(1) Let ξ be a primitive n-th root of unity in k. The k-algebra homomorphism

$$f : k[T]/(T^n - 1) \to k \text{ defined by } T \mapsto \xi$$

gives rise to a k-basis $(1, f, \ldots, f^{n-1})$ of $\mathrm{Hom}_k(k[T]/(T^n - 1), k)$ as k-vector space. One can check that

$$k^{\mathbb{Z}/n} \to \mathrm{Hom}_k(k[T]/(T^n - 1), k),$$
$$(a_1, \ldots, a_n) \mapsto a_1 + a_2 f + \ldots + a_n f^{n-1}$$

is not only an isomorphism of k-algebras, but also of the respective Hopf algebra structures which define the group scheme.

(2) Even in this case $\mathbb{Z}/n$ is still reduced while μ_n is not.

□

We already noted that an action of the group scheme $\mathbb{Z}/n$ is equivalent to an action of the constant simplicial presheaf with value the underlying set of $\mathbb{Z}/n$, but it is not true that this constant simplicial presheaf coincides with the simplicial presheaf $\widehat{\mathbb{Z}/n}$ represented by group scheme $\mathbb{Z}/n$.

Lemma 2.1.8. *The étale simplicial sheaf $\widehat{\mathbb{Z}/n}$ takes the values $\widehat{\mathbb{Z}/n}(U) = \mathbb{Z}/n^{\pi_0(U)}$ for $U \in \mathcal{S}\mathrm{m}/k$.*

Proof. The étale topology is subcanonical [Mil80, p.51] so $\widehat{\mathbb{Z}/n}$ is an étale sheaf and we have $\widehat{\mathbb{Z}/n}(U) = \mathrm{Hom}_{\mathcal{S}\mathrm{m}/k}(U, \mathbb{Z}/n) \cong \mathrm{Hom}_{\mathcal{S}\mathrm{et}}(\pi_0(U), \mathbb{Z}/n)$. □

Lemma 2.1.9. *The sheafification $a_{\mathrm{Zar}}(\mathbb{Z}/n)$ is isomorphic to the sheaf $\widehat{\mathbb{Z}/n}$.*

Proof. Knowing about Zariski sheafification of constant presheaves this is a corollary of the lemma above. □

Now we introduce two concepts of stabilizers. Their difference will be responsible for many distinctions in the following.

Definition 2.1.10. Let G be a group scheme acting on a scheme X and $x : \kappa(x) \to X$ be a point of X. The scheme theoretic stabilizer G_x is defined by the pullback diagram

(2.6)
$$\begin{tikzcd} G_x \arrow[r] \arrow[d] & G \times X \arrow[d, "(\alpha_X, \mathrm{pr}_X)"] \\ \kappa(x) \arrow[r, "\Delta \circ x"] & X \times X. \end{tikzcd}$$

In general an action $\alpha_X : G \times X \to X$ of a group object G on some object X is called free if the morphism

$$(\alpha_X, \mathrm{pr}_X) : G \times X \to X \times X$$

is a monomorphism. By the characterization of locally finite type monomorphisms [Gro67, Proposition 17.2.6] this implies that an action in $\mathcal{S}\mathrm{m}/k$ is free if and only if all isotropy groups are trivial in the sense that $G_x \xrightarrow{\cong} \mathrm{Spec}(\kappa(x))$ is an isomorphism for all $x \in X$.

The category $G\mathcal{S}\mathrm{m}/k$ of G-objects in $\mathcal{S}\mathrm{m}/k$ inherits limits from $\mathcal{S}\mathrm{m}/k$. The scheme underlying the limit of a diagram $D : I \to G\mathcal{S}\mathrm{m}/k$ is the limit of the underlying schemes and the group action on the limit is assembled as follows. One may regard the collection of group actions attached to D as a natural transformation $G \times |D| \to |D|$ of functors $I \to \mathcal{S}\mathrm{m}/k$, where $|D|$ is the diagram of underlying schemes of D. This natural transformation induces a morphism $G \times \lim |D| \cong \lim G \times |D| \to \lim |D|$ which gives the group action on the limit.

Lemma 2.1.11. *Let X, Y be in $G\mathcal{S}\mathrm{m}/k$ such that the action on one of them is free. Then the product $X \times Y$ in $G\mathcal{S}\mathrm{m}/k$ carries a free G-action.*

Proof. Let X be equipped with a free G-action. To see that the product action α induces a monomorphism

$$(\alpha, \mathrm{pr}_{23}) : G \times X \times Y \to X \times Y \times X \times Y$$

one observes that the composition

$$\mathrm{pr}_{134} \circ (\alpha, \mathrm{pr}_{23}) : G \times X \times Y \to X \times X \times Y$$

equals the morphism $(\alpha_X, \mathrm{pr}_X, \mathrm{id}_Y)$, which as a pairing of the monomorphisms $(\alpha_X, \mathrm{pr}_X)$ and id_Y is itself a monomorphism. Thus, the first morphism $(\alpha, \mathrm{pr}_{23})$ in the composition is also mono. □

There is a forgetful functor $U : G\mathcal{S}\mathrm{m}/k \to |G| - \mathcal{T}\mathrm{op}$. As we only consider finite constant group schemes G, we disregard the difference between G and its underlying space $|G|$ here. Applying U to the diagram (2.6) we obtain a morphism $i : G_x \to S_x$ into the pullback

$$\begin{tikzcd}
G_x \arrow[dr, dashed] & & \\
& S_x \arrow[r] \arrow[d] & G \times UX \arrow[d] \\
& * \arrow[r, "Ux"] & UX \times UX.
\end{tikzcd}$$

where S_x is the set theoretic stabilizer of x.

Lemma 2.1.12. *Let G be a finite constant group acting on a scheme X and let $x \in X$. Then there is an inclusion of subgroups $G_x \leq S_x \leq G$.*

Proof. We know that for an element x in the underlying set UX of the scheme X we have

$$S_x = \{g \in G \mid \text{the morphism } g : UX \to UX \text{ satisfies } gx = x\}$$

and in the same way we can describe (the underlying set of) G_x as

$$G_x = \{g \in S_x \mid \text{the induced morphism } g : \mathcal{O}_{X,x} \to \mathcal{O}_{X,x} \text{ equals } \mathrm{id}_{\mathcal{O}_{X,x}}\}.$$

□

Example 2.1.13. For the conjugation action of $\mathbb{Z}/2$ on $\mathrm{Spec}(\mathbb{C})$ the two notations of a stabilizer differ. We have $G_* = 0$ for the scheme theoretic stabilizer and $S_* = \mathbb{Z}/2$ for the set theoretic stabilizer.

Lemma 2.1.14. *Let G be a finite constant group and let $f : X \to Y$ be an étale morphism in $G\mathcal{S}\mathrm{m}/k$ such that for all $y \in Y$ there is an element $x \in X$ with $f^* : \kappa(y) \xrightarrow{\cong} \kappa(x)$. If there is such an x with the additional property that $S_x = S_{f(x)}$ then f induces an isomorphism of the respective scheme theoretic stabilizers.*

Proof. Since f is equivariant, we have an inclusion of the underlying subgroups $G_x \leq G_{f(x)}$ for all $x \in X$. Let $y \in Y$ and let $x \in X$ be as above. We want to show that f induces an isomorphism of schemes from G_x to G_y. Let g be an element in the underlying set of G_y. From the assumptions we know that g is then also an element in the set theoretic stabilizer S_x. Consider the commutative square

$$\begin{tikzcd}
\mathcal{O}_{Y,y} \arrow[r, "g_y=\mathrm{id}"] \arrow[d, "f_x"] & \mathcal{O}_{Y,y} \arrow[d, "f_x"] \\
\mathcal{O}_{X,x} \arrow[r, "g_x"] & \mathcal{O}_{X,x}.
\end{tikzcd}$$

We need to see that the action g_x induced by g on the local ring of X at x is trivial, i.e. $g_x = \mathrm{id}$, to conclude that the underlying subgroups G_x and G_y coincide. Since f_x induces an isomorphism on residue fields, it follows from Nakayama's Lemma that f_x is itself surjective (cf. [Lan02, Proposition X.4.5.]). So, f_x is an epimorphism and we cancel it in $f_x = g_x \circ f_x$ to obtain $g_x = \mathrm{id}$ and hence $G_x = G_y$ for the underlying subgroups of G. Finally, we may again apply that f induces an isomorphism between $\kappa(y)$ and $\kappa(x)$ to obtain that f also induces an isomorphism

$$G_x = |G_x| \times \kappa(x) \xrightarrow{\cong} |G_y| \times \kappa(y) = G_y$$

of the scheme theoretic isotropy groups. □

Let k be a field of characteristic 0 and let $G\mathcal{S}\mathrm{m}/k$ be the category of G-equivariant separated smooth k-schemes with G-equivariant morphisms. Much power in classical equivariant topology is obtained from adjunctions connecting equivariant to non-equivariant questions, e.g. the two adjunctions with the functor from spaces to G-spaces, which adds a trivial G-action. Due to the usual problems with quotients in algebraic geometry it seems to be difficult to carry both of the mentioned adjunctions to a motivic setup. In [Voe01] Voevodsky chose to carry over the adjunction

$$(-)/G : \mathrm{G}\mathcal{T}\mathrm{op} \rightleftarrows \mathcal{T}\mathrm{op} : (-)_{\mathrm{trivial}},$$

but we essentially follow [HKO11b] who build up their theory with an adjunction analogous to the classical adjunction

$$(-)_{\mathrm{trivial}} : \mathcal{T}\mathrm{op} \rightleftarrows \mathrm{G}\mathcal{T}\mathrm{op} : (-)^G.$$

For any k-scheme X there is the trivial G-scheme

$$tr(X) = (X, G \times X \xrightarrow{\pi_X} X) \tag{2.7}$$

over k. Mapping X to $tr(X)$ gives embeddings $\mathcal{S}\mathrm{ch}_\mathrm{k} \subset G\mathcal{S}\mathrm{ch}_\mathrm{k}$ and $\mathcal{S}\mathrm{m}/k \subset G\mathcal{S}\mathrm{m}/k$. If $X \in G\mathcal{S}\mathrm{ch}_\mathrm{k}$, one defines the functor

$$h_{X^G} : \mathcal{S}\mathrm{ch}_\mathrm{k}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}, Y \mapsto \mathrm{Hom}_{G\mathcal{S}\mathrm{ch}_\mathrm{k}}(tr(Y), X).$$

It is natural to ask for the representability of h_{X^G} and one is inclined to denote a representing object by X^G. The following theorem answers this question and supports the notation.

Theorem 2.1.15. *Let G be a finite constant group scheme over k and let $X \in G\mathcal{S}\mathrm{ch}_\mathrm{k}$. Then there exists a G-invariant closed subscheme X^G of X with a trivial G-action, representing h_{X^G}.*

Proof. Let $\{U_i\}_{i \in I}$ be the family of all closed G-invariant subschemes of X on which G acts trivially and let J_i be the quasi-coherent ideal of $\mathcal{O}_X$ corresponding to U_i. Let $J := \cap_i J_i$ be the intersection of $\mathcal{O}_X$ modules and denote by X^G the closed subscheme of X corresponding to the ideal sheaf J. Then X^G is G-invariant and has a trivial G-action as it is shown in [Fog73, Theorem 2.3]. □

Remark 2.1.16. Theorem 2.1.15 has a notable history. It is stated in more general terms as [DG70, Exp. VIII, Théorème 6.4]. Fogarty still tried to loose the assumptions on G in [Fog73, Theorem 2.3], but his published proof contains a gap which can not be closed, as shown in [Wri76]. However, in this special case of a finite constant group scheme Fogarty's proof also holds.

Definition 2.1.17. Let U, V be in $G\mathcal{S}\mathrm{m}/k$. A G-morphism $f : U \to V$ is called strongly étale if the induced map φ/G on categorical quotients (cf. [MFK94, Definition 0.5]) is étale and the diagram

$$\begin{tikzcd} U \arrow[r, "\pi_U"] \arrow[d, "\varphi"'] & U/G \arrow[d, "\varphi/G"] \\ V \arrow[r, "\pi_V"'] & V/G \end{tikzcd} \tag{2.8}$$

is cartesian.

Lemma 2.1.18. *Let $\varphi : U \to V$ be a strongly étale morphism in $G\mathcal{S}\mathrm{m}/k$. Then the diagram*

$$\begin{tikzcd} U^G \arrow[r, hook, "i"] \arrow[d] & U \arrow[d, "\varphi"] \\ V^G \arrow[r, hook] & V \end{tikzcd} \tag{2.9}$$

is cartesian.

Proof. Applying the right adjoint $(-)^G$ to diagram (2.8) one obtains a pullback diagram

$$\begin{tikzcd} U^G \arrow[r] \arrow[d] & U/G \arrow[d, "\varphi/G"] \\ V^G \arrow[r, "\pi_V \circ i"'] & V/G \end{tikzcd}$$

and hence the composition of (2.9) & (2.8) is a pullback, so in particular the square (2.9) is. □

To set up an algebraic G-fixed points functor $G\mathcal{S}\mathrm{m}/k \to \mathcal{S}\mathrm{m}/k$ in our context the following theorem is of high importance.

Theorem 2.1.19 (Luna's étale slice theorem)**.** *Let G be a reductive algebraic group acting on an affine variety X. Let $x \in X$ be such that X is smooth at x and the orbit Gx is closed. Then there exists a locally closed smooth subvariety V of X and an étale G_x-equivariant morphism $\psi : V \to T_xV$, such that*

(1) V is affine and contains x,
(2) V is G_x-invariant,
(3) the image of the G-morphism $\phi : G \times_{G_x} V \to V$ induced by the G-action on X is a saturated open subset U of X and
(4) the restriction $\phi : G \times_{G_x} V \to U$ is strongly étale,
(5) $T_xX \cong T_xV \oplus T_x(Gx)$,
(6) the image of ψ is a saturated open subset W of T_xV and
(7) the restriction $\psi : V \to W$ is strongly étale.

Proof. [Lun73, Théorème du slice étale, p.97] and [Dré00, Theorem 5.3 & 5.4]. □

Lemma 2.1.20. *Let G be a finite constant group scheme and let X be a smooth separated k-scheme with a G-action. Then the G-fixed points X^G of X are a closed smooth subscheme of X.*

Proof. Let $\{U_i\}_i$ be an open affine cover of X. Since G is finite and X separable the intersection

$$V_i := \bigcap_{g\in G} gU_i$$

is affine open and we may assume without loss of generality that X is affine.

To show that $X^G_{\bar{k}}$ is regular, let $x \in X^G_{\bar{k}}$ be a closed point. The orbit of x is closed and we may apply Theorem 2.1.19 to obtain strongly étale morphisms

$$U \overset{\varphi}{\leftarrow} V \overset{\psi}{\rightarrow} W$$

where U, V, W are as in the theorem. Thus, by Lemma 2.1.18 the morphisms φ^G and ϕ^G are étale and since the maps $U^G \to X^G$ and $W^G \to (T_xV)^G$ are open immersions we get that x is a regular point of X^G if and only if $\psi(x)$ is a regular point of $(T_xV)^G$, which is regular as a linear subspace (cf. Remark 2.1.22) of T_xV. $\square$

The above lemma establishes the adjunction

$$(-)_G : \mathcal{S}\mathrm{m}/k \rightleftarrows G\mathcal{S}\mathrm{m}/k : (-)^G \tag{2.10}$$

which is of crucial importance for the following stable equivariant motivic homotopy theory. We will also consider H-fixed points functors $(-)^H : G\mathcal{S}\mathrm{m}/k \to \mathcal{S}\mathrm{m}/k$, for proper subgroups H of G, defined as first restricting the G-action to an H-action and then applying the above construction. For a G-space X, the spaces X^H have a canonical action of the Weyl group of H in G, but we do not take care of this action for the moment.

Definition 2.1.21. A representation of G in $G\mathcal{S}\mathrm{m}/k$ is an affine space $\mathbb{A}^n$ equipped with a k-linear G-action, i.e. an action induced by a homomorphism $G \to GL(\mathbb{A}^n)$. The regular representation of G in $G\mathcal{S}\mathrm{m}/k$ is given by

$$\mathbb{A}[G] = \mathrm{Spec}(k[X_g | g \in G])$$

where the G-action is induced by the group homomorphism

$$G \to \mathrm{Aut}_k(k[X_g | g \in G]), h \mapsto (X_g \mapsto X_{hg}).$$

Remark 2.1.22. A linear algebraic group is called linearly reductive if every rational representation is completely reducible. It is the statement of Maschke's Theorem [Lan02, Theorem XVIII.1.2] that a finite group is linearly reductive if the characteristic of k does not divide the group order. Linear reductivity is needed to control the stabilization (cf. Proposition 4.1.11) in our chosen framework for stable equivariant motivic homotopy theory. For most of the later statements we will have other reasons to restrict ourselves to a base field k of characteristic zero. It should just be mentioned that the requirement of a linearly reductive group would still allow any linear algebraic group G, such that the connected component of the unit G^0 is a torus and $\mathrm{char}(k) \nmid [G : G^0]$ by Nagata's Theorem [Koh11, Theorem 2.4].

Example 2.1.23. Let X in $G\mathcal{S}\mathrm{m}/k$ be a scheme with a free G-action. Then $X^G = \emptyset$, since there is not closed G-invariant subscheme with a trivial G-action. Actually, for any closed G-invariant subscheme U the diagram

$$\begin{tikzcd}
G \times U \arrow[r, hook] \arrow[d] & G \times X \arrow[d, hook] \\
U \times U \arrow[r] & X \times X
\end{tikzcd}$$

commutes, and hence the left vertical morphism is mono and the induced action on U is free. But a trivial action (2.7) on U is not free, if G has at least to elements $e \neq g$, because the morphism (α_U, pr_U) equalizes the parallel pair

$$U \underset{(\{g\},\mathrm{id})}{\overset{(\{e\},\mathrm{id})}{\rightrightarrows}} G \times U \xrightarrow{(\alpha_U, pr_U)} U \times U$$

even though the parallel morphisms do not coincide. This example implies that the simplicial presheaf EG^H is equal to $\widetilde{\emptyset}$ for all $e < H \leq G$, where EG is the total space of the geometric universal bundle defined in Definition 5.1.7. For more examples on fixed-points of schemes we refer to Example 3.2.7 and Lemma 4.3.4.

2.2. **Grothendieck Topologies.** Let $\mathcal{C}$ be a category. A sieve in $\mathcal{C}$ is a full subcategory S, which is closed under taking domains, i.e. for any morphism $f : X \to Y$ in $\mathcal{C}$ such that Y belongs to S, the domain X also belongs to S. A sieve in $\mathcal{C}/X$ is called a *sieve on* X.

Definition 2.2.1. A Grothendieck topology on a category $\mathcal{C}$ is given by a collection $J(X)$ of sieves on X for every object X in $\mathcal{C}$, such that the following properties are satisfied:

(T1) For any morphism $f : Y \to X$ in $\mathcal{C}$ and any sieve S in $J(Y)$ the pullback sieve
$$f^*S := \{\varphi : Z \to Y \mid f \circ \varphi \in S\}$$
is in $J(X)$.

(T2) Given sieves S, T on X, such that S is in $J(X)$ and for any $f : Y \to X \in S$ the pullback f^*T is in $J(Y)$, then T is in $J(X)$.

(T3) The sieve $\mathcal{C}/X$ is in $J(X)$.

A sieve on an object X in $\mathcal{C}$ is called a covering sieve for the topology if it is in $J(X)$. A pair $(\mathcal{C}, T)$ consisting of a Grothendieck topology T on a category $\mathcal{C}$ is called a *site*.

A family $\mathcal{U} = \{f_i : U_i \to X\}_{i \in I}$ of morphisms in $\mathcal{C}$ generates a sieve

$$\langle \mathcal{U} \rangle := \{f : Z \to X \text{ in } \mathcal{C} \mid \exists i \in I, g \in \mathrm{Hom}_{\mathcal{C}}(Z, U_i) : f = f_i \circ g\} \tag{2.11}$$

on X. A family $\mathcal{U}$ is called a covering family for a given topology T, if the sieve $\langle \mathcal{U} \rangle$ generated by $\mathcal{U}$ is a covering sieve in T.

A bit more natural than the axioms of a Grothendieck topology when compared to the axioms of a classical topological space are the axioms of a sort of basis for a Grothendieck topology, a so-called pretopology.

Definition 2.2.2. A pretopology on a category $\mathcal{C}$ with pullbacks is given by a collection $Cov(X)$ of families of morphisms $\{U_i \to X\}_i$ with codomain X, such that the following properties are satisfied:

(P1) For any morphism $f : X \to Y$ in $\mathcal{C}$ and any family $\{\alpha_i : U_i \to Y\}_i$ in $Cov(Y)$ the family $\{\alpha_i' : U_i \times_Y X \to X\}_i$ of pullbacks of the α_i along f is in $Cov(X)$.
(P2) For any family $\{U_i \to X\}_{i\in I}$ in $Cov(X)$ and families $\{U_{ij} \to U_i\}_{j\in J_i}$ in $Cov(U_i)$ the composition $\{U_{ij} \to U_i \to X\}_{i\in I, j\in J_i}$ is in $Cov(X)$.
(P3) Isomorphisms are always coverings.

A pretopology with covering families $Cov(X)$ on $\mathcal{C}$ generates a topology T on $\mathcal{C}$ by defining that a sieve S on X is a covering sieve in T if and only if there is family R in $Cov(X)$, such that $R \subset S$.

Example 2.2.3. Of the many important sites in algebraic geometry we will now introduce the three most important for our work on motivic homotopy below, namely the Zariski-, étale, and Nisnevich topologies on $\mathcal{S}\mathrm{m}/k$.

Let Z denote the pretopology in which all covering families are given by jointly surjective families $\{\alpha_i : U_i \to X\}_i$ of morphisms in $\mathcal{S}\mathrm{m}/k$ where all the α_i are open immersions. Jointly surjective means, that the induced morphism $\coprod_i U_i \to X$ is surjective. The topology generated by Z is called the Zariski topology on $\mathcal{S}\mathrm{m}/k$. Analogously, the topology generated be jointly surjective étale morphisms is called the étale topology on $\mathcal{S}\mathrm{m}/k$. Between these two topologies, there is Nisnevich's completely decomposable topology [Nis89] which is the topology generated by the pretopology Nis. A jointly surjective family $\{\alpha_i : U_i \to X\}_i$ of morphisms in $\mathcal{S}\mathrm{m}/k$ is a covering of X in Nis if all the α_i are étale and for any $x \in X$ there is an element u in some U_i, such that $\alpha_i : U_i \to X$ induces an isomorphism $\kappa(x) \to \kappa(u)$.

It may happen that different pretopologies generate the same topology. The following proposition will be helpful to keep track of generating pretopologies for a given topology.

Proposition 2.2.4. *Let $\mathcal{C}$ be a category and $X \in \mathcal{C}$. Let E be a pretopology on $\mathcal{C}$ and let T be the topology on $\mathcal{C}$ generated by E. Let $J_E(X)$ be the collection of sieves on X generated by the families in E and let $J_T(X)$ be the collection of covering sieves in T. A sieve R on X is in $J_T(X)$ if and only if there is a sieve R' in $J_E(X)$, such that $R' \subset R$.*

Proof. This is [AGV72, Proposition II.1.4] □

A geometric morphism $f : \mathrm{Shv}(\mathcal{C}, T_1) \to \mathrm{Shv}(\mathcal{D}, T_2)$ between two (Grothendieck) topoi consists of an adjunction

$$f^* : \mathrm{Shv}(\mathcal{D}, T_2) \rightleftarrows \mathrm{Shv}(\mathcal{C}, T_1) : f_*$$

such that the *inverse image* functor f^* preserves finite limits.

Definition 2.2.5. A point in topos $\mathcal{E}$ is a geometric morphism $\mathcal{S}\mathrm{et} \to \mathcal{E}$.

A point in a topological space X is the same a morphism from the one-point space to X. The same is true for points of a topos, in the following sense. There is a site associated to any topological space, which has the open subsets as objects, the inclusions as morphisms, and jointly surjective families as coverings. The site associated to the one-point space therefore has

objects: $\emptyset$ and $\{*\}$,
morphisms: $\mathrm{id}_{\emptyset}$, $i : \emptyset \to \{*\}$ and id_*, and
coverings: $Cov(\emptyset) = \{\{\mathrm{id}_{\emptyset}\}\}$ and $Cov(\{*\}) = \{\{\mathrm{id}_*\}, \{i, \mathrm{id}_*\}\}$.

We observe that the category of sheaves on the site associated to the one-point space is equivalent to the category of sets. This interpretation of sets is supported by the fact that for any Grothendieck topos $\mathrm{Shv}(\mathcal{C})$, there is a unique geometric morphism

$$\mathrm{Shv}(\mathcal{C}) \to \mathcal{S}\mathrm{et}.$$

A family $\{U_i \to X\}_i$ is called a covering family for a topology if the sieve generated by it is a covering sieve in this topology. In particular, for a topology generated by a given pretopology there may be more covering families than just those defining the pretopology. For a topology on a category with pullbacks the collections of all its covering families defines a pretopology. The following proposition states that the conservativity of some set of points for a site can be decided on this finest pretopology. More precisely:

Proposition 2.2.6. *Let $(\varphi_i)_{i\in I}$ be a family of fiber functors on a site $\mathcal{C}$. The family $(\rho_i)_{i\in I}$ of associated points in* $\mathrm{sShv}(\mathcal{C})$ *is conservative if and only if for all families $(X_j \to X)_{j\in J}$, such that for all $i \in I$ the family $(\varphi_i(X_j) \to \varphi_i(X))_{j\in J}$ is surjective, the family $(X_j \to X)_{j\in J}$ is a covering family of X in $\mathcal{C}$.*

Proof. This is a [AGV72, Prop. 6.5.a] □

Remark 2.2.7. Let $\mathcal{C}$ be a site. A functor $F : \mathcal{C} \to \mathcal{S}\mathrm{et}$ is called a fiber functor (or continuous flat functor) if the left Kan extension $Lan_Y F : \mathrm{Pre}(\mathcal{C}) \to \mathcal{S}\mathrm{et}$ commutes with finite limits and F takes covering families to jointly surjective families of maps. The word 'associated' in the statement above refers to the equivalence of the category of points in $\mathrm{Shv}(\mathcal{C})$ and the category of fiber functors on $\mathcal{C}$ [AGV72, (6.2.1.1)].

Our crucial application of Proposition 2.2.6 will appear as Lemma 3.3.3, but the following two examples are useful consequences as well.

Example 2.2.8. In Example 2.2.3 we have defined the Zariski topology on $\mathcal{S}\mathrm{m}/k$. Let X in $\mathcal{S}\mathrm{m}/k$ and let $x \in X$ be an element of the underlying topological space $|X|$ of X. For a presheaf F on $\mathcal{S}\mathrm{m}/k$ define

$$F(\mathcal{O}_{X,x}) := \operatorname*{colim}_{U\in Nbh(X,x)} F(U),$$

where $Nbh(X,x)$ is the category with objects the scheme morphisms $f : U \to X$ with $x \in f(|u|)$ that appear in some covering of X and morphisms $m : V \to U$ those scheme morphisms that appear in some refinement and such that x is still in the image of $f \circ m$. A cofinality argument shows that for a Zariski sheaf F on X we recover the stalk $F_x = F(\mathcal{O}_{X,x})$ of F at x. Thus, we know that the family

$$\{p_x : \mathrm{Shv}(X_{\mathrm{Zar}}) \to \mathcal{S}\mathrm{et} \mid p_x(F) = F(\mathcal{O}_{X,x})\}_{\{x \in X\}}$$

is a conservative family of points for the site $(\mathcal{S}\mathrm{m}/k)_{\mathrm{Zar}}/X \simeq (\mathcal{S}\mathrm{m}/\mathrm{X})_{\mathrm{Zar}}$. With Proposition 2.2.6 we find that the family $\{p_x\}_{\{x \in X \in \mathcal{S}\mathrm{m}/k\}}$, now indexed over (a small skeleton of) $\mathcal{S}\mathrm{m}/k$, is a conservative set of points for the big Zariski site, since the proposition reduces this issue to the pullback sites $(\mathcal{S}\mathrm{m}/\mathrm{X})_{\mathrm{Zar}}$.

Example 2.2.9. A closer examination would show that all arguments in the example above rely on general abstract nonsense, even the fact that mapping to Zariski stalks gives points. Hence, the procedure may be generalized to any site by adjusting the definition of $Nbh(X,x)$ according to the respective covering families (cf. [AGV72, IV.6.8]). We define

$$F(\mathcal{O}^h_{X,x}) := \underset{U \in Nis(X,x)}{\mathrm{colim}} F(U),$$

where $Nis(X,x)$ is the category of Nisnevich neighborhoods of (X,x), defined the same way as $Nbh(X,x)$ in Example 2.2.8 but using the Nisnevich topology. The superscript h in $\mathcal{O}^h_{X,x}$ refers to the *henselization* of the Zariski local ring $\mathcal{O}_{X,x}$ [Gro67, 18.6].

We already have introduced geometric morphisms $\mathrm{Shv}(\mathcal{D}) \to \mathrm{Shv}(\mathcal{C})$ of topoi and we close this subsection with a few statements about how a functor $\mathcal{C} \to \mathcal{D}$ between Grothendieck sites may induce a geometric morphism. As is turns out in the next subsection, any geometric morphism is a Quillen adjunction and so this concepts will find some applications in Section 3, e.g. in Lemma 3.1.4 or in the characterization of equivariant weak equivalences in Section 3.3.

Definition 2.2.10. A functor $F : \mathcal{C} \to \mathcal{D}$ between two Grothendieck sites is a *morphism of sites* if

(1) for any sheaf $G : \mathcal{D}^{op} \to \mathcal{S}\mathrm{et}$ the composition

$$\mathcal{C}^{op} \xrightarrow{F^{op}} \mathcal{D}^{op} \xrightarrow{G} \mathcal{S}\mathrm{et}$$

is a sheaf on $\mathcal{C}$ and

(2) the functor $F^* : \mathrm{Shv}(\mathcal{C}) \to \mathrm{Shv}(\mathcal{D})$ left adjoint to precomposition with F^{op} commutes with finite limits.

We will apply the following criterion to show that a given functor is a morphism of sites:

Proposition 2.2.11. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between two Grothendieck sites, where the topology on $\mathcal{C}$ is given by a pretopology. Suppose that F commutes with finite limits, that $\mathcal{C}$ is subcanonical, and that F maps covering families defining the pretopology on $\mathcal{C}$ to covering families for the topology on $\mathcal{D}$. Then F is morphism of sites.*

Proof. This is [AGV72, III.Proposition 1.6. & (IV.4.9.2)] . □

Example 2.2.12. Let $K : k$ be a finite separable field extension. The small Nisnevich site $\mathrm{Spec}(K)_{Nis}$ has all the étale finite type morphisms $X \to \mathrm{Spec}(K)$ as its objects. By [MM92, VII.Theorem 5] composition with $\mathrm{Spec}(K) \to \mathrm{Spec}(k)$ defines a geometric morphism $\mathrm{Shv}(\mathrm{Spec}(K)_{Nis}) \to \mathrm{Shv}(\mathcal{S}\mathrm{m}/k_{Nis})$. It is well known, that the objects of $\mathrm{Spec}(K)_{Nis}$ (by definition the same as those of $\mathrm{Spec}(K)_{et}$) are finite disjoint unions of spectra of the form $\mathrm{Spec}(L)$, where $L : K$ is a finite seperable extension. A morphism $\coprod_i \mathrm{Spec}(L_i) \to \mathrm{Spec}(K)$ is a cover in $\mathrm{Spec}(K)_{Nis}$ if and only if one of the L_i is equal to K. Thus, the evaluation of a presheaf $P \in \mathrm{Pre}(\mathrm{Spec}(K)_{Nis})$ is isomorphic to the evaluation of the associated sheaf and it follows that evaluation at K is point in $\mathrm{Shv}(\mathcal{S}\mathrm{m}/k_{Nis})$, associated to the geometric morphism

$$\mathcal{S}\mathrm{et} \xrightarrow{ev_K} \mathrm{Pre}(\mathrm{Spec}(K)_{Nis}) \xrightarrow{i} \mathrm{Shv}(\mathrm{Spec}(K)_{Nis}) \xrightarrow{\iota_{K:k}} \mathrm{Shv}(\mathcal{S}\mathrm{m}/k_{Nis}).$$

2.3. **Local Homotopy Theory.** All of the abstract homotopy theory in this thesis is based on so-called local homotopy theory of simplicial presheaves on site, which is a powerful fusion of topos theory and the combinatorial (model for) homotopy theory given by the homotopy theory of simplicial sets. To be not too excessive we are not going to develop this theory to a large extent, but instead we point to Jardine's original work [Jar87] and to his excellent textbook [Jar10] as our main references for this section and recollect just a few essentials with an emphasis on later applications.

Definition 2.3.1. Let $\mathcal{C}$ be a site with enough points. A morphism $f : X \to Y$ in $\mathrm{sPre}(\mathcal{C})$ is called a *local weak equivalence* if for all points $p : \mathcal{S}\mathrm{et} \to \mathrm{Shv}(\mathcal{C})$ the morphism of stalks $p^*(f) : p^*(X) \to p^*(Y)$ is a weak equivalence of simplicial sets.

Remark 2.3.2. The notion of point (cf. Definition 2.2.5) depends on the topology on $\mathcal{C}$ and gives a morphism $p^* : \mathrm{Shv}(\mathcal{C}) \to \mathcal{S}\mathrm{et}$. What we mean by $p^* : \mathrm{sPre}(\mathcal{C}) \to \mathrm{s}\mathcal{S}\mathrm{et}$ in the above is the left Kan extension of the fiber functor $p^*|_Y : \mathcal{C} \to \mathcal{S}\mathrm{et}$ to a functor $\mathrm{Pre}(\mathcal{C}) \to \mathcal{S}\mathrm{et}$ which then induces the functor $p^* : \mathrm{sPre}(\mathcal{C}) \to \mathrm{s}\mathcal{S}\mathrm{et}$ on $Fun(\Delta^{op}, -)$. This explanation of local weak equivalences does not only require a conservative set of points, but also a subcanonical topology. None of this is really necessary (cf. [Jar96]), but the situation below will always be at least as good as that.

The following example provides a discussion of the Čech nerve of a covering and how it gives a local weak equivalence.

Example 2.3.3. Any equivalence relation $\sim$ on a set S defines a groupoid $C(\sim)$ with objects the elements of S and a single morphism $s \to t$ if $s \sim t$. Further, any map $f : S \to T$ of sets defines an equivalence relation $\sim_f$ of 'having the same image under f' on S. We denote the associated groupoid by $C(f)$. It is well known that the nerve $BC(f)$ of a groupoid $C(f)$ is a Kan simplicial set and one computes that

$$\begin{aligned} \pi_0(BC(f)) &= S/\sim_f \\ &\cong f(S), \text{ and} \\ \pi_i(BC(f), s) &= 0, \text{ for all } i \geq 1, s \in S, \end{aligned}$$

since - concerning π_1 - there is always only exactly one morphism between to objects in the same path component of $BC(f)$ and since the nerve is 2-coskeletal and has no higher homotopy groups. So that considering $f(S)$ as a constant simplicial set we have a weak equivalence $BC(f) \to f(S)$. This construction generalizes to an objectwise construction for morphisms $f : X \to Y$ of presheaves and gives a simplicial presheaf $BC(f)$ with $BC(f)(U) = BC(f_U)$ and an objectwise weak equivalence $BC(f) \to im(f)$.

If we now consider a site $\mathcal{C}$ with enough points and f is a local epimorphism (i.e. an epimorphism at all stalks), then the inclusion $im(f) \hookrightarrow Y$ is a stalkwise isomorphism and hence a local weak equivalence. Thus, for a local epimorphism f we have a local weak equivalence $BC(f) \to Y$. By Proposition 2.2.6 any covering family $\mathcal{U} = \{U_i \to Y\}_i$ gives a local epimorphism $\coprod_i U_i \to Y$ of the represented presheaves and so by the above a local weak equivalence

$$\check{C}(\mathcal{U}) := BC(\coprod_i U_i \to Y) \xrightarrow{\sim} Y$$

from the so-called *Čech nerve* $\check{C}(\mathcal{U})$ of the covering $\mathcal{U}$ to Y.

The following lemma is an easy consequence of the example above.

Lemma 2.3.4. *Let $\mathcal{U} = \{U_i \to X\}$ be a covering in the site $\mathcal{C}$, let $Y := \coprod_i U_i$ and let Z denote the coequalizer of the diagram*

$$Y \times_X Y \rightrightarrows Y$$

in the category of presheaves on $\mathcal{C}$ and consider Z as a discrete simplicial presheaf. There is an objectwise weak equivalence $\check{C}(\mathcal{U}) \to Z$.

Proof. This is not more than a rephrasement of the observation that Z is the presheaf $\pi_0(\check{C}(\mathcal{U}))$ and the Example 2.3.3. □

After this short recollection of first results on local weak equivalences, we come to review another feature given in the main theorem - Theorem 2.3.5 - of this subsection. The model structure in Theorem 2.3.5 has the property of being simplicial, which means that given simplicial presheaves X, Y in $\mathrm{sPre}(\mathcal{C})$, there is a simplicial set of morphisms $\mathrm{s}\mathcal{S}\mathrm{et}(X, Y)$, natural in X and Y and compatible with the model structure in a sense specified below. In simplicial degree n this hom-object is given by

$$\mathrm{s}\mathcal{S}\mathrm{et}(X, Y)_n = \mathrm{Hom}_{\mathrm{sPre}(\mathcal{C})}(X \times \Delta^n, Y),$$

where the simplicial set Δ^n is considered as a constant simplicial presheaf. The homotopical compatibility is codified into the so-called axiom (SM7), which says that:

For a cofibration $j : A \to B$ in $\mathrm{sPre}(\mathcal{C})$ and a fibration $q : X \to Y$ in $\mathrm{sPre}(\mathcal{C})$ the induced morphism

$$\mathrm{s}\mathcal{S}\mathrm{et}(B, X) \to \mathrm{s}\mathcal{S}\mathrm{et}(A, X) \times_{\mathrm{s}\mathcal{S}\mathrm{et}(A,Y)} \mathrm{s}\mathcal{S}\mathrm{et}(B, Y) \tag{SM7}$$

of simplicial sets is a Kan fibration which is acyclic if j or q are. A model structure on $\mathrm{sPre}(\mathcal{C})$ is called simplicial if it satisfies the axiom (SM7). That the model structures we consider below are simplicial with respect to this enrichment has a deep influence on the corresponding homotopy theory, for example on the bigrading of the long exact sequences of weighted stable equivariant homotopy groups of a cofiber sequence (4.4). Although one might expect some equivariant artifacts to show up in the case $\mathcal{C} = G\mathcal{S}\mathrm{m}/k$, this standard enrichment leads to a very convenient theory in many aspects.

The following theorem is [Jar10, Theorem 5.8] and defines the *local injective model structure.*

Theorem 2.3.5. *Let $\mathcal{C}$ be a small site. There is a cofibrantly generated, proper, and simplicial model structure on* $\mathrm{sPre}(\mathcal{C})$ *(resp.* $\mathrm{sShv}(\mathcal{C})$*) with local weak equivalences as weak equivalences and monomorphisms as cofibrations.*

As local weak equivalences in $\mathrm{sPre}(\mathcal{C})$ and $\mathrm{sShv}(\mathcal{C})$ are determined by the topology on $\mathcal{C}$ it is no surprise that a geometric morphism $\mathrm{Shv}(\mathcal{D}) \to \mathrm{Shv}(\mathcal{C})$ is a Quillen adjunction. Actually, the left adjoint $\mathrm{sShv}(\mathcal{C}) \to \mathrm{sShv}(\mathcal{D})$ preserves cofibrations and local weak equivalences [Jar10, Lemma 5.20]. If the geometric morphism is induced by a morphism of sites, there is a variant of this statement which we record as a lemma for better citing.

Lemma 2.3.6. *Let $F : \mathcal{C} \to \mathcal{D}$ be a morphism of sites. Then*

$$F^* : \mathrm{sPre}(\mathcal{C}) \rightleftarrows \mathrm{sPre}(\mathcal{D}) : F_*$$

is a Quillen adjunction with respect to the local injective model structures.

Proof. This follows from the paragraph above: since F^* still preserves monomorphisms and since sheafification preserves the stalks it follows that F^* does also preserve local weak equivalence of simplicial presheaves. $\square$

3. Unstable Equivariant Motivic Homotopy Theory

In the first subsection we recall the setup of unstable equivariant motivic homotopy theory of Hu, Kriz, and Ormsby [HKO11b]. We show by a counterexample, that it does not have the property of recognizing weak equivalences fixed-pointwise. In the second subsection we introduce a topology which has this property. We investigate how these two topologies relate. In the third subsection the recognition property for the H-Nisnevich topology is proven. The fourth subsection gives a few constructions of functors and shows their compatibility with the respective homotopy theories. In the last subsection we recall the result of Krishna and Østvær that equivariant algebraic K-theory is representable in a motivic setup based on the equivariant Nisnevich topology and we show that it is not representable in the H-Nisnevich topology.

3.1. **The Equivariant Nisnevich Topology.** A lesson to be learned from non-equivariant motivic homotopy theory is that one should start with a reasonable topology on the site of geometric objects. The meaning of reasonable was a delicate question in ordinary motivic homotopy theory and it turns out that even with the experience of knowing the Nisnevich topology and all the advantages it enjoys it is still a difficult problem to find a convenient equivariant analogue.

Definition 3.1.1. A morphism $f : Y \to X$ in $G\mathcal{S}\mathrm{m}/k$ is called equivariant Nisnevich cover (in the sense of [HKO]) if it is étale and for all $x \in X$ there is a $y \in Y$ with $f(y) = x$, such that f induces an isomorphism $k(x) \to k(y)$ of residue fields and an isomorphism $S_y \to S_x$ of set theoretic stabilizers.

We will later see that this topology has several good properties, e.g. being subcanonical and induced by complete, regular, and bounded cd-structure. For the beginning we just define this local homotopy theory associated to this site.

Proposition 3.1.2. *There is a local injective model structure on* $\mathrm{sPre}(G\mathcal{S}\mathrm{m}/k)$ *(resp.* $\mathrm{sShv}(G\mathcal{S}\mathrm{m}/k)$*) with respect to the equivariant Nisnevich topology defined above. This model structure is simplicial proper and cofibrantly generated and therefore allows Bousfield localization.*

Proof. This is an instance of Jardine's more general Theorem 2.3.5. $\square$

Remark 3.1.3. As already mentioned in the introduction of this section there are many alternative topologies on $G\mathcal{S}\mathrm{m}/k$. For example, one could consider a topology which is blind for the group action, that means where a family is covering if and only if the underlying family of schemes is a Nisnevich covering.

Other alternatives are the isovariant topology which is discussed in Serpé's paper [Ser10] or Voevodsky's suggestion discussed in [Voe01, Section 3.1]. We will only investigate one other alternative, a fixed-point-wise defined topology of the next subsection, in this work.

Lemma 3.1.4. *There is a Quillen adjunction*

$$tr : \mathrm{sPre}(\mathcal{S}\mathrm{m}/k) \rightleftarrows \mathrm{sPre}(G\mathcal{S}\mathrm{m}/k) : (-)^G$$

with the respective local model structures.

Proof. The functor $tr : \mathcal{S}\mathrm{m}/k \to G\mathcal{S}\mathrm{m}/k$ commutes with limits by the discussion before Lemma 2.1.11 and thus by Proposition 2.2.11 is a morphism of sites. Then Lemma 2.3.6 implies that the Kan-extension gives a Quillen adjunction. □

I owe a debt of gratitude to Ben Williams for the following example. It shows that local weak equivalences with respect to the equivariant Nisnevich topology can not be detected by the family $\{(-)^H\}_{H\leq G}$ of fixed-point functors.

Example 3.1.5. The following happens in $\mathbb{Z}/2$-$\mathcal{S}\mathrm{m}/\mathbb{C}$: Let Y in $G\mathcal{S}\mathrm{m}/k$ be given by the disjoint union $\mathbb{G}_m \coprod \mathbb{G}_m$ be equipped with the $\mathbb{Z}/2$ action permuting the summands. Let $X = \mathbb{G}_m = \mathrm{Spec}(k[T, T^{-1}])$ carry the $\mathbb{Z}/2$ action induced by $T \mapsto -T$. We define a G-equivariant morphism

$$p := \mathrm{id} \coprod \sigma : Y \to X,$$

where σ is the non-trivial automorphism acting on X. Note that the fixed-point morphisms

$$\begin{aligned} p^e &= \mathbb{G}_m \coprod \mathbb{G}_m \xrightarrow{\mathrm{id} \coprod \mathrm{id}} \mathbb{G}_m \text{ and} \\ p^G &= \mathrm{id}_\emptyset \end{aligned}$$

are Nisnevich covers in the usual non-equivariant sense. Now, consider the coequalizer diagram

$$\begin{tikzcd} Y \times_X Y \arrow[r, shift left] \arrow[r, shift right] & Y \arrow[r] \arrow[dr, "p"'] & W \arrow[d, dashed, "h"] \\ & & X \end{tikzcd}$$

The map h is not a local weak equivalence in the equivariant Nisnevich topology and p is not a cover in that topology. The reason is that the generic point of $X = \mathbb{G}_m$ inherits an action and does not lift to Y: There is a map

$$(t \mapsto -t) \circlearrowright \mathrm{Spec}(\mathbb{C}(t)) \to X = \mathbb{G}_m,$$

but the value of the point (cf. Example 2.2.12) at Y and W is $\emptyset$ since

$$\emptyset = \mathrm{Hom}_G(\mathbb{C}(t), Y) \twoheadrightarrow \mathrm{Hom}_G(\mathbb{C}(t), W).$$

Hence, h is not a local weak equivalence for the equivariant Nisnevich topology and by Lemma 2.3.4 the morphism p can not be a covering for this topology. This example has to be seen in contrast to the equivariant topology defined in Section 3.2.

The next and last step towards an unstable equivariant motivic homotopy theory is a left Bousfield localization with respect to the projection morphisms

$$\{X \times \mathbb{A}^1 \to X \mid X \in G\mathcal{S}\mathrm{m}/k\}$$

where $\mathbb{A}^1$ is considered in $G\mathcal{S}\mathrm{m}/k$ with the trivial G-action. We call the resulting model structure the $\mathbb{A}^1$-local model structure on $\mathrm{sPre}(G\mathcal{S}\mathrm{m}/k)$ based on the equivariant Nisnevich topology, even though this precision is rarely necessary.

While the localization explicitly just contracts the trivial 1-dimensional representation all other representations are $\mathbb{A}^1$-locally contractible as well.

Lemma 3.1.6. *Let V be a representation of G in $\mathcal{S}\mathrm{m}/k$ and $X \in G\mathcal{S}\mathrm{m}/k$. Then the projection morphism $X \times V \to X$ represents an $\mathbb{A}^1$-local weak equivalence in* $\mathrm{sPre}(G\mathcal{S}\mathrm{m}/k)$*. More generally, any equivariant vector bundle is an $\mathbb{A}^1$-local weak equivalence in* $\mathrm{sPre}(G\mathcal{S}\mathrm{m}/k)$*.*

Proof. In fact, any equivariant geometric vector bundle $E \to X$ in $G\mathcal{S}\mathrm{m}/k$ is a strict $\mathbb{A}^1$-homotopy equivalence and thus an $\mathbb{A}^1$-local weak equivalence by [MV99, Lemma 2.3.6]. □

Next, we state a specialization of a result from [KØ10] telling that the equivariant Nisnevich topology is generated by a cd-structure in the sense of [Voe00]. These results have useful applications in descent questions.

A cd-structure on site $\mathcal{C}$ is a collection of squares which is closed under isomorphism. We keep the language of stacks from [KØ10] for the following definition as we are just focused on recollecting the forthcoming proposition.

Definition 3.1.7. An equivariant Nisnevich square (eN-square) is a cartesian square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ B \arrow[r, "i"] & Y \end{tikzcd}$$

in $G\mathcal{S}\mathrm{m}/k$, such that i induces an open immersion of quotient stacks $[B/G] \to [Y/G]$ with complement Z and for the reduced closed substack Z of $[Y/G]$ the map étale p induces an isomorphism $Z \times_{[Y/G]} [X/G] \to Z$.

Proposition 3.1.8. *The equivariant Nisnevich topology is generated by the complete, regular, and bounded cd-structure defined by the eN-squares.*

Proof. This is a corollary of [KØ10, Theorem 8.4]. □

Voevodsky's cd-structures are a valuable generalization of the Brown-Gersten property for the Zariski topology to more general topologies. When a topology τ is generated by a good cd-structure this allows a nice characterization of sheaves, reasoning about bounds for the sheaf cohomological dimension and last but not least a feasible criterion for descent, which is essentially the reason why we stated the proposition above. More precisely, a simplicial presheaf F on site $(\mathcal{C}, \tau)$ satisfies descent with respect to τ if and only if F maps distinguished squares to homotopy pullback squares [MV99, Proposition 1.16.].

3.2. **The H-Nisnevich Topology.** Next, we define a topology on $G\mathcal{S}\mathrm{m}/k$ in a way that automatically extends the functors $(-)^H$ to adjunctions of topoi. This will provide a recognition of local weak equivalences by their H-fixed points.

Definition 3.2.1. Let τ be the collection of all sieves on the objects $X \in G\mathcal{S}\mathrm{m}/k$ such that S is a sieve in τ if and only if for all subgroups $H \leq G$ the sieve generated by S^H is a covering sieve for the Nisnevich topology on $\mathcal{S}\mathrm{m}/k$. We call τ the H-Nisnevich-topology. A sieve on X in τ is called an H-sieve and the collection of all H-sieves on X is denoted by $\tau(X)$.

Lemma 3.2.2. *The collection τ defines a Grothendieck topology on $G\mathcal{S}\mathrm{m}/k$.*

Recall from (2.11) that we denote by $\langle S \rangle$ the sieve generated by a family S of morphisms with a common codomain. To prove the above lemma, we firstly prove another statement:

Lemma 3.2.3. *Let S be an H-sieve on $X \in G\mathcal{S}\mathrm{m}/k$. The sieves $\langle (f^*S)^H \rangle$ and the pullback $(f^H)^* \langle S^H \rangle$ on X^H in $\mathcal{S}\mathrm{m}/k$ coincide.*

Proof. We have on the one hand

$$\begin{aligned}
\langle (f^*S)^H \rangle &= \langle (\{ Z \xrightarrow{\varphi} Y \mid f \circ \varphi \in S \})^H \rangle \\
&= \langle \{ Z^H \xrightarrow{\varphi^H} Y^H \mid f \circ \varphi \in S \} \rangle \\
&= \left\{ W \xrightarrow{\psi} Y^H \in \mathcal{S}\mathrm{m}/k \;\middle|\; \begin{array}{l} \exists \varphi : Z \to Y \in G\mathcal{S}\mathrm{m}/k, \\ \xi : W \to Z^H \in \mathcal{S}\mathrm{m}/k : \\ \psi = \varphi^H \circ \xi \wedge f \circ \varphi \in S \end{array} \right\}
\end{aligned}$$

and on the other hand

$$
\begin{aligned}
(f^H)^*\langle S^H\rangle &= (f^H)^* \left\{ W \xrightarrow{\psi} X^H \;\middle|\; \begin{array}{l} \exists \varphi : Z \to X \in G\mathcal{S}\mathrm{m}/k, \\ \xi : W \to Z^H \in \mathcal{S}\mathrm{m}/k : \\ \psi = \varphi^H \circ \xi \wedge \varphi \in S \end{array} \right\} \\
&= \left\{ W \xrightarrow{\psi} Y^H \in \mathcal{S}\mathrm{m}/k \;\middle|\; \begin{array}{l} \exists \varphi : Z \to X \in G\mathcal{S}\mathrm{m}/k, \\ \xi : W \to Z \in \mathcal{S}\mathrm{m}/k : \\ f^H \circ \psi = \varphi^H \circ \xi \wedge \varphi \in S \end{array} \right\}
\end{aligned}
$$

Now, comparing the respective descriptions we immediately observe the inclusion

$$\langle (f^*S)^H\rangle \subseteq (f^H)^*\langle S^H\rangle.$$

Let $\psi : W \to Y^H$ be in $(f^H)^*\langle S^H\rangle$. Then there exist morphisms $\varphi : Z \to X \in S$ and $\xi : W \to Z^H \in \mathcal{S}\mathrm{m}/k$ such that the square

$$
\begin{tikzcd}
W \arrow[r, "\psi"] \arrow[d, "\xi"'] & Y^H \arrow[d, "f^H"] \\
Z^H \arrow[r, "\varphi^H"] & X^H
\end{tikzcd}
$$

commutes. Now the pullback $\bar{\varphi} : P \to Y$ of φ along f in $G\mathcal{S}\mathrm{m}/k$ has the property that $\bar{\varphi}^H$ is still the pullback of φ^H along f^H, since $(-)^H$ is right adjoint. Therefore, ψ factors via the induced morphism over $\bar{\varphi}^H$. Moreover, since φ is in the sieve S it follows that $\bar{f} \circ \varphi = f \circ \bar{\varphi}$ is in S, so $\bar{\varphi}$ satisfies all conditions to guarantee that ψ is in $\langle (f^*S)^H\rangle$. □

Proof of Lemma 3.2.2. Let X in $G\mathcal{S}\mathrm{m}/k$.

(T1) Let S be an H-sieve on X and $f : Y \to X \in G\mathcal{S}\mathrm{m}/k$. Then, for all $H \leq G$

$$\langle (f^*S)^H\rangle = (f^H)^*\langle S^H\rangle \qquad \text{by Lemma 3.2.3}$$

is a Nisnevich sieve and therefore f^*S is an H-sieve.

(T2) Let S, T be sieves on X. Let S be an H-sieve such that for all $f \in S$ the pullback f^*T is an H-sieve. Hence, for all $f \in S$ we have again by Lemma 3.2.3 and our assumption that $\langle (f^*T)^H\rangle = (f^H)^*\langle T^H\rangle$ is a Nisnevich sieve on X^H. An arbitrary $g \in \langle S^H\rangle$ factors into $g = f^H \circ h$ for some $f \in S$, so that

$$\langle (g^*T)^H\rangle = (f^H \circ h)^*\langle T^H\rangle = h^*\left((f^H)^*\langle T^H\rangle\right)$$

is a Nisnevich sieve by the pullback axiom of the Nisnevich topology.

(T3) Since $\mathrm{id}_X \in G\mathcal{S}\mathrm{m}/k/X$, we have that
$$\langle (G\mathcal{S}\mathrm{m}/k/X)^H \rangle \supseteq \langle \mathrm{id}_{X^H} \rangle = \mathcal{S}\mathrm{m}/k/X^H$$
is a Nisnevich sieve for all $H \leq G$.

$\square$

Lemma 3.2.4. *A family $\{f_i : U_i \to X\}_i$ of morphisms in $G\mathcal{S}\mathrm{m}/k$ is a covering family in the H-Nisnevich-topology if and only if for all $H \leq G$ the family $\{f_i^H : U_i^H \to X^H\}_i$ is a Nisnevich cover.*

Proof. The family $\{f_i\}_i$ is an H-cover if and only if $\langle \{f_i\}_i \rangle$ is an H-sieve, if and only if for all $H \leq G$ we have that $\langle \langle \{f_i\}_i \rangle^H \rangle$ is a Nisnevich sieve. By arguments similar to the proof of Lemma 3.2.3 we note that $\langle \langle \{f_i\}_i \rangle^H \rangle = \langle \{f_i^H\}_i \rangle$, so the assumption is true if and only if for all $H \leq G$ we have that $\langle \{f_i^H\}_i \rangle$ is a Nisnevich sieve, which in turn is equivalent to the assertion that $\{f_i^H\}$ is Nisnevich cover. $\square$

The above lemma states that H-covers consist of G-equivariant étale maps such that for any point x there is a point y above x with isomorphic residue field and whose isotropy group is large enough. More precisely we have the following comparison result with the equivariant Nisnevich topology from Definition 3.1.1.

Lemma 3.2.5. *A morphism $f : X \to Y$ in $G\mathcal{S}\mathrm{m}/k$ is a cover in the H-Nisnevich topology (an H-cover) if and only if it is true that f is étale (as a morphism of schemes), for every point y in Y there is a x in X, such that f induces an isomorphism of residue fields, and*

(*) *also induces an isomorphism $G_x \xrightarrow{\cong} G_y$ of scheme theoretic stabilizers.*

Proof. We start with the direction which is used in the corollary below: Assume that $f : X \to Y \in G\mathcal{S}\mathrm{m}/k$ is a morphism such that f^e is Nisnevich in $\mathcal{S}\mathrm{m}/k$ and f induces an isomorphism on scheme theoretic isotropy.

In the commutative diagram

$$\begin{tikzcd}
X^H \arrow[dr, dashed, "i"] \arrow[drr, "\iota_X^H"] \arrow[ddr, "f^H"'] & & \\
& X \times_Y Y^H \arrow[r, "j"'] \arrow[d, "f'"] & X \arrow[d, "f"] \\
& Y^H \arrow[r, "\iota_Y^H"] & Y
\end{tikzcd}$$

the morphisms ι_X^H and ι_Y^H are closed immersions, hence so are j and i. From the isotropy condition (*) it follows that f^H is surjective, so that by dimension X^H is a union of irreducible components of $X \times_Y Y^H$ and thus i and also f^H are étale. If for any $y \in Y^H$ an element $x \in X$ is given with the property that f induces isomorphisms of the respective residue fields and scheme theoretic stabilizers, then x is X^H and therefore f^H is Nisnevich.

Conversely, let f^H be a Nisnevich cover in $\mathcal{S}\mathrm{m}/k$ for all subgroups $H \leq G$. Given an element $y \in Y$ say with $G_y = H \times \kappa(y)$, then y is in Y^H and there is an element x in X^H, such that f induces an isomorphism from $\kappa(y)$ to $\kappa(x)$. Since x is in X^H we know

$$G_x = K \times \kappa(x) \geq H \times \kappa(x) \cong H \times \kappa(y) = G_y$$

and the equivariance of f implies $G_x \leq G_y$, so that f induces an isomorphism on scheme theoretic isotropy. □

Corollary 3.2.6. *Every equivariant Nisnevich cover is an H-cover.*

Proof. This follows from the above lemma combined with Lemma 2.1.14. □

The following example reminds one to be careful while thinking about isotropy groups and fixed points.

Example 3.2.7. Let $L : k$ be a finite Galois extension and $G = \mathrm{Gal}(L : k)$. The induced G-action on $\mathrm{Spec}(L)$ has empty fixed points $\mathrm{Spec}(L)^G = \emptyset$. This is since $\mathrm{Spec}(L)^G$ is by construction a closed subscheme of $\mathrm{Spec}(L)$ and

$$\mathrm{Hom}_{\mathcal{S}\mathrm{m}/k}(\mathrm{Spec}(L), \mathrm{Spec}(L)^G) \cong \mathrm{Hom}_{G\mathcal{S}\mathrm{m}/k}(\mathrm{Spec}(L)_{\mathrm{tr}}, \mathrm{Spec}(L)) = \emptyset.$$

The set-theoretic stabilizer S_* of the unique point $*$ is obviously the whole group G, but the scheme theoretic stabilizer is trivial, that is $G_* = \mathrm{Spec}(L)$, since the action is free and hence the left vertical arrow in the pullback diagram

$$\begin{tikzcd}
G_* \arrow[r] \arrow[d, "\cong"'] & G \times \mathrm{Spec}(L) \arrow[d, "\Psi"', "\cong"] \\
\mathrm{Spec}(L) \arrow[r, "\Delta"] & \mathrm{Spec}(L) \times \mathrm{Spec}(L)
\end{tikzcd}$$

is an isomorphism as well.

Lemma 3.2.8. *The H-Nisnevich topology is subcanonical, i.e. representable presheaves are sheaves on $G\mathcal{S}\mathrm{m}/k$.*

Proof. Let $\{Z_i \xrightarrow{\iota_i} Z\}_i$ be H-Nisnevich covering and let $U : G\mathcal{S}\mathrm{m}/k \to \mathcal{S}\mathrm{m}/k$ the forgetful functor. U is faithful and as a (trivial) fixed point functor U takes the chosen covering to a Nisnevich covering in $\mathcal{S}\mathrm{m}/k$. Hence, the bottom row in the diagram

$$\begin{tikzcd}
\mathrm{Hom}_G(Z,X) \arrow[r] \arrow[d] & \prod \mathrm{Hom}_G(Z_i,X) \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \prod \mathrm{Hom}_G(Z_i \times_Z Z_j, X) \arrow[d] \\
\mathrm{Hom}_k(UZ,UX) \arrow[r] & \prod \mathrm{Hom}_k(UZ_i,UX) \arrow[r, shift left] \arrow[r, shift right] & \prod \mathrm{Hom}_k(UZ_i \times_{UZ} UZ_j), UX)
\end{tikzcd}$$

is an equalizer and all vertical arrows are injective. A family $(\sigma_i)_i$ in the product $\prod \mathrm{Hom}_G(Z_i, X)$ which is equalized by the double arrow is mapped to a family in $\prod \mathrm{Hom}_k(UZ_i, UX)$ which is also equalized and therefore comes from a morphism g in $\mathrm{Hom}_k(UZ, UX)$. To see that g is equivariant we have to show that the square labeled with '?' commutes in the following diagram.

$$\begin{tikzcd}
G \times \coprod Z_i \arrow[r, "\mathrm{id}_G \times \coprod f_i"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & G \times Z \arrow[r, "\mathrm{id}_G \times g"] \arrow[d, "\alpha_Z"] \arrow[dr, phantom, "?"] & G \times X \arrow[d, "\alpha_X"] \\
\coprod Z_i \arrow[r, "\coprod \iota_i"] & Z \arrow[r, "g"'] & X
\end{tikzcd}$$

First note that all ι_i and $g \circ \iota_i$ are equivariant. The square in question commutes since both the outer rectangle and the left square commute, and since $\mathrm{id}_G \times \coprod f_i$ is an epimorphism. $\square$

Corollary 3.2.9. *The equivariant Nisnevich topology is also subcanonical.*

Lemma 3.2.10. *For all $H \leq G$, the H fixed points functor $(-)^H : G\mathcal{S}\mathrm{m}/k \to \mathcal{S}\mathrm{m}/k$ is continuous map of sites.*

Proof. [AGV72, III.Proposition 1.6.] $\square$

Lemma 3.2.11. *The adjunction* (2.10) *extends via left Kan extension of $(-)^G$ to an adjunction*

$$((-)^G)_* : \mathrm{sShv}(G\mathcal{S}\mathrm{m}/k) \rightleftarrows \mathrm{sShv}(\mathcal{S}\mathrm{m}/k) : R^G, \tag{3.1}$$

where the right adjoint is composition with $(-)^G$.

Proof. Consider the situation

$$\begin{tikzcd}
G\mathcal{S}\mathrm{m}/k \arrow[rr, "Y"] \arrow[dr, "(-)^G"'] & & \mathrm{sPre}(G\mathcal{S}\mathrm{m}/k) \arrow[d, dashed, shift left, "R"] \\
& \mathcal{S}\mathrm{m}/k \arrow[r, "Y"'] & \mathrm{sPre}(\mathcal{S}\mathrm{m}/k) \arrow[u, dashed, shift left, "L"]
\end{tikzcd}$$

where L is the left Kan extension of $Y \circ (-)^G$ along the horizontal Yoneda embedding Y and R is the right adjoint of L. The right adjoint R is given by composition with $(-)^G$, which is a continuous map of sites and so R restricts to a functor R'

$$\begin{tikzcd} \mathrm{sPre}(G\mathcal{S}\mathrm{m}/k) \arrow[r, shift left, "a_1"] \arrow[d, shift right, "L"'] & \mathrm{sShv}(G\mathcal{S}\mathrm{m}/k) \arrow[l, shift left, "i_1"] \\ \mathrm{sPre}(\mathcal{S}\mathrm{m}/k) \arrow[r, shift left, "a_2"] \arrow[u, shift right, "R"'] & \mathrm{sShv}(\mathcal{S}\mathrm{m}/k) \arrow[l, shift left, "i_2"] \arrow[u, "R'"'] \end{tikzcd}$$

of sheaves with respect to the Nisnevich (resp. H-Nisnevich) topology. We have

$$\begin{aligned} \mathrm{Hom}_{\mathrm{sShv}(G)}(X, R'Y) &\cong \mathrm{Hom}_{\mathrm{sPre}(G)}(i_1X, i_1R'Y) \\ &\cong \mathrm{Hom}_{\mathrm{sPre}(G)}(i_1X, Ri_2Y) \\ &\cong \mathrm{Hom}_{\mathrm{sPre}}(Li_1X, i_2Y) \\ &\cong \mathrm{Hom}_{\mathrm{sShv}}(a_2Li_1X, Y) \end{aligned}$$

and so $((-)^G)_* := a_2Li_1$ is right adjoint to R^G. □

From now on we will mostly leave sheaves aside and focus on a theory of presheaves. The few statements about sheaves we collected so far were just given to allow a study of points for this H-Nisnevich topology on $G\mathcal{S}\mathrm{m}/k$ in the next subsection.

For any subgroup $H \leq G$, we define the H-fixed points functor

$$(-)^H : \mathrm{sPre}(G\mathcal{S}\mathrm{m}/k) \to \mathrm{sPre}(\mathcal{S}\mathrm{m}/k) \tag{3.2}$$

as the composite

$$\begin{tikzcd} \mathrm{sPre}(G\mathcal{S}\mathrm{m}/k) \arrow[r, "\mathrm{res}_H"] & \mathrm{sPre}(H\mathcal{S}\mathrm{m}/k) \arrow[r, "(-)^H"] & \mathrm{sPre}(\mathcal{S}\mathrm{m}/k) \\ X \arrow[r, mapsto] & X(G \times_H -), & \end{tikzcd}$$

where res_H is the restriction functor or forgetful functor. Note that precomposing with the induction functor $G \times_H -$ coincides with the left Kan extension of $\mathrm{res}_H : G\mathcal{S}\mathrm{m}/k \to H\mathcal{S}\mathrm{m}/k$. Hence we could have equivalently defined H-fixed points as a left Kan extension in one step.

Remark 3.2.12. The functor $(-)^G : \mathrm{sPre}(G\mathcal{S}\mathrm{m}/k) \to \mathrm{sPre}(\mathcal{S}\mathrm{m}/k)$ is also right adjoint which can be seen as follows. On the scheme level we have the adjunction $(-)_G : \mathcal{S}\mathrm{m}/k \rightleftarrows G\mathcal{S}\mathrm{m}/k : (-)^G$ with the left adjoint given by the trivial G-action functor $(-)_G$. The right adjoint R to left Kan extension of $(-)_G$ along the obvious Yoneda embedding is given by precomposition with $(-)_G$ and hence commutes with colimits. Further, for a representable sheaf $\widetilde{X}$ we evaluate

$$R(\widetilde{X})(U) \cong \mathrm{Hom}_{G\mathcal{S}\mathrm{m}/k}(U_G, X) \cong \mathrm{Hom}_{\mathcal{S}\mathrm{m}/k}(U, X^G) = \widetilde{X^G}(U)$$

and note that R and $(-)^G$ coincide on representables and therefore are equal. The same arguments work to show that $\mathrm{res}_H : \mathrm{sPre}(G\mathcal{S}\mathrm{m}/k) \to \mathrm{sPre}(H\mathcal{S}\mathrm{m}/k)$ is also right adjoint and we eventually note that the H-fixed points functor $(-)^H$: $\mathrm{sPre}(G\mathcal{S}\mathrm{m}/k) \to \mathrm{sPre}(\mathcal{S}\mathrm{m}/k)$ from (3.2) is a left and right adjoint functor, for all $H \leq G$.

3.3. **Characterization of Weak Equivalences.** Recall that a point x in a topos T is a geometric morphism $x : \mathcal{S}\mathrm{et} \to T$ or equivalently, by Freyd's Theorem [Mac71, Theorem V.6.2], a functor $x^* : T \to \mathcal{S}\mathrm{et}$ which commutes with colimits and finite limits. In this subsection $G\mathcal{S}\mathrm{m}/k$ is equipped with the H-Nisnevich topology by default.

Denote by $\mathcal{H}\mathrm{ensel} := \{x^* : F \mapsto F(\mathrm{Spec}(\mathcal{O}^h_{X,x}) \mid x \in X\}_X$ the set of functors indexed over all X in a small skeleton of $\mathcal{S}\mathrm{m}/k$. This gives a conservative family of points for the Nisnevich topology on $\mathcal{S}\mathrm{m}/k$ (cf. Example 2.2.9).

Lemma 3.3.1. *As the Nisnevich site on $\mathcal{S}\mathrm{m}/k$ has enough points a morphism f in $\mathrm{sPre}(\mathcal{S}\mathrm{m}/k)$ is local weak equivalence if and only if it is a stalkwise weak equivalence of simplicial sets.*

Proof. This is a combination of Example 2.2.9 and our discussion of local weak equivalences in Definition 2.3.1. □

Lemma 3.3.2. *Let x^* be a point in $\mathrm{sShv}(\mathcal{S}\mathrm{m}/k)$. Then the composition $x^* \circ (-)^H$ is a point in $\mathrm{sShv}(G\mathcal{S}\mathrm{m}/k)$. Hence, if $f \in \mathrm{sPre}(G\mathcal{S}\mathrm{m}/k)$ is a local weak equivalence, then f^H is a local weak equivalence in $\mathrm{sPre}(\mathcal{S}\mathrm{m}/k)$.*

Proof. By Remark 3.2.12 the left Kan extension

$$(-)^H : \mathrm{sPre}(G\mathcal{S}\mathrm{m}/k) \to \mathrm{sPre}(\mathcal{S}\mathrm{m}/k)$$

is also a right adjoint and therefore preserves limits. As a left adjoint it preserves colimits and hence $x^* \circ (-)^H$ is a point in $\mathrm{sShv}(G\mathcal{S}\mathrm{m}/k)$. Thus, for any local weak equivalence $f \in \mathrm{sPre}(G\mathcal{S}\mathrm{m}/k)$ the morphism $x^* f^H$ is weak equivalence of simplicial sets, so f^H is local weak equivalence in $\mathrm{sPre}(\mathcal{S}\mathrm{m}/k)$. □

Lemma 3.3.3. *The set of functors $\mathrm{sShv}(G\mathcal{S}\mathrm{m}/k) \to \mathcal{S}\mathrm{et}$ given by*

$$\left\{x^* \circ (-)^H \mid H \leq G, x^* \in \mathcal{H}\mathrm{ensel}\right\}$$

is a conservative family of points in $\mathrm{sShv}(G\mathcal{S}\mathrm{m}/k)$ (for the H-Nisnevich topology).

Proof. Let $\mathfrak{X} := (f_j^H : X_j \to X)_{j \in J}$ be a family of morphisms in $G\mathcal{S}\mathrm{m}/k$ such that

$$\left(x^*(X_j^H \xrightarrow{f_j^H} X^H)\right)_{j \in J}$$

is surjective for all Nisnevich points $x^* \in \mathcal{H}\mathrm{ensel}$ and $H \leq G$. Then by Proposition 2.2.6, $(f_j^H : X_j^H \to X^H)_{j \in J}$ is a Nisnevich covering in $\mathcal{S}\mathrm{m}/k$. Hence, $\mathfrak{X}$ is a H-Nisnevich covering. □

The following is also an immediate consequence.

Corollary 3.3.4. *A morphism $f \in \mathrm{sPre}(G\mathcal{S}\mathrm{m}/k)$ is a local weak equivalence if and only if for all subgroups $H \leq G$ the morphism f^H is a local weak equivalence in* $\mathrm{sPre}(\mathcal{S}\mathrm{m}/k)$.

Corollary 3.3.5. *For all subgroups $H \leq G$, the adjunction*

$$(-)^H : \mathrm{sPre}(G\mathcal{S}\mathrm{m}/k) \rightleftarrows \mathrm{sPre}(\mathcal{S}\mathrm{m}/k) : R_H$$

is a Quillen adjunction for the local injective model structures.

Proof. We have just concluded that $(-)^H$ preserves local weak equivalences. Because of being right adjoint (and the fact that both categories have pullbacks) the functor $(-)^H$ also preserves monomorphisms, i.e. local injective cofibrations. □

To achieve the same result for $\mathbb{A}^1$-local weak equivalences we cite a result of Hirschhorn which takes care of the Bousfield localization on both sides of a Quillen adjunction.

Proposition 3.3.6. *Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be a Quillen pair and let K be a class of morphisms in $\mathcal{C}$. Denote by $L_K\mathcal{C}$, resp. $L_{\mathbb{L}_F K}\mathcal{D}$, the left Bousfield localization of $\mathcal{C}$ with respect to K, resp. of $\mathcal{D}$ with respect to the image of K under the left derived of F. Then $F : L_K\mathcal{C} \rightleftarrows L_{\mathbb{L}_F K}\mathcal{D} : G$ remains a Quillen pair.*

Proof. [Hir03, Theorem 3.3.20] □

Lemma 3.3.7. *Let $H, K \leq G$. The composition $(-)^K \circ (-)_H : \mathrm{sPre}(\mathcal{S}\mathrm{m}/k) \to \mathrm{sPre}(\mathcal{S}\mathrm{m}/k)$ equals some coproduct of identities. In particular, the H-fixed points functors $(-)^H$ are right Quillen functors in a Quillen adjunction*

$$(-)_H : \mathrm{sPre}(\mathcal{S}\mathrm{m}/k) \leftrightarrows \mathrm{sPre}(G\mathcal{S}\mathrm{m}/k) : (-)^H$$

with respect to the local injective model structures.

Proof. Both functors commute with colimits, so we only need to check the statement for representables. We have

$$((\widetilde{Y})_H)^K \cong \left(\widetilde{G/H \times Y}\right)^K \cong (\widetilde{G/H)^K} \times Y \cong \coprod_{(G/H)^K} \widetilde{Y}.$$

Furthermore, the functors $(-)^K$ detect local weak equivalences by Corollary 3.3.4 and a (finite) coproduct of local weak equivalences is a local weak equivalence. Eventually, to check that $(-)_H$ preserves monomorphisms recall that $(-)_H$ is the left Yoneda extension of $G/H \times - : \mathcal{S}\mathrm{m}/k \to G\mathcal{S}\mathrm{m}/k$ which preserves all finite limits. Left Kan extensions of flat functors preserve finite limits and in particular monomorphisms. □

Lemma 3.3.8. *For every subgroup $H \leq G$, the H-fixed points functor $(-)^H$ is a right Quillen functor in the adjunction*

$$(-)_H : \mathrm{sPre}(\mathcal{S}\mathrm{m}/k) \leftrightarrows \mathrm{sPre}(G\mathcal{S}\mathrm{m}/k) : (-)^H$$

with respect to the $\mathbb{A}^1$-local injective model structures.

Proof. By Proposition 3.3.6 the Quillen adjunction

$$(-)_H : \mathrm{sPre}(\mathcal{S}\mathrm{m}/k) \leftrightarrows \mathrm{sPre}(G\mathcal{S}\mathrm{m}/k) : (-)^H$$

of Lemma 3.3.7 descents to a Quillen adjunction

$$L_K \,\mathrm{sPre}(\mathcal{S}\mathrm{m}/k) \underset{(-)^H}{\overset{(-)_H}{\rightleftarrows}} L_{\mathbb{L}_{(-)_H} K} \,\mathrm{sPre}(G\mathcal{S}\mathrm{m}/k)$$

of left Bousfield localizations, where K is the class of morphisms represented by $\{X \times \mathbb{A}^1 \to X \mid X \in \mathcal{S}\mathrm{m}/k\}$ and $\mathbb{L}_{(-)_H} K$ is the image of that class under the total left derived of $(-)_H$. The latter is a (proper) subclass of the class of morphisms represented by $\{X \times \mathbb{A}^1 \to X \mid X \in G\mathcal{S}\mathrm{m}/k\}$ which is used to $\mathbb{A}^1$-localize on the equivariant side. Hence, the identity gives a left Quillen functor

$$L_{\mathbb{L}_{(-)_H} K} \,\mathrm{sPre}(G\mathcal{S}\mathrm{m}/k) \to \mathrm{sPre}(G\mathcal{S}\mathrm{m}/k)$$

where the right hand side carries the $\mathbb{A}^1$-local injective model structure. Composing the two Quillen adjunctions we obtain the conclusion. $\square$

Proposition 3.3.9. *A morphism $f \in \mathrm{sPre}(G\mathcal{S}\mathrm{m}/k)$ is an $\mathbb{A}^1$-local weak equivalence if and only if for all subgroups $H \leq G$ the morphism f^H is an $\mathbb{A}^1$-local weak equivalence in $\mathrm{sPre}(\mathcal{S}\mathrm{m}/k)$.*

Proof. By Proposition 3.3.6 the functors $(-)^H$ are left Quillen functors of the $\mathbb{A}^1$-local injective model structures. Thus, it follows by Ken Brown's Lemma [Hov99, Lemma 1.1.12] that $(-)^H$ *preserves* $\mathbb{A}^1$-local weak equivalences.

Conversely, suppose that $f : X \to Y$ in $\mathrm{sPre}(G\mathcal{S}\mathrm{m}/k)$ is a map such that for all subgroups H of G, the morphism $f^H \in \mathrm{sPre}(\mathcal{S}\mathrm{m}/k)$ is an $\mathbb{A}^1$-local weak equivalence. Let r be a fibrant replacement functor in the $\mathbb{A}^1$-local injective structure on $\mathrm{sPre}(G\mathcal{S}\mathrm{m}/k)$. Then $(-)^H$ takes the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\sim_{\mathbb{A}^1}"'] & Y \arrow[d, "\sim_{\mathbb{A}^1}"] \\ rX \arrow[r, "rf"] & rY \end{tikzcd}$$

to the diagram

$$\begin{tikzcd} X^H \arrow[r, "f^H", "\sim_{\mathbb{A}^1}"'] \arrow[d, "\sim_{\mathbb{A}^1}"'] & Y^H \arrow[d, "\sim_{\mathbb{A}^1}"] \\ (rX)^H \arrow[r, "(rf)^H"] & (rY)^H \end{tikzcd}$$

where all the arrows decorated with $\sim_{\mathbb{A}^1}$ are $\mathbb{A}^1$-local weak equivalences. Hence $(rf)^H$ is an $\mathbb{A}^1$-local weak equivalence between objects which are $\mathbb{A}^1$-locally injective fibrant by Lemma 3.3.8. Therefore, $(rf)^H$ is a local weak equivalence for all H and it follows by Corollary 3.3.4 that rf is local weak equivalence and so f is an $\mathbb{A}^1$-local weak equivalence. □

3.4. **Quillen Functors.** While characterizing $\mathbb{A}^1$-local weak equivalence we have observed in Lemma 3.3.3 that all the H-fixed points functors are Quillen adjunctions with respect to the local injective model structures. In this subsection we check and establish a few more Quillen adjunctions between spaces, G-spaces, and G-spectra. The treatment of spectra happens in anticipation of Section 4 where we introduce stable homotopy theory more thoroughly. The few statements about Quillen adjunctions for spectra at the end of this subsection just collect some immediate consequences concerning a levelwise model structure.

We have already made use of the following lemma in the proof of Proposition 3.3.9, where we have noticed this fact as a consequence of Corollary 3.3.5 and Proposition 3.3.6.

Lemma 3.4.1. *Let H be a subgroup of G. The H-fixed points adjunction*

$$(-)^H : \mathrm{sPre}(G\mathcal{S}\mathrm{m}/k) \rightleftarrows \mathrm{sPre}(\mathcal{S}\mathrm{m}/k) : R_H$$

is a Quillen adjunction with respect to the $\mathbb{A}^1$-local injective model structure on both sides.

It is also worth mentioning that the corresponding statements hold for the universal model categories with local weak equivalences and its $\mathbb{A}^1$-localizations.

Lemma 3.4.2. *Let H be a subgroup of G. The H-fixed points adjunction*

$$(-)^H : \mathrm{sPre}(G\mathcal{S}\mathrm{m}/k) \rightleftarrows \mathrm{sPre}(\mathcal{S}\mathrm{m}/k) : R_H$$

is a Quillen adjunction with respect to the local projective model structure on both sides. Furthermore, the adjunction is also a Quillen adjunction with respect to the $\mathbb{A}^1$-local projective model structure on both sides.

Proof. The right adjoint R_H is given by precomposition with $(-)^H : G\mathcal{S}\mathrm{m}/k \to \mathcal{S}\mathrm{m}/k$ so R_H clearly preserves projective fibrations and objectwise weak equivalences. Therefore its left adjoint $(-)^H : \mathrm{sPre}(G\mathcal{S}\mathrm{m}/k) \to \mathrm{sPre}(\mathcal{S}\mathrm{m}/k)$ preserves projective (acyclic) cofibrations. Since local projective cofibrations are just the projective cofibrations the statement follows from the fact that $(-)^H$ also preserves local weak equivalences by Corollary 3.3.4.

The $\mathbb{A}^1$-local statement is now an immediate consequence of the first part and Lemma 3.4.1. □

For the purpose of constructing a stable equivariant motivic homotopy theory we need to consider pointed simplicial presheaves $\mathrm{sPre}_\cdot(G\mathcal{S}\mathrm{m}/k)$ on $G\mathcal{S}\mathrm{m}/k$, i.e. presheaves with values in pointed simplicial sets. All model structures on unpointed simplicial presheaves carry over to model structures on pointed ones by the adjunction

$$(-)_+ : \mathrm{sPre}(G\mathcal{S}\mathrm{m}/k) \rightleftarrows \mathrm{sPre}_\cdot(G\mathcal{S}\mathrm{m}/k) : U,$$

where $(-)_+$ adds a disjoint basepoint and U is the forgetful functor. A morphism $f \in \mathrm{sPre}_\cdot$ of pointed simplicial presheaves is then defined to be a weak equivalence (resp. (co-)fibration) if $U(f)$ is a weak equivalence (resp. (co-)fibration) in the corresponding model structure on unpointed presheaves. This defines a model structure on $\mathrm{sPre}_\cdot$ [Hov99, Proposition 1.1.8] and unpointed Quillen adjunctions extend to pointed ones [Hov99, Proposition 1.3.5]. A little care is just necessary for extending Quillen equivalences (cf. [Hov99, Proposition 1.3.17]).

Recall that a (sequential) spectrum $X \in \mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathrm{sPre}_\cdot(G\mathcal{S}\mathrm{m}/k), \mathbb{T}_G \wedge -)$ consists of a sequence (X_n, σ_n) of spaces $X_n \in \mathrm{sPre}_\cdot(G\mathcal{S}\mathrm{m}/k)$ and bonding maps

$$\sigma_n : \mathbb{T}_G \wedge X_n \to X_{n+1}.$$

The suspension spectrum functor

$$\Sigma^\infty : \mathrm{sPre}_\cdot(G\mathcal{S}\mathrm{m}/k) \to \mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathrm{sPre}_\cdot(G\mathcal{S}\mathrm{m}/k), \mathbb{T}_G \wedge -)$$

is given by $\Sigma^\infty(X)_n = \mathbb{T}_G^n \wedge X$ and identity bonding maps.

Lemma 3.4.3. *The adjunction*

$$\Sigma^\infty : \mathrm{sPre}_\cdot(G\mathcal{S}\mathrm{m}/k) \rightleftarrows \mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathrm{sPre}_\cdot(G\mathcal{S}\mathrm{m}/k), \mathbb{T}_G \wedge -) : \Omega^\infty$$

is a Quillen adjunction with respect to the levelwise and the stable structure on spectra.

Proof. In the projective levelwise model structure on $\mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathrm{sPre}_\cdot(G\mathcal{S}\mathrm{m}/k), \mathbb{T}_G \wedge -)$, the fibrations and weak equivalences are defined levelwise. Thus, Ω^∞ is clearly a right Quillen functor.

The 'stable' statement is a corollary of the above paragraph since the stable structure is a left Bousfield localization of the levelwise one. □

As in ordinary stable equivariant homotopy theory there are two reasonable fixed-point functors. One, which satisfies the expected adjunction with the push forward of non-equivariant spectra and another one, the *geometric fixed-points functor* Φ^G, which applies levelwise the fixed-point functor of spaces.

Lemma 3.4.4. *The geometric fixed points functor*

$$\Phi^G : \mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathrm{sPre}_\cdot(G\mathcal{S}\mathrm{m}/k), \mathbb{T}_G \wedge -) \to \mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathrm{sPre}_\cdot(\mathcal{S}\mathrm{m}/k), T \wedge -)$$

is a right Quillen functor with respect to the levelwise model structure.

Proof. The prolongation of the fixed-point functor on spaces to spectra and Quillen property are discussed later in Section 4 after the discussion of the stable model category. It is just clear at this point that we obtain a detecting Quillen functor Φ^G for the levelwise model structure on $\mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathrm{sPre}_\cdot(G\mathcal{S}\mathrm{m}/k), \mathbb{T}_G \wedge -)$. $\square$

Let A be a based G-space and E be G-spectrum. Then one defines the G-spectrum $A \wedge E$ by

$$(A \wedge E)_n = A \wedge E_n \text{ and } \sigma_n^{A \wedge E} : \mathbb{T}_G \wedge A \wedge E_n \xrightarrow{t} A \wedge \mathbb{T}_G \wedge E_n \xrightarrow{1 \wedge \sigma_n^E} A \wedge E_{n+1}.$$

The endofunctor $A \wedge -$ on $\mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathrm{sPre}_\cdot(G\mathcal{S}\mathrm{m}/k))$ has a right adjoint $F(A, -)$ given by

$$F(A, X)_n = \underline{\mathrm{Hom}}(A, X_n)$$

and the bonding maps are defined by the commutativity of the diagram

$$\begin{tikzcd} T \wedge \underline{\mathrm{Hom}}(A, X_n) \arrow[rr] \arrow[dr] & & \underline{\mathrm{Hom}}(A, X_{n+1}) \\ & \underline{\mathrm{Hom}}(A, T \wedge X_n) \arrow[ur, "\sigma_{n*}"] & \end{tikzcd}$$

where the left diagonal morphism is adjoint to evaluation morphism

$$A \wedge T \wedge \underline{\mathrm{Hom}}(A, X_n) \cong T \wedge A \wedge \underline{\mathrm{Hom}}(A, X_n) \xrightarrow{\mathrm{id}_T \wedge \epsilon} T \wedge X_n.$$

Next, we consider the objects of $G\mathcal{S}\mathrm{m}/k$ as objects in the classifying topos $\mathcal{B}G$, i.e. the functor

$$i : G\mathcal{S}\mathrm{m}/k \to {}^G\mathrm{sPre}(\mathcal{S}\mathrm{m}/k), (X, \alpha) \mapsto (\mathrm{Hom}_{\mathcal{S}\mathrm{m}/k}(-, X), \alpha_*)$$

into the G-objects of $\mathrm{sPre}(\mathcal{S}\mathrm{m}/k)$. Let

$$ext : \mathrm{sPre}(G\mathcal{S}\mathrm{m}/k) \to {}^G\mathrm{sPre}(\mathcal{S}\mathrm{m}/k) \tag{3.3}$$

be the left Kan extension of i and define

$$-/G : \mathrm{sPre}(G\mathcal{S}\mathrm{m}/k) \to \mathrm{sPre}(\mathcal{S}\mathrm{m}/k) \text{ by } X \mapsto \mathrm{colim}(ext(X))$$

where we understand $ext(X)$ as diagram $G \to \mathrm{sPre}(\mathcal{S}\mathrm{m}/k)$.

Remark 3.4.5. It is well known that category of G-objects in a topos, where G is a group object in the topos, is itself a topos again [AGV72, IV.2.5], namely the classifying topos of G. In many places some kind of equivariant homotopy theory is developed for this topos [Gui, Ste10] and Carlsson and Joshua use this approach in their recent preprints [CJ11a, CJ11b] to set up some equivariant motivic homotopy theory. It is not impossible that both topoi $\mathrm{sPre}(G\mathcal{S}\mathrm{m}/k)$ and ${}^G\mathrm{sPre}(\mathcal{S}\mathrm{m}/k)$ provide an equally convenient basis for homotopy theory, but a comparison is a bit non-trivial. The category $G\mathcal{S}\mathrm{m}/k$ of original interest embeds into both and the unit of the adjunction

$$ext : \mathrm{sPre}(G\mathcal{S}\mathrm{m}/k) \rightleftarrows {}^G\mathrm{sPre}(\mathcal{S}\mathrm{m}/k) : int$$

is an isomorphism on representables. However, ext is not fully faithful on the whole category as this would imply an equality of functors tr_g and $int \circ tr_e$ which is in contradiction to Lemma 5.1.11 and the discussion directly above of it.

Lemma 3.4.6. *The adjunction*

$$ext(EG \times -)/G : \mathrm{sPre}(G\mathcal{S}\mathrm{m}/k) \rightleftarrows \mathrm{sPre}(\mathcal{S}\mathrm{m}/k) : F(EG, int \circ tr(-))$$

is a Quillen adjunction with respect to the $\mathbb{A}^1$*-local injective model structures.*

Proof. We will show that $(EG \times -)/G$ preserves monomorphisms and takes $\mathbb{A}^1$-local equivariant weak equivalences, even in the H-Nisnevich sense, to ordinary $\mathbb{A}^1$-local weak equivalences.

Firstly, we handle the monomorphisms. The externalization functor ext is the left Kan extension of a functor denoted by i above which is similar to a Yoneda embedding and preserves limits by the discussion about limits directly before Lemma 2.1.11. Thus, i is a flat functor and its left Kan extension ext preserves finite limits and in particular monomorphisms. That colim, and thus $-/G = \mathrm{colim} \circ ext$, also preserves monomorphisms in this situation is not too difficult to check by hand. Alternatively, one may apply Isbell and Mitchell's [IM76, Lemma 2.2].

Secondly, $EG \times -$ takes a weak equivalence to a weak equivalence and

$$\mathrm{colim} : {}^G\mathrm{sPre}(\mathcal{S}\mathrm{m}/k) \rightleftarrows \mathrm{sPre}(\mathcal{S}\mathrm{m}/k) : \Delta$$

is a Quillen functor pair for the projective model structure on the left hand side, also sometimes called the coarse model structure. It is therefore sufficient to show that $ext(EG \times -)$ takes weak equivalences to projective $\mathbb{A}^1$-local weak equivalences between projectively cofibrant objects. Objects in the image of $ext(EG \times -)$ always have a free G-action and thus are cofibrant, so we just have to check that ext maps weak equivalences to weak equivalences. If $f \in \mathrm{sPre}(G\mathcal{S}\mathrm{m}/k)$ is a weak equivalence, then $ext(f)$ is an objectwise $\mathbb{A}^1$-local weak equivalence if and only if $ev_*(ext(f))$ is an $\mathbb{A}^1$-local weak equivalence. Now $ev_* \circ ext$ coincides with the trivial fixed points functor $(-)^e$, which is easily checked on representables, and we know from Proposition 3.3.9 that f^e is an $\mathbb{A}^1$-local weak equivalence. $\square$

3.5. **Representing Equivariant Algebraic K-Theory.** This subsection starts with a recollection of equivariant algebraic K-theory following Thomason [Tho87]. The main result of this subsection shows that equivariant algebraic K-theory does not satisfy descent with respect to topologies that, like the H-Nisnevich topology, contain certain morphisms as coverings. We also state a result of Krishna and Østvær [KØ10] which indicates that the equivariant Nisnevich topology from Subsection 3.1 allows K-theory to satisfy descent. Finally, we discuss the effect of our non-descent result on the K-theory descent property of the isovariant Nisnevich topology as it is investigated in [Ser10].

Definition 3.5.1. Let X be in $G\mathcal{S}\mathrm{m}/k$. A quasi-coherent G-module (F, φ) on X is given by a quasi-coherent $\mathcal{O}_X$-module F and an isomorphism

$$\varphi : \alpha_X^* F \xrightarrow{\cong} pr_2^* F$$

of $\mathcal{O}_{G\times X}$-modules, such that the cocycle condition

$$(pr_{23}^* \varphi) \circ ((\mathrm{id} \times \alpha_X)^* \varphi) = (m \times \mathrm{id})^* \varphi$$

is satisfied. F is called coherent (resp. locally free) if it is coherent (resp. locally free) as an $\mathcal{O}_X$-module.

Coherent G-modules on some X in $G\mathcal{S}\mathrm{m}/k$ form an abelian category $M(G, X)$ and locally free coherent G-modules (G-equivariant vector bundles) form an exact subcategory $P(G, X)$. To these exact categories we associate the simplicial nerve $BQM(G, X)$ (resp. $BQP(G, X)$) of Quillen's Q-construction [Qui73, §2] on the category. Finally, denote by $\mathcal{G}(G, X) = \Omega BQM(G, X)$ and $K(G, X) = \Omega BQP(G, X)$ the K-theory spectra (or infinite loop spaces) associated to the exact categories of coherent G-modules on X and to those that are locally free. In his fundamental work Thomason already shows that for a separated noetherian regular G-scheme X the inclusion of categories induces an equivalence $K(G, X) \xrightarrow{\sim} \mathcal{G}(G, X)$ [Tho87, Theorem 5.7] and that hence for such an X the equivariant K-theory satisfies homotopy invariance in the sense that the projection induces an equivalence

$$K(G, X) \to K(G, X \times \mathbb{A}^n)$$

even with respect to any linear G-action on $\mathbb{A}^n$ [Tho87, Corollary 4.2].

By the origin of the use of word *motivic* in this area of mathematics, or in other words by Grothendieck's idea of what it should mean to associate a motive to a scheme, it should be considered a fundamental test for any candidate of a motivic homotopy category, whether it allows representability for a sufficient amount of cohomological theories or not. One obstacle for a theory F to be representable in $\mathcal{H}(k, G)$ is that it has to satisfy descent with respect to the topology used to define the local model structure. This is a kind of homotopical sheaf condition which implies the compatibility of the theory F with local weak equivalences. More precisely:

Definition 3.5.2. An objectwise fibrant simplicial presheaf F on a site $\mathcal{C}$ *satisfies Čech descent* with respect to the topology on $\mathcal{C}$ if for any covering family $\{U_i \to X\}_i$ in $\mathcal{C}$ the morphism

$$F(X) \longrightarrow \operatorname{holim}(\textstyle\prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \times_X U_j) \substack{\rightarrow\\[-1ex]\rightarrow\\[-1ex]\rightarrow} \ldots)$$

is a weak equivalence of simplicial sets. An arbitrary simplicial presheaf is said to satisfy Čech descent if an objectwise fibrant replacement of it does.

It is a straight reformulation of this definition that a simplicial presheaf F satisfies Čech descent if and only if for any covering $\mathcal{U} = \{U_i \to X\}_i$ and an injective fibrant replacement F' of F the induced map

$$\mathrm{s}\mathcal{S}\mathrm{et}(X, F') \to \mathrm{s}\mathcal{S}\mathrm{et}(\check{C}(\mathcal{U}), F')$$

is a weak equivalence of simplicial sets. By Example 2.3.3 it is suggested that this could unmask the local injective model structure of Theorem 2.3.5 as a Bousfield localization of the (objectwise) injective model structure at Čech nerves. In fact a slight generalization of this is true, replacing Čech nerves by hypercovers [DHI04, Theorem 1.1].

In [KØ10, Theorem 5.4] Krishna and Østvær show that presheaf of K-theory of perfect complexes on Deligne-Mumford stacks satisfies descent with respect to a version of the Nisnevich topology. Restricting the results on Deligne-Mumford stacks to the subcategory of G-schemes, the topology restricts to the equivariant Nisnevich topology and the main results of a sequel work by Krishna [Kri12, Theorem 1.2& 1.3] indicate that this could imply descent of equivariant K-theory for the equivariant Nisnevich topology (cf. [KØ10, Remark 7.10]), at least up to some invertation, e.g. rationally.

However, the rest of this section is devoted to showing that equivariant K-theory does not satisfy descent with respect to certain topologies, including the H-Nisnevich topology.

Lemma 3.5.3. *Let $L : k$ be a finite Galois extension with Galois group G. Let $X = \mathrm{Spec}(L)$ with the Galois action and let $Y = G \times \mathrm{Spec}(L)$, with the action* *entirely on G. Then all vertical maps in the diagram of pullback squares*

$$
\begin{tikzcd}
& Y \times_X Y \arrow[rr] \arrow[dl] \arrow[dd] & & Y \arrow[dl] \arrow[dd] \\
Y \arrow[rr, "f"] \arrow[dd] & & X \arrow[dd] & \\
& \mathrm{Spec}(L \otimes_k L) \arrow[rr] \arrow[dl] & & \mathrm{Spec}(L) \arrow[dl] \\
\mathrm{Spec}(L) \arrow[rr] & & \mathrm{Spec}(k) &
\end{tikzcd}
\tag{3.4}
$$

are Galois coverings, i.e. G-torsors.

Proof. The G-actions on the top square are free, because a finite Galois extension gives an isomorphism $\oplus_G L \cong L \otimes_k L$. Further, the vertical maps are geometric quotients and the conclusion is then by [MFK94, Proposition 0.9]. $\square$

Proposition 3.5.4. *Let $L : k$ be a finite Galois extension with Galois group G such that the Brauer group $\mathrm{Br}(k)$ has torsion away from the characteristic of k. If τ is a topology on $G\mathcal{S}\mathrm{m}/k$ such that the map*

$$f : G \times \mathrm{Spec}(L)_{\circlearrowright \mathrm{tr}} \to \mathrm{Spec}(L)_{\circlearrowright \mathrm{gal}} \qquad \textit{from } (3.4)$$

is a covering in τ, then equivariant algebraic K-theory does not satisfy descent with respect to τ.

Proof. Suppose equivariant algebraic K-theory satisfies descent with respect (to τ and hence) to f, i.e.

$$K(G, \mathrm{Spec}(L)_{\circlearrowright \mathrm{gal}}) \to \mathrm{holim}_{n \geq 0} K(G, \check{C}(f)_n) \tag{3.5}$$

is a weak equivalence. By a result on the equivariant K-theory of G-torsors [Mer05, Proposition 3] we have $K(G, \mathrm{Spec}(L)_{\circlearrowright \mathrm{Gal}}) \simeq K(\mathrm{Spec}(k))$. Combining this Proposition 3 with Lemma 3.5.3 we also have a weak equivalence

$$K(\mathrm{Spec}(L \otimes_k L)) \xrightarrow{\simeq} K(G, G \times \mathrm{Spec}(L) \times_{\mathrm{Spec}(k)} G \times \mathrm{Spec}(L)).$$

This gives a levelwise weak equivalence

$$
\begin{tikzcd}
K(G, G \times \mathrm{Spec}(L)) \arrow[r, shift left] \arrow[r, shift right] & K(G, Y \times_X Y) \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \cdots \\
K(L) \arrow[u, "\simeq"] \arrow[r, shift left] \arrow[r, shift right] & K(L \otimes_k L) \arrow[u, "\simeq"] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \cdots \arrow[u, "\simeq"]
\end{tikzcd}
\tag{3.6}
$$

of cosimplicial spectra. Now, by assumption on the field extension $L : k$, we have

$$L \otimes_k L \cong \bigoplus_G L = \mathrm{Map}(G, L)$$

and since there is a weak equivalence

$$K(\bigoplus_G L) \xrightarrow{\simeq} \prod_G K(L) = Map(G, K(L))$$

there is also a levelwise weak equivalence

$$\begin{tikzcd} K(L) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\mathrm{id}"'] & K(L \otimes_k L) \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d, "\simeq"'] & K(L \otimes_k L \otimes_k L) \arrow[d, "\simeq"'] & \cdots \\ K(L) \arrow[r, shift left] \arrow[r, shift right] & \mathrm{Map}(G, K(L)) \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \mathrm{Map}(G \times G, K(L)) & \cdots \end{tikzcd} \tag{3.7}$$

of cosimplicial spectra. Combining (3.6) and (3.7) we obtain a weak equivalence

$$\begin{aligned} \mathrm{holim}_{n\geq 0} K(G, \check{C}(f)_n) &\simeq \mathrm{holim}(K(L) \rightrightarrows \mathrm{Map}(G, K(L)) \ldots) \\ &\simeq Map(\underset{n}{\mathrm{hocolim}}\, \check{C}(G \to *)_n, K(L)) \\ &\simeq Map(EG, K(L)) \\ &= K(L)^{hG}. \end{aligned}$$

By comparing the homotopy fixed-point spectral sequences with the descent spectral sequence

$$\begin{array}{cl} H^s(\mathrm{Gal}(L : k), \pi K(L)) & \Rightarrow \pi_{t-s}\left(K(L)^{hG}\right) \\ \cong & \\ H^s_{et}(\mathrm{Spec}(k), \widetilde{\pi} K) & \Rightarrow K^{et}_{t-s}(k) \end{array},$$

we know that the homotopy fixed-points $K(L)^{hG}$ are equivalent to the étale K-theory spectrum $K^{et}(k)$ of k, so it follows from (3.5) that the morphism

$$K(k) \xrightarrow{\simeq} K^{et}(k) \tag{3.8}$$

is a weak equivalence. We can compute completions of the right hand side by the descent spectral sequence [Mit97, Corollary 1.5], which gives

$$K^{et}_0(k)^\wedge_l \cong K_0(k)^\wedge_l \oplus H^2_{et}(k, \mathbb{Z}_l(1)).$$

By [Mil11, Theorem 3.14] there is an isomorphism from the Brauer group $\mathrm{Br}(k)$ to the étale cohomology group $H^2_{et}(k, \mathbb{G}_m)$. In particular, the l-torsion of $\mathrm{Br}(k)$ survives to $H^2_{et}(k, \mathbb{Z}_l(1))$, so in this case the latter summand is non-zero. This is a contradiction to the equivalence of (3.8) and hence to our initial assumption that equivariant algebraic K-theory satisfies descent with respect to τ. $\square$

Remark 3.5.5.

(a) The techniques and arguments used in the proof above are fairly standard for work in relation to the Quillen-Lichtenbaum conjectures [Lic73, Qui73] which assert that there is an Atiyah-Hirzebruch style spectral sequence

$$E_2^{p,q} = H^p_{et}(X, \mathbb{Z}_l(\frac{q}{2})) \Rightarrow K_{p-q}(X) \otimes \mathbb{Z}_l$$

from étale cohomology to K-theory. If K-theory satisfied descent with respect to the étale topology, then this spectral sequence would be an implication. Anyhow, in general K-theory does not satisfy étale descent without any restriction on the degree of the K-groups, but still there are many positive results regarding versions of étale descent for K-theory and the Quillen-Lichtenbaum conjectures [Tho85, RW99, Øst03, Voe08].

(b) The proposition above provides a counterexample to the main theorem of [Ser10] that equivariant K-theory satisfies 'isovariant' descent. In loc. cit. a parametrized version of scheme-theoretic isotropy is introduced as G_X, where X is a G-scheme, and defined as the pullback

$$\begin{tikzcd} G_X \arrow[r] \arrow[d] & G \times X \arrow[d, "\alpha \times \mathrm{pr}_X"] \\ X \arrow[r, "\Delta"] & X \times X. \end{tikzcd}$$

Now Serpé calls a family $\{U_i \to X\}_i$ in $G\mathcal{S}\mathrm{m}/k$ an isovariant Nisnevich cover if the underlying family of schemes is a Nisnevich cover and for all $U_i \to X$ the induced morphism $G_{U_i} \to G_X$ furnishes a pullback square

$$\begin{tikzcd} G_{U_i} \arrow[r] \arrow[d] & G_X \arrow[d] \\ U_i \arrow[r] & X. \end{tikzcd} \tag{3.9}$$

The singleton $\{f : G \times \mathrm{Spec}(L)_{\circlearrowleft \mathrm{tr}} \to \mathrm{Spec}(L)_{\circlearrowleft \mathrm{gal}}\}$ defines an isovariant Nisnevich cover. This is because firstly the G-actions on domain and codomain are free. On the domain this follows from Lemma 2.1.11 and on the codomain freeness is implied by the assumption $L \otimes_k L \cong \oplus_G L$. Therefore, the corresponding commutative square of type (3.9) is a pullback square. Secondly, f is a Nisnevich covering, since the components of $G \times \mathrm{Spec}(L)$ map to $\mathrm{Spec}(L)$ along the elements of the Galois group.

Eventually, $\{f\}$ is not only a counterexample to the proof of [Ser10, Proposition 2.7], since f/G is the canonical map $\mathrm{Spec}(L) \to \mathrm{Spec}(k)$ which is not a Nisnevich cover, but it is also a counterexample to the descent assertion [Ser10, Theorem 4.2].

(c) It follows also that equivariant algebraic K-theory does not satisfy descent with respect to the H-Nisnevich topology, since for example
$$\{\mathbb{Z}/2 \times \mathrm{Spec}(\mathbb{C})_{\circlearrowright\mathrm{tr}} \to \mathrm{Spec}(\mathbb{C})_{\circlearrowright\mathrm{gal}}\}$$
is H-Nisnevich cover in $\mathbb{Z}/2$-$\mathcal{S}\mathrm{m}/\mathbb{R}$.

4. Stable Equivariant Motivic Homotopy Theory

In this section we construct an equivariant stable homotopy category based on the unstable homotopy theories of Section 3. Non-equivariantly these motivic stabilization processes have been elaborated using (symmetric) sequential spectra by Jardine [Jar00], using enriched functors by Dundas, Röndigs and Østvær [DRØ03b], and using S-modules by Hu [Hu03]. Based on the work of Jardine an equivariant stabilization has been worked out by Hu, Kriz and Ormsby [HKO11b]. We start with recollecting Jardine's framework and we treat the case of the H-Nisnevich topology in parallel. Later in this section we will introduce weighted stable homotopy groups and give characterizations of the equivariant stable weak equivalence as recognized by two naturally emerging families of stable fixed-point functors.

4.1. **The Stable Model Category.**

The definition representation spheres below already aims towards a stable equivariant homotopy theory. Analogously to the work of Mandell [Man04] in classical topology, Hu, Kriz, and Ormsby consider sequential (symmetric) spectra in loc. cit. with respect to smashing with the regular representation sphere.

Definition 4.1.1. Let $V \in G\mathcal{S}\mathrm{m}/k$ be a representation of G. We define the representation sphere S^V to be the quotient
$$V/(V-0)$$
in $\mathrm{sPre}(G\mathcal{S}\mathrm{m}/k)$. For the special case of the regular representation we introduce the notation
$$\mathbb{T}_G := S^{\mathbb{A}[G]}.$$

Remark 4.1.2. At this point it becomes visible why we have already discussed the requirement of linear reductivity in Remark 2.1.22. A splitting of the representation V causes a splitting of the representation sphere:
$$S^{V\oplus W} \cong S^V \wedge S^W.$$
Clearly, the reason to invert the regular representation sphere is to invert smashing with all representation spheres and therefore it should be emphasized again that the group G has to be linearly reductive for this approach to make sense.

However, there are models for stable homotopy theory based on enriched functors [Lyd98, Blu06, DRØ03a] instead of sequential spectra. This allows a more flexible stabilization and in a very recent preprint [CJ11a] Carlsson and Joshua apply this technique to stabilize a slightly different approach to equivariant motivic homotopy theory without being restricted to linearly reductive groups.

The category $\mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathcal{C}, Q)$ of sequential spectra in a model category $\mathcal{C}$ with respect to a left Quillen functor $Q : \mathcal{C} \to \mathcal{C}$ consists of objects

$$(X_n, \sigma_n)_{n \in \mathbb{N}},$$

where the X_n's are objects in $\mathcal{C}$ and $\sigma_n : Q(X_n) \to X_{n+1}$ are morphisms in $\mathcal{C}$, the so-called bonding maps. The morphisms in $\mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathcal{C}, Q)$ are given by sequences of morphisms in $\mathcal{C}$ which commute with the respective bonding maps.

There is the usual Yoga of model structures for stable homotopy theory in the sense of spectra in general model categories (cf. [Hov01]) that also applies to the equivariant and non-equivariant stable motivic homotopy theory as developed below. We depict our procedure in the following diagram, where in the top row the relevant categories of equivariant motivic spaces, sequential and symmetric spectra and their standard Quillen adjunctions show up. Below the top row, various model structures appear and are connected by arrows.

(4.1)

$$\begin{tikzcd}
\mathrm{sPre}_{\cdot}(G\mathcal{S}\mathrm{m}/k) \arrow[r, shift left, "\Sigma^\infty"] & \mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathrm{sPre}_{\cdot}(G\mathcal{S}\mathrm{m}/k), \mathbb{T}_G \wedge -) \arrow[l, shift left, "\Omega^\infty"] \arrow[r, shift left, "V"] & \mathcal{S}p^{\Sigma}(\ldots) \arrow[l, shift left, "U"] \\
\text{(1) local injective} \arrow[d] & \text{(3) levelwise} \arrow[d] & \text{levelwise} \arrow[d] \\
\text{(2) } \mathbb{A}^1\text{-local injective} \arrow[ur] \arrow[urr, bend left] & \text{(4) stable} & \text{stable}
\end{tikzcd}$$

Here, we choose to start with the local injective model structure (1) on pointed simplicial presheaves, in which the cofibrations are given by monomorphisms and weak equivalences are the local weak equivalences after forgetting the basepoint. The vertical arrows mean Bousfield localization, in this case at the class

$$\left\{ X \wedge \mathbb{A}^1_+ \to X \mid X \in \mathrm{sPre}_{\cdot}(G\mathcal{S}\mathrm{m}/k) \right\}$$

which gives the $\mathbb{A}^1$-local injective model structures (2). These model structures can be lifted to projective levelwise model structures on sequential $\mathbb{T}_G$-spectra [Jar00, Lemma 2.1] (3), which can now again be localized at the class of stable equivalences to result in the stable model structures (4).

Fortunately, compared with Hovey's general setup, we are in the good situation of [Hov01, Theorem 4.9] and thus we may proceed as Jardine in [Jar00] to define stable weak equivalences.

Lemma 4.1.3. *The adjunction*

$$\mathbb{T}_G \wedge - : \mathrm{sPre}_{\cdot}(G\mathcal{S}\mathrm{m}/k) \rightleftarrows \mathrm{sPre}_{\cdot}(G\mathcal{S}\mathrm{m}/k) : \Omega_{\mathbb{T}_G}$$

prolongates canonically to an adjunction

$$\Sigma'_{\mathbb{T}_G} : \mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathrm{sPre}_{\cdot}(G\mathcal{S}\mathrm{m}/k), \mathbb{T}_G \wedge -) \rightleftarrows \mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathrm{sPre}_{\cdot}(G\mathcal{S}\mathrm{m}/k), \mathbb{T}_G \wedge -) : \Omega'_{\mathbb{T}_G}$$

called fake suspension adjunction.

Proof. Use the identity transformation on $(\mathbb{T}_G \wedge -)^2$ to prolongate $\mathbb{T}_G \wedge -$ and compose unit and counit of the adjunction to obtain a natural transformation

$$\mathbb{T}_G \wedge (\Omega_{\mathbb{T}_G}(-)) \to \Omega_{\mathbb{T}_G}(\mathbb{T}_G \wedge -))$$

which prolongates $\Omega_{\mathbb{T}_G}$ to the right adjoint. □

Remark 4.1.4. The above lemma is originally [Hov01, Corollary 1.6] in the general situation. Note that there is no twisting of the smash factors involved in the bonding maps, which is why the resulting suspension is called fake suspension in contrast to the suspension defined in (4.2).

Definition 4.1.5. Let R denote a levelwise fibrant replacement functor. A morphism $f \in \mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathrm{sPre}_{\cdot}(G\mathcal{S}\mathrm{m}/k), \mathbb{T}_G \wedge -)$ is called a stable equivalence if

$$(\Omega' \circ \mathrm{sh})^{\infty} R(f)$$

is a levelwise equivalence.

For Jardine's machinery to work, we need to assure that the object $\mathbb{T}_G$ which is used for suspending fulfills a technical property, which then implies a good behavior of the right adjoint to smashing with $\mathbb{T}_G$.

Lemma 4.1.6. *The object* $\mathbb{T}_G \in \mathrm{sPre}_{\cdot}(G\mathcal{S}\mathrm{m}/k)$ *is compact in the sense of* [Jar00, 2.2].

Proof. The analog statement about the presheaf quotient $\mathbb{A}^1/(\mathbb{A}^1 \setminus 0)$ in Jardine's work is [Jar00, Lemma 2.2]. All the arguments in the proof are statements about the flasque model structure on simplicial presheaves on a general site [Isa05]. The only thing used about about schemes is that an inclusion of schemes gives a monomorphism of the represented presheaves, which is true for an inclusion of equivariant schemes like $(\mathbb{A}[G] \setminus 0) \hookrightarrow \mathbb{A}[G]$ as well. □

Theorem 4.1.7. *Let T be a compact object in* $\mathrm{sPre}_{\cdot}(G\mathcal{S}\mathrm{m}/k)$. *There is a proper simplicial model structure on the associated category* $\mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathrm{sPre}_{\cdot}(G\mathcal{S}\mathrm{m}/k), T \wedge -)$ *of T-spectra with stable weak equivalences and stable fibrations.*

Proof. This is [Jar00, Theorem 2.9] □

Definition 4.1.8. Let X in $\mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathrm{sPre}_{\cdot}(G\mathcal{S}\mathrm{m}/k), \mathbb{T}_G \wedge -)$. We define the suspension $\Sigma_{\mathbb{T}_G} X$ by $\Sigma_{\mathbb{T}_G} X_n = \mathbb{T}_G \wedge X_n$ with bonding maps

$$\sigma_{\Sigma X} : \mathbb{T}_G \wedge \mathbb{T}_G \wedge X_n \xrightarrow{\tau \wedge \mathrm{id}_{X_n}} \mathbb{T}_G \wedge \mathbb{T}_G \wedge X_n \xrightarrow{\sigma_X} \mathbb{T}_G \wedge X_{n+1}$$

where $\tau : \mathbb{T}_G \wedge \mathbb{T}_G \to \mathbb{T}_G \wedge \mathbb{T}_G$ denotes the twist of the two smash factors. The right adjoint to $\Sigma_{\mathbb{T}_G}$ is also levelwise given by the internal hom $\Omega_{\mathbb{T}_G}$, i.e. $\Omega_{\mathbb{T}_G}(X)_n = \Omega_{\mathbb{T}_G}(X_n)$ with bonding maps adjoint to

$$X_n \wedge \mathbb{T}_G \xrightarrow{\tau} \mathbb{T}_G \wedge X_n \xrightarrow{\sigma_X} X_{n+1}.$$

Together these two functors give the suspension adjunction

$$\Sigma_{\mathbb{T}_G} : \mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathrm{sPre}_{\cdot}(G\mathcal{S}\mathrm{m}/k), \mathbb{T}_G \wedge -) \rightleftarrows \mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathrm{sPre}_{\cdot}(G\mathcal{S}\mathrm{m}/k), \mathbb{T}_G \wedge -) : \Omega_{\mathbb{T}_G}. \tag{4.2}$$

To be able to untwist the levelwise smashing inside the definition of the functor $\mathbb{T}_G \wedge -$ an important condition appears to be the symmetry of $\mathbb{T}_G$.

Lemma 4.1.9. *There is an $\mathbb{A}^1$-homotopy in* $\mathrm{sPre}_{\cdot}(G\mathcal{S}\mathrm{m}/k)$ *between the cyclic permutation of the smash factors*

$$\mathbb{T}_G \wedge \mathbb{T}_G \wedge \mathbb{T}_G \to \mathbb{T}_G \wedge \mathbb{T}_G \wedge \mathbb{T}_G$$

and the identity.

Proof. This is [HKO11b, Lemma 2] for the $\mathbb{A}^1$-local model structure with respect to the equivariant Nisnevich topology, but the topology on $G\mathcal{S}\mathrm{m}/k$ does not matter for this statement to hold. □

A consequence, which is also true in the more general situation of Hovey's [Hov01, Theorem 9.3], is that smashing with $\mathbb{T}_G$ is invertible in the stable model.

Theorem 4.1.10. *The suspension adjunction* (4.2) *is a Quillen equivalence with respect to the stable model structure.*

Proof. Let Y be fibrant and $f : \mathbb{T}_G \wedge X \to Y$ in $\mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathrm{sPre}_{\cdot}(G\mathcal{S}\mathrm{m}/k), \mathbb{T}_G \wedge -)$. By [Jar00, Corollary 3.16]

$$ev : \mathbb{T}_G \wedge \Omega_{\mathbb{T}_G} Y \to Y$$

is a stable equivalence, so we may deduce from the commutative diagram

$$\begin{tikzcd} & \mathbb{T}_G \wedge \Omega_{\mathbb{T}_G} Y \arrow[d, "\sim"', "ev"] \\ \mathbb{T}_G \wedge X \arrow[ur, "\mathbb{T}f^\sharp"] \arrow[r, "f"] & Y \end{tikzcd}$$

that f is a stable equivalence if and only if $\mathbb{T}f$ is stable equivalence, which is by [Jar00, Corollary 3.18] if and only if the adjoint morphism $f^\sharp$ is a stable equivalence. □

Proposition 4.1.11. *Let V be a representation of G. Then the adjunction*

$$- \wedge S^V : \mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathrm{sPre}_{\cdot}(G\mathcal{S}\mathrm{m}/k), \mathbb{T}_G \wedge -) \rightleftarrows \mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathrm{sPre}_{\cdot}(G\mathcal{S}\mathrm{m}/k), \mathbb{T}_G \wedge -) : \Omega^V$$

is a Quillen equivalence.

Proof. Smashing with S^V is a left Quillen functor. There exists a representation W such that $V \oplus W \cong n\mathbb{A}^G$ is a n-fold sum of the regular representation. Now one can show using the theorem above that $\Omega^{n\mathbb{T}_G} \circ S^W$ is 'Quillen inverse' to S^V. $\square$

In Definition 4.1.5 a morphism $f : X \to Y$ of equivariant spectra was defined to be a stable equivalence if $\mathrm{colim}_i (sh\Omega'_{\mathbb{T}_G})^i R(f)$ is a levelwise equivalence of equivariant spectra. Equivalently, for all $m, n \in \mathbb{N}$ and all $H \leq G$ the induced maps of all sectionwise n-th homotopy groups in level m of the H-fixed points are isomorphisms, i.e.

$$f_* : \underset{i}{\mathrm{colim}}[G/H \wedge S^n \wedge \mathbb{T}^i_G, X_{m+i}|_U] \to \underset{i}{\mathrm{colim}}[G/H \wedge S^n \wedge \mathbb{T}^i_G, Y_{m+i}|_U] \tag{4.3}$$

is an isomorphism of groups for all $U \in \mathcal{S}\mathrm{m}/k$.

The standard simplicial enrichment of local homotopy theory on $\mathrm{sPre}(\mathcal{C})$ gives us another splitting of $\mathbb{T}_G$.

Lemma 4.1.12. *There is an isomorphism $\mathbb{T}_G \cong S^1 \wedge (\mathbb{A}[G] - 0)$ in the unstable equivariant homotopy category.*

Proof. Recall that $\mathbb{T}_G \cong \mathbb{A}[G]/(\mathbb{A}[G]-0)$ where $\mathbb{A}[G]$ is pointed by 1 and consider the diagram

$$\begin{tikzcd}
\partial\Delta[1] \wedge (\mathbb{A}[G] - 0) \arrow[r, hook] \arrow[d, hook] & \mathbb{A}[G] \arrow[r, "\sim"] \arrow[d] & * \arrow[d] \\
\Delta[1] \wedge (\mathbb{A}[G] - 0) \arrow[r] \arrow[d, "\sim"'] & P \arrow[r] \arrow[d] & S^1 \wedge (\mathbb{A}[G] - 0) \\
* \arrow[r] & \mathbb{T}_G &
\end{tikzcd}$$

consisting of push out squares. The two morphisms decorated with a tilde are $\mathbb{A}^1$-local weak equivalences. The vertical one being

$$\Delta[1] \wedge (\mathbb{A}[G] - 0) \xrightarrow{p \wedge \mathrm{id}} \Delta[0] \wedge (\mathbb{A}[G] - 0) = *$$

and the horizontal one by Lemma 3.1.6. Further, both morphisms to the push out P are cofibrations and hence by left properness there is a zig-zag

$$\mathbb{T}_G \xleftarrow{\sim} P \xrightarrow{\sim} S^1 \wedge (\mathbb{A}[G] - 0)$$

of weak equivalences. $\square$

Continuing from (4.3) we compute that f is stable equivalence if and only if the induced map

$$\underset{i}{\mathrm{colim}}[G/H \wedge S^{n+i} \wedge (\mathbb{A}[G]-0)^i, X_{m+i}|_U] \to \underset{i}{\mathrm{colim}}[G/H \wedge S^{n+i} \wedge (\mathbb{A}[G]-0)^i, Y_{m+i}|_U]$$

is an isomorphism. This leads naturally to the following definition.

Definition 4.1.13. Let X in $\mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathrm{sPre}_{\cdot}(G\mathcal{S}\mathrm{m}/k), \mathbb{T}_G \wedge -)$. The *weighted stable homotopy groups* $\pi^H_{s,t}X$ are defined to be the presheaf of groups on $\mathcal{S}\mathrm{m}/k$ given by

$$\pi^H_{s,t}(X)(U) = \operatorname*{colim}_{i\geq 0}[G/H \wedge S^{s+i} \wedge (\mathbb{A}[G]-0)^{t+i} \wedge U_+, X_i]$$

Lemma 4.1.14. *A morphism $f : X \to Y$ of equivariant spectra is a stable equivalence if and only if it induces isomorphisms*

$$\pi^H_{s,t}(f) : \pi^H_{s,t}(X) \xrightarrow{\cong} \pi^H_{s,t}(Y)$$

for all $s, t \in \mathbb{Z}$ and $H \leq G$.

Proof. This is the analog of [Jar00, Lemma 3.7] □

Cofiber and Fiber Sequences. Recall from Theorem 4.1.7 and Proposition 4.1.11 that we consider $\mathcal{S}\mathrm{p}^{\mathbb{N}}(G\mathcal{S}\mathrm{m}/k)$ as a proper stable model category. The theory of cofiber and fiber sequences is therefore quite convenient. Given a morphism $f : X \to Y$ of equivariant spectra the homotopy cofiber (resp. homotopy fiber) is defined by the homotopy push out (resp. homotopy pullback) square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ * \arrow[r] & hocofib(f) \end{tikzcd} \qquad \begin{tikzcd} hofib(f) \arrow[r] \arrow[d] & * \arrow[d] \\ X \arrow[r, "f"] & Y. \end{tikzcd}$$

The simplicial structure on $\mathcal{S}\mathrm{p}^{\mathbb{N}}(G\mathcal{S}\mathrm{m}/k)$ provided by Theorem 4.1.7 implies that there is a stable weak equivalence

$$hocofib(X \to *) \simeq S^1 \wedge X.$$

At this point we omit a thorough introduction of the triangulated structure on the stable homotopy category $\mathcal{SH}(k, G)$ via $S^1/(\mathbb{A}[G]-0)$-bispectra and (co-) fiber sequences which works out perfectly analogous to what is developed in Jardine's Section 3.3 of [Jar00]. Instead, we just state the following important consequence.

Lemma 4.1.15. *Given a cofiber sequence*

$$X \xrightarrow{f} Y \to hocofib(f)$$

of equivariant spectra, there is a long exact sequence of presheaves of groups

$$\ldots \to \pi^G_{s,t}(X) \to \pi^G_{s,t}(Y) \to \pi^G_{s,t}(hocofib(f)) \to \pi^G_{s-1,t}(X) \to \ldots \tag{4.4}$$

4.2. **Naive G-Spectra and Change of Universe.** For a smooth connection between stable equivariant and non-equivariant homotopy theories it is convenient to introduce naive G-spectra, a natural intermediate. We mirror some results from the topological theory, where our reference is [LMS86, Section II].

Definition 4.2.1. An object in $\mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathrm{sPre}_{\cdot}(G\mathcal{S}\mathrm{m}/k), T\wedge -)$ is called a (sequential) naive G-spectrum. We consider the category $\mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathrm{sPre}_{\cdot}(G\mathcal{S}\mathrm{m}/k), T\wedge -)$ of naive G-spectra as endowed with the stable model structure analogous to (4.1), i.e. take the $\mathbb{A}^1$-local injective model structure with respect to the H-Nisnevich topology on $\mathrm{sPre}_{\cdot}(G\mathcal{S}\mathrm{m}/k)$ and localize the levelwise (projective) model structure on $\mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathrm{sPre}_{\cdot}(G\mathcal{S}\mathrm{m}/k), T\wedge -)$ along stable equivalences.

We will usually continue to call an object E in $\mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathrm{sPre}_{\cdot}(G\mathcal{S}\mathrm{m}/k), \mathbb{T}_G\wedge -)$ an equivariant spectrum or G-spectrum, but to emphasize the distinction E is sometimes called a genuine G-spectrum.

Given a non-equivariant spectrum X in $\mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathrm{sPre}_{\cdot}(\mathcal{S}\mathrm{m}/k)$ we may apply the canonical prolongation of the trivial G-action functor (4.9)

$$(-)_{tr} : \mathrm{sPre}(\mathcal{S}\mathrm{m}/k) \to \mathrm{sPre}(G\mathcal{S}\mathrm{m}/k)$$

on X to obtain a naive G spectrum X_{tr}. Let E be any naive G-spectrum and define a genuine G-spectrum i_*E by $(i_*E)_n = \widetilde{\mathbb{T}}_G^n \wedge E_n$ with bonding maps

$$\mathbb{T}_G \wedge i_*E_n \cong \widetilde{\mathbb{T}}_G \wedge T \wedge i_*E_n \xrightarrow{\mathrm{id}\wedge\sigma_n} \widetilde{\mathbb{T}}_G^{n+1} \wedge E_{n+1}.$$

The resulting functor i_* from naive to genuine G-spectra has a right adjoint i^*, which is defined by $(i^*E)_n = \underline{\mathrm{Hom}}_G(\widetilde{\mathbb{T}}_G^n, E_n)$ with bonding maps

$$T\wedge i^*E_n \to i^*E_{n+1} = \underline{\mathrm{Hom}}_G(\widetilde{\mathbb{T}}_G^{n+1}, E_{n+1}) \text{ adjoint to}$$
$$\widetilde{\mathbb{T}}_G^{n+1} \wedge T \wedge i^*E_n \cong \mathbb{T}_G \wedge \widetilde{\mathbb{T}}_G^n \wedge \underline{\mathrm{Hom}}_G(\widetilde{\mathbb{T}}_G^n, E_n) \xrightarrow{ev} \mathbb{T}_G \wedge E_n \xrightarrow{\sigma_n} E_{n+1}.$$

This way, we have defined a *change of universe* adjunction

$$i_* : \mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathrm{sPre}_{\cdot}(G\mathcal{S}\mathrm{m}/k), T\wedge -) \rightleftarrows \mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathrm{sPre}_{\cdot}(G\mathcal{S}\mathrm{m}/k), \mathbb{T}_G \wedge -) : i^*.$$

The name is derived from an account to classical stable equivariant topology based on coordinate-free spectra, where spectra are indexed on a universe with a trivial G-action in the naive case and indexed on a universe of arbitrary representations in the genuine case.

Lemma 4.2.2. *The change of universe adjunction (i_*, i^*) is a Quillen adjunction with respect to the stable model structures.*

Proof. The pair (i_*, i^*) is a Quillen adjunction with respect to the levelwise model structures. Let X be a stably fibrant genuine G-spectrum, in particular we have weak equivalences

$$X_n \xrightarrow{\sim} \underline{\mathrm{Hom}}_G(\mathbb{T}_G, X_{n+1})$$

of $\mathbb{A}^1$-locally fibrant simplicial presheaves for every n. The right Quillen functor $\underline{\mathrm{Hom}}_G(\widetilde{\mathbb{T}}^n_G, -)$ preserves them and we compute

$$\begin{aligned} i^*X_n \cong \underline{\mathrm{Hom}}_G(\widetilde{\mathbb{T}}^n_G, X_n) &\simeq \underline{\mathrm{Hom}}_G(\widetilde{\mathbb{T}}^n_G, \underline{\mathrm{Hom}}_G(\mathbb{T}_G, X_{n+1})) \\ &\cong \underline{\mathrm{Hom}}_G(\widetilde{\mathbb{T}}^n_G \wedge \mathbb{T}_G, X_{n+1}) \\ &\cong \underline{\mathrm{Hom}}_G(T, \underline{\mathrm{Hom}}_G(\widetilde{\mathbb{T}}^{n+1}_G, X_{n+1})) = (\Omega_T i^*X)_n \end{aligned}$$

and note that i^*X is a stably fibrant naive G-spectrum [Jar00, Lemma 2.7]. Further, the adjunction (i_*, i^*) is compatible with the simplicial enrichments and we combine this with the (SM7)-style characterization of stable equivalences [Jar00, Corollary 2.12]: Let W be a stably fibrant and levelwise-injective fibrant genuine G-spectrum and let $f : X \to Y$ be a trivial cofibration of naive G-spectra. The diagram

$$\begin{tikzcd} s\mathcal{S}\mathrm{et}(i_*Y, W) \arrow[r, "i_*f^*"] \arrow[d, "\cong"] & s\mathcal{S}\mathrm{et}(i_*X, W) \arrow[d, "\cong"] \\ s\mathcal{S}\mathrm{et}(Y, i^*W) \arrow[r, "\sim", "f^*"'] & s\mathcal{S}\mathrm{et}(X, i^*W) \end{tikzcd}$$

commutes and therefore i_*f is stable equivalence (and a cofibration). $\square$

The forgetful functor $(-)^e : \mathrm{sPre}(G\mathcal{S}\mathrm{m}/k) \to \mathrm{sPre}(\mathcal{S}\mathrm{m}/k)$ (the e-fixed points functor) also has a canonical prolongation

$$(-)^e : \mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathrm{sPre}_\cdot(G\mathcal{S}\mathrm{m}/k), T \wedge -) \to \mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathrm{sPre}_\cdot(\mathcal{S}\mathrm{m}/k), T \wedge -)$$

and for a (genuine) G-spectrum E, we call E^e (resp. $(i^*E)^e$) the underlying non-equivariant spectrum of E.

Lemma 4.2.3. *Let E be a naive G-spectrum. The unit morphism*

$$E \to i^*i_*E$$

is a non-equivariant stable equivalence.

Proof. Let X be a naive G-equivariant suspension spectrum. Consider the commutative diagram

$$\begin{tikzcd} X^e \arrow[r] \arrow[d, "\sim"'] & i^*i_*X^e \arrow[d, "\sim"] \\ R^\infty X^e \arrow[r] & R^\infty i^*i_*X^e \end{tikzcd} \tag{4.5}$$

of non-equivariant spectra. We compare domain and codomain of the lower horizontal morphism. The level n in the domain is given by

$$\begin{aligned} R^\infty X_n^e &= \operatorname*{colim}_{j\geq 0} \underline{\mathrm{Hom}}(T^j, X_{j+n}^e) \\ &= \operatorname*{colim}_{j\geq 0} \underline{\mathrm{Hom}}(T^j, T^j \wedge X_n^e) \end{aligned}$$

while for the codomain we need a few transformations to compute

$$\begin{aligned} R^\infty i^* i_* X_n^e &= \operatorname*{colim}_{j\geq 0} \underline{\mathrm{Hom}}(T^j, i^* i_* X_{j+n}^e) \\ &= \operatorname*{colim}_{j\geq 0} \underline{\mathrm{Hom}}(T^j, i^* i_* X_{j+n}^e) \\ &= \operatorname*{colim}_{j\geq 0} \underline{\mathrm{Hom}}(T^j, \underline{\mathrm{Hom}}_G(\widetilde{\mathbb{T}}_G^{j+n}, \widetilde{\mathbb{T}}_G^{j+n} \wedge X_{j+n})^e) \\ &= \operatorname*{colim}_{j\geq 0} \underline{\mathrm{Hom}}_G(G_+ \wedge T^j \wedge \widetilde{\mathbb{T}}_G^{j+n}, \widetilde{\mathbb{T}}_G^{j+n} \wedge X_{j+n}) \end{aligned}$$

and replace $G_+ \wedge \widetilde{\mathbb{T}}_G^{j+n}$ by the weakly equivalent $G_+ \wedge T^{(j+n)(|G|-1)}$. The equivariant weak equivalence is given by $G_+ \wedge Y^e \to G_+ \wedge Y, (g, x) \mapsto (g, g \cdot x)$ in $\mathrm{sPre}_\cdot(G\mathcal{S}\mathrm{m}/k)$. We continue

$$\begin{aligned} &\simeq \operatorname*{colim}_{j\geq 0} \underline{\mathrm{Hom}}_G(G_+ \wedge T^{j+(j+n)(G-1)}, \widetilde{\mathbb{T}}_G^{j+n} \wedge X_{j+n}) \\ &= \operatorname*{colim}_{j\geq 0} \underline{\mathrm{Hom}}(T^{j+(j+n)(G-1)}, (\widetilde{\mathbb{T}}_G^{j+n} \wedge X_{j+n})^e) \\ &= \operatorname*{colim}_{j\geq 0} \underline{\mathrm{Hom}}(T^{j+(j+n)(G-1)}, T^{j+(j+n)(G-1)} \wedge X_n^e). \end{aligned}$$

Thus, the (filtered and hence homotopy) colimit in the codomain is taken over a cofinal system for the colimit in the domain. Therefore the lower horizontal morphism is a levelwise equivalence in diagram (4.5).

Now let X be an arbitrary naive G-spectrum. X is stably equivalent to the colimit

$$\operatorname{colim}(\Sigma_T^\infty X_0 \to \Sigma_T^\infty X_1[-1] \to \Sigma_T^\infty X_1[-1] \to \ldots)$$

shifted suspension spectra. By the same arguments as in [Jar00, Lemma 4.29], basically because stable weak equivalences are closed under filtered colimits [Jar00, Lemma 3.12], the conclusion follows from the first part of this proof. □

Not only the forgetful functor $(-)^e$ has a canonical prolongation, but also its space level adjoint functor $ind = G_+ \wedge -$ has prolongates canonically due to the twisting isomorphism $G_+ \wedge T \wedge X \cong T \wedge G_+ \wedge X$ to naive G-spectra.

Lemma 4.2.4. *The adjunction*

$$ind : \mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathrm{sPre}_\cdot(\mathcal{S}\mathrm{m}/k), T \wedge -) \rightleftarrows \mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathrm{sPre}_\cdot(G\mathcal{S}\mathrm{m}/k), T \wedge -) : res = (-)^e$$

is a Quillen adjunction with respect to the stable model structures.

Proof. First, note that (ind, res) is a Quillen adjunction for the levelwise model structures by Lemma 3.3.8 and that res preserves levelwise equivalences. Since we have

$$\begin{aligned} res(R^\infty X) &= res\left(\operatorname*{colim}_{n\geq 0} \underline{\mathrm{Hom}}_G(T^n, X_n)\right) \\ &\cong \operatorname*{colim}_{n\geq 0} res\left(\underline{\mathrm{Hom}}_G(T^n, X_n)\right) \\ &\cong \operatorname*{colim}_{n\geq 0} \underline{\mathrm{Hom}}(T^n, res(X_n)) = R^\infty res(X) \end{aligned}$$

it follows that res also preserves stable equivalences. Together with a characterization of stably fibrant objects [Jar00, Lemma 2.7& 2.8] a similar computation reveals that res preserves stably fibrant objects. As the stable model structures are left Bousfield localizations of the levelwise ones, it is sufficient to show that ind maps trivial cofibrations to stable equivalences. So let $f : X \to Y$ be a trivial cofibration in $\mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathrm{sPre}_{\cdot}(\mathcal{S}\mathrm{m}/k), T \wedge -)$ and let W be a stably fibrant and injective-levelwise fibrant object in $\mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathrm{sPre}_{\cdot}(G\mathcal{S}\mathrm{m}/k), T \wedge -)$. We make use of the simplicial structure and observe that the diagram

$$\begin{tikzcd} \mathrm{s}\mathcal{S}\mathrm{et}(ind(Y), W) \arrow[r, "ind(f)^*"] \arrow[d, "\cong"] & \mathrm{s}\mathcal{S}\mathrm{et}(ind(X), W) \arrow[d, "\cong"] \\ \mathrm{s}\mathcal{S}\mathrm{et}(Y, res(W)) \arrow[r, "\sim", "f^*"'] & \mathrm{s}\mathcal{S}\mathrm{et}(X, res(W)) \end{tikzcd}$$

commutes and that $res(W)$ is still stably fibrant and 'injective'. Thus, $ind(f)$ is stable equivalence [Jar00, Corollary 2.12]. $\square$

Lemma 4.2.5. *Let $d : E \to F$ be a non-equivariant stable equivalence of naive G-spectra and let X be stably equivalent to an induced naive G-spectrum. Then the map*

$$d_* : [X, E] \xrightarrow{\cong} [X, F]$$

is an isomorphism.

Proof. Due to naturality the diagram

$$\begin{tikzcd} {[X, E]} \arrow[r, "\cong"] \arrow[d, "d_*"] & {[ind(D), E]} \arrow[r, "\cong"] \arrow[d, "d_*"] & {[D, E^e]} \arrow[d, "d_*^e"] \\ {[X, F]} \arrow[r, "\cong"] & {[ind(D), F]} \arrow[r, "\cong"] & {[D, F^e]} \end{tikzcd}$$

commutes, where the maps decorated with '$\cong$' are isomorphisms by Lemma 4.2.4 and the assumption of a stable equivalence between X and $ind(D)$. Further, we assume the d^e is a stable equivalence, hence $(d^e)_*$ and d_* are isomorphisms. □

Proposition 4.2.6. *Let X be stably equivalent to an induced naive G-spectrum and let E be any naive G-spectrum. Then there is an isomorphism*

$$i_* : [X, E] \xrightarrow{\cong} [i_* X, i_* E].$$

Proof. By Lemma 4.2.3 and Lemma 4.2.5 the morphism i_* is a composition of isomorphisms

$$i_* : [X, E] \xrightarrow{\eta_E} [X, i^* i_* E] \cong [i_* X, i_* E].$$

□

With the same arguments as for Lemma 4.2.4 all the other induction/restriction adjunctions

$$ind_G^H : \mathrm{sPre}_\cdot(H\mathcal{S}\mathrm{m}/k) \rightleftarrows \mathrm{sPre}(G\mathcal{S}\mathrm{m}/k) : res_H^G$$

prolongate to Quillen adjunctions between the respective naive equivariant categories as well. This is also true for the fixed-point functors and we record the following lemma for the study of fixed-point functor of genuine G-spectra in the next subsection.

Lemma 4.2.7. *For all $H \leq G$, the canonically prolongated adjunction*

$$(-)_H : \mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathrm{sPre}_\cdot(\mathcal{S}\mathrm{m}/k), T \wedge -) \rightleftarrows \mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathrm{sPre}_\cdot(G\mathcal{S}\mathrm{m}/k), T \wedge -) : (-)^H$$

is a Quillen adjunction with respect to the stable model structure on both sides.

Proof. Again, note that $((-)_H, (-)^H)$ is a Quillen adjunction for the levelwise model structures. Let $f : X \to Y$ be a stable acyclic cofibration of non-equivariant spectra. We have to show that f_H is stable equivalence of naive G-spectra or equivalently that for all $n \in \mathbb{N}$ and $K \leq G$ the morphism $R^\infty(f_H)_n^K$ is an $\mathbb{A}^1$-local weak equivalence. Since we have

$$\begin{aligned}
\underline{\mathrm{Hom}}(T^i, X^K) &= \mathrm{Hom}_{\mathrm{sPre}}(T^i \wedge \widetilde{(\ )}_+, X^K) \\
&\cong \mathrm{Hom}_{\mathrm{sPre}(G)}(T^i \wedge G/K_+ \wedge \widetilde{(\ )}_+, X) \\
&\cong \underline{\mathrm{Hom}}_G(T^i, X)^K
\end{aligned}$$

we see that $(R^\infty(f_H)_n)^K \cong R^\infty((f_H)^K)_n$ and the statement follows from Lemma 3.3.8. □

4.3. **Characterization of Stable Weak Equivalences.** In this section we define two fixed point functors

$$(4.6)\qquad (-)^H : \mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathrm{sPre}_{\cdot}(G\mathcal{S}\mathrm{m}/k), \mathbb{T}_G \wedge -) \to \mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathrm{sPre}_{\cdot}(\mathcal{S}\mathrm{m}/k), T \wedge -)$$

$$(4.7)\qquad \Phi^H : \mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathrm{sPre}_{\cdot}(G\mathcal{S}\mathrm{m}/k), \mathbb{T}_G \wedge -) \to \mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathrm{sPre}_{\cdot}(\mathcal{S}\mathrm{m}/k), T \wedge -)$$

from G-spectra to non-equivariant spectra for any subgroup $H \leq G$. The situation is pretty much the same as in classical stable equivariant homotopy theory, where the (Lewis-May) fixed point functor $(-)^H$ has the expected left adjoint, but is rather abstract and the *geometric fixed point functor* Φ^H is the levelwise extension of the unstable fixed point functor. We show that both families of fixed-point functors detect motivic equivariant stable weak equivalences. This means that we obtain two stable versions of Proposition 3.3.9.

The Lewis-May fixed points. For a non-equivariant T-spectrum E we define the push forward E_{fixed} to a genuine G-spectrum by the composition
(4.8)

$$\begin{tikzcd}[column sep=small]
\mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathrm{sPre}_{\cdot}(\mathcal{S}\mathrm{m}/k), T \wedge -) \arrow[rr, "(-)_{fixed}"] \arrow[dr, "(-)_{tr}"'] & & \mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathrm{sPre}_{\cdot}(G\mathcal{S}\mathrm{m}/k), \mathbb{T}_G \wedge -) \\
& \mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathrm{sPre}_{\cdot}(G\mathcal{S}\mathrm{m}/k), T \wedge -), \arrow[ur, "i_*"'] &
\end{tikzcd}$$

that is X_{fixed} is the genuine G-equivariant spectrum defined by

$$(X_{fixed})_n = \widetilde{\mathbb{T}_G}^n \wedge (X_n)_{tr}$$

where $\widetilde{\mathbb{T}_G}$ is the representation sphere associated to the reduced regular representation and $(X_n)_{tr}$ is the image of X_n under the left adjoint functor $(-)_{tr}$ from the adjunction

$$(4.9)\qquad (-)_{tr} : \mathrm{sPre}_{\cdot}(\mathcal{S}\mathrm{m}/k) \rightleftarrows \mathrm{sPre}_{\cdot}(G\mathcal{S}\mathrm{m}/k) : (-)^G$$

of left Kan extensions, cf. (2.7). The bonding maps of X_{fixed} are defined by

$$\begin{tikzcd}
\mathbb{T}_G \wedge \widetilde{\mathbb{T}_G}^{\wedge n} \wedge tr(X_n) \arrow[r, dashed] \arrow[d, "\cong"', "\tau"] & \widetilde{\mathbb{T}_G}^{\wedge n+1} \wedge tr(X_{n+1}) \\
\widetilde{\mathbb{T}_G} \wedge \widetilde{\mathbb{T}_G}^{\wedge n} \wedge T \wedge tr(X_n) \arrow[ur, "\mathrm{id} \wedge \sigma_n"'] &
\end{tikzcd}$$

Since not only $(-)_{tr}$, but by Lemma 4.2.7 the whole family of fixed-point adjunctions canonically prolongates to Quillen adjunctions

$$(-)_H : \mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathrm{sPre}_{\cdot}(\mathcal{S}\mathrm{m}/k), T \wedge -) \rightleftarrows \mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathrm{sPre}_{\cdot}(G\mathcal{S}\mathrm{m}/k), T \wedge -) : (-)^H$$

we may compose adjoints and make the following definition.

Definition 4.3.1. Let X be a genuine G-equivariant spectrum. We define the (Lewis-May) H-fixed points of X by

$$X^H := (i^* X)^H.$$

Lemma 4.3.2. *The adjunction*

$$(-)_{fixed} : \mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathrm{sPre}_{\cdot}(\mathcal{S}\mathrm{m}/k), T \wedge -) \rightleftarrows \mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathrm{sPre}_{\cdot}(G\mathcal{S}\mathrm{m}/k), \mathbb{T}_G \wedge -) : (-)^G$$

as well as the other H-fixed point adjunctions are Quillen adjunctions with respect to the stable model structures.

Proof. The Lewis-May fixed point adjunctions are compositions of Quillen adjunctions by Lemma 4.2.2 (change of universe) and Lemma 4.2.7 (naive fixed points). □

Proposition 4.3.3. *Let $f : X \to Y$ be a morphism in $\mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathrm{sPre}(G\mathcal{S}\mathrm{m}/k))$. Then the following are equivalent*

(1) f is a stable weak equivalence.
(2) For all subgroups $H \leq G$, the morphism f^H is a stable equivalence of non-equivariant spectra.

Proof. The morphism f is a stable equivalence of G-spectra if and only if it induces isomorphisms on all weighted stable homotopy groups $\pi^H_{s,t}$. We compute

$$[G/H \wedge S^{s+j} \wedge (\mathbb{A}[G] - 0)^{t+j}, X_j]^G \cong [G/H \wedge S^{s+j} \wedge (\mathbb{G}_m)^{t+j} \wedge \widetilde{\mathbb{T}_G}^{t+j}, X_i]^G$$

where we use Lemma 4.1.12 and the splitting $\mathbb{T}_G = T \wedge \widetilde{\mathbb{T}_G} \simeq S^1 \wedge \mathbb{G}_m \wedge \widetilde{\mathbb{T}_G}$, so that we can (cofinally) replace $\mathbb{A}[G] - 0$ by $\mathbb{G}_m \wedge \widetilde{\mathbb{T}_G}$ and obtain

$$\begin{aligned}
&\cong [G/H \wedge S^{s+j} \wedge (\mathbb{G}_m)^{t+j}, \Omega^{t+j}_{\widetilde{\mathbb{T}_G}} X_j]^G \\
&\cong [G/H \wedge S^{s+j} \wedge (\mathbb{G}_m)^{t+j}, i^* X[-t]_j]^G \\
&\cong [S^{s+j} \wedge (\mathbb{G}_m)^{t+j}, i^* X[-t]_j]^H \\
&\cong [S^{s+j} \wedge (\mathbb{G}_m)^{t+j}, i^* X[-t]^H_j]
\end{aligned}$$

So that equivalently f^H induces isomorphisms on non-equivariant weighted stable homotopy groups and hence is a stable equivalence for all $H \leq G$. □

The geometric fixed points. We will need the following lemma to extend the adjunction of Corollary 3.2.11 from unstable to stable homotopy theories.

Lemma 4.3.4. *The G-fixed points of the regular representation sphere are canonically isomorphic to the Tate object T, i.e.*

$$(\mathbb{T}_G)^G \cong T$$

Proof. The regular representation $\mathbb{A}[G]$ decomposes into a sum $\oplus_{i=1}^n m_i U_i$ of inequivalent irreducible representations U_i. Let U_1 be the trivial representation, which splits off canonically due to the norm element $\Sigma_{g\in G} g$ in the finite group case. Then we have

$$U_i^G \cong \begin{cases} \mathbb{A}^1 & \text{if } i = 1 \\ 0 & \text{else,} \end{cases}$$

because non-trivial fixed-points would give a G-invariant submodule and hence a G-invariant complement (by Maschke's Theorem in our case). □

Corollary 4.3.5. *There is a canonical natural isomorphism*

$$(- \wedge T) \circ (-)^G \to (-)^G \circ (- \wedge \mathbb{T}_G)$$

of functors $\mathrm{sPre}_\cdot(G\mathcal{S}\mathrm{m}/k) \to \mathrm{sPre}_\cdot(\mathcal{S}\mathrm{m}/k)$ *and hence a prolongation of the adjunction* (3.1) *to an adjunction*

$$\Phi^G : \mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathrm{sPre}_\cdot(G\mathcal{S}\mathrm{m}/k), (- \wedge \mathbb{T}_G)) \rightleftarrows \mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathrm{sPre}_\cdot(\mathcal{S}\mathrm{m}/k), (- \wedge T)).$$

Proof. The left Kan extension $(-)^G$ from Corollary 3.2.11 preserves smash products, since it is also right adjoint by Remark 3.2.12.

Therefore the isomorphism from the lemma above gives a natural isomorphism

$$T \wedge (-)^G \cong \mathbb{T}_G^G \wedge (-)^G \cong (\mathbb{T}_G \wedge -)^G.$$

From this natural transformation $\tau : ((-)^G \wedge T) \xrightarrow{\cong} (- \wedge \mathbb{T}_G)^G$ one obtains a prolongation of $(-)^G$ by $(X_\cdot)_n^G = (X_n)^G$ with bonding maps

$$\begin{tikzcd} T \wedge X_n^G \arrow[r, dashed] \arrow[d, "\tau_{X_n}"'] & X_{n+1}^G \\ (\mathbb{T}_G \wedge X_n)^G \arrow[ur, "\sigma_n^G"'] & \end{tikzcd}$$

To prolongate the right adjoint R^G of $(-)^G$ one needs a natural transformation

$$\mathbb{T}_G \wedge R^G(-) \to R_G(T \wedge -),$$

but using the adjunction and in particular the counit ϵ we obtain natural morphisms

$$(\mathbb{T}_G \wedge R^G(-))^G \cong T \wedge (R^G(-))^G \xrightarrow{\mathrm{id} \wedge \epsilon} T \wedge -.$$

The prolongations are still adjoint. □

Remark 4.3.6. For a finite group G the norm element $\Sigma_{g\in G} g \in \mathbb{A}[G]$ gives a canonical splitting $\mathbb{A}[G] \cong \mathbb{A}^1 \times \widetilde{\mathbb{A}[G]}$ of the trivial part of the regular representation. Therefore, we have a canonical morphism from the Tate object T with a trivial action to the regular representation sphere $\mathbb{T}_G$ which factors for any $H \leq G$ as

$$\begin{tikzcd} T \arrow[rr] \arrow[dr, dashed, "c_H"'] & & \mathbb{T}_G \\ & \mathbb{T}_G^H \arrow[ur] & \end{tikzcd}$$

This canonical morphism c_H gives a natural transformation

$$T \wedge (-)^H \xrightarrow{c_H} \mathbb{T}_G^H \wedge (-)^H \cong (\mathbb{T}_G \wedge -)^H$$

which leads to a prolongation of the H-fixed points to a functor

$$\Phi^H : \mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathrm{sPre}_\cdot(G\mathcal{S}\mathrm{m}/k), \mathbb{T}_G \wedge -) \to \mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathrm{sPre}_\cdot(\mathcal{S}\mathrm{m}/k), T \wedge -). \tag{4.10}$$

Lemma 4.3.7. *Let $X \in \mathrm{sPre}_\cdot(G\mathcal{S}\mathrm{m}/k)$ and let Y be a genuine equivariant G-spectrum. For all subgroups $H \leq G$, we have*

$$\Phi^H(X \wedge Y) = X^H \wedge \Phi^H(Y).$$

In particular, Φ^G is compatible with suspension spectra in the sense that

$$\Phi^G(\Sigma^\infty_{\mathbb{T}_G} X) = \Sigma^\infty_T X^G.$$

Proof. The geometric fixed points functor Φ^H is a prolongation and smashing with a space is defined as a levelwise smash product, thus the first statement follows from the compatibility of the space level fixed point functors with smash products. For the second statement additionally use Lemma 4.3.4. □

One adds a disjoint basepoint to the unique morphism $EG \to *$ and then takes the homotopy cofiber of the suspension spectra in $\mathcal{S}\mathrm{p}^{\mathbb{N}}(G\mathcal{S}\mathrm{m}/k)$ to acquire the cofiber sequence

$$EG_+ \to S^0 \xrightarrow{p} \widetilde{EG}, \tag{4.11}$$

which is of fundamental importance in equivariant homotopy theory.

Lemma 4.3.8. *The unreduced suspension $\widetilde{EG}$ defined by the cofiber sequence*

$$EG_+ \to S^0 \to \widetilde{EG}$$

is non-equivariantly contractible.

Proof. The space EG is non-equivariantly contractible by Lemma 5.1.2, hence the morphism $EG_+ \to S^0$ of spectra is stable weak equivalence of the underlying non-equivariant spectra. Applying [Jar00, Lemma 3.7] twice to the long exact sequence of underlying T spectra

$$\ldots \to \pi_{t+1,s}(\widetilde{EG}) \to \pi_{t,s}(EG_+) \xrightarrow{\cong} \pi_{t,s}(S^0) \to \pi_{t,s}(\widetilde{EG}) \to \ldots$$

we see that $\widetilde{EG}$ is contractible. □

Lemma 4.3.9. *Let $f : X \to Y$ be a non-equivariant stable equivalence of equivariant motivic spectra. Then*

$$\mathrm{id} \wedge f : EG_+ \wedge X \to EG_+ \wedge Y$$

is an equivariant stable equivalence.

Proof. We consider the cofiber sequence

$$X \xrightarrow{f} Y \to hocofib(f) =: Z$$

and assume that Z is non-equivariantly contractible. Let $Z \to Z'$ be a stably fibrant replacement in $\mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathrm{sPre}_{\cdot}(G\mathcal{S}\mathrm{m}/k), \mathbb{T}_G \wedge -)$. Then Z' is levelwise non-equivariantly contractible and $EG_+ \wedge Z$ is stably equivalent to $EG_+ \wedge Z'$. But $EG_+ \wedge Z'$ is even equivariantly levelwise contractible and hence so is $EG_+ \wedge Z$. □

For a comparison of geometric and Lewis-May fixed points, we introduce the following generalization of EG. A *family of subgroups* of G is defined to be a set $\mathcal{F}$ of subgroups of G, such that $\mathcal{F}$ is closed under taking subgroups and conjugation. Given such a family $\mathcal{F}$, there might exist a G-representation $V = V_{\mathcal{F}}$ with the property that

$$V^H \text{ is } \begin{cases} > 0 & \text{if } H \in \mathcal{F}, \\ 0 & \text{if } H \notin \mathcal{F}. \end{cases} \tag{4.12}$$

On the other hand, given a G-representation V, the set of subgroups with defining property (4.12) is a family of subgroups. We consider the cofiber sequence

$$(V - 0)_+ \to S^0 \to S^V$$

and observe that the fixed points $(S^V)^H$ are computed by the diagram

$$\begin{tikzcd} (V-0)^H \arrow[r] \arrow[d] & V^H \arrow[d] \\ * \arrow[r] & (S^V)^H. \end{tikzcd}$$

Thus, $(S^V)^H$ is S^0 for subgroups H which are not in $\mathcal{F}$ and otherwise $(S^V)^H$ is equal to $S^{2r,r}$, for some $r > 0$. Denote by $E\mathcal{F}$ the infinite smash product $\mathrm{colim}_{j\geq 0}(V-0)^{\wedge j}$ and by $\widetilde{E\mathcal{F}}$ the infinite smash product $\mathrm{colim}_{j\geq 0}(S^V)^{\wedge j}$. It follows that $\widetilde{E\mathcal{F}}^H$ is S^0 if H is not in $\mathcal{F}$. For a subgroup $H \in \mathcal{F}$ the H-fixed points are an infinite smash of positive dimensional spheres and therefore contractible. In particular, we note that for the family $\mathcal{P}$ of all proper subgroups of G, the reduced regular representation gives an adequate representation and the fixed points of the unreduced suspension $\widetilde{E\mathcal{P}}$ are given by

$$\widetilde{E\mathcal{P}}^H \simeq \begin{cases} * & \text{if } H < G \\ S^0 & \text{if } H = G. \end{cases}$$

Lemma 4.3.10. *The evaluation morphism*

$$(\widetilde{E\mathcal{P}} \wedge X)^G \to \Phi^G(X)$$

is a levelwise equivalence of non-equivariant spectra.

Proof. We compute that

$$\begin{aligned}(\widetilde{E\mathcal{P}} \wedge X)^G_n &= \underline{\mathrm{Hom}}_G(\widetilde{\mathbb{T}}^n_G, \widetilde{E\mathcal{P}} \wedge X_n)^G \\ &\cong \mathrm{s}\mathcal{S}\mathrm{et}_G(\widetilde{\mathbb{T}}^n_G \wedge (-)_{tr}, \widetilde{E\mathcal{P}} \wedge X_n)\end{aligned}$$

where $\widetilde{\mathbb{T}}^n_G$ is a homotopy colimit of equivariant cells $G/H_+ \wedge S^{p_H,q_H}$ and therefore

$$\begin{aligned}&\cong \operatorname*{holim}_{H \leq G} \mathrm{s}\mathcal{S}\mathrm{et}_G(G/H_+ \wedge S^{p_H,q_H} \wedge (-)_{tr}, \widetilde{E\mathcal{P}} \wedge X_n) \\ &\cong \operatorname*{holim}_{H \leq G} \mathrm{s}\mathcal{S}\mathrm{et}(S^{p_H,q_H} \wedge (-)_{tr}, \widetilde{E\mathcal{P}}^H \wedge X_n^H)\end{aligned}$$

All the non-initial holim-factors corresponding to proper subgroups are contractible and since $\widetilde{\mathbb{A}[G]}$ has no trivial subrepresentation we have $(p_G, q_G) = (0, 0)$, so that

$$\begin{aligned}&\simeq \mathrm{s}\mathcal{S}\mathrm{et}(S^0 \wedge (-)_{tr}, \widetilde{E\mathcal{P}}^G \wedge X_n^G) \\ &\cong X_n^G.\end{aligned}$$

□

We are now ready for a characterization of equivariant stable equivalences by their geometric fixed points. The topological analogue of the following theorem seems to be surprisingly rare in the literature. The author has learned the topological statement from a recent paper of Ragnarsson [Rag10, Proposition 3.1]. However, the proof below is essentially a transformation of the one given in [Sch11, Theorem 6.13].

Proposition 4.3.11. *Let $f : X \to Y$ be a morphism in $\mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathrm{sPre}_{\cdot}(G\mathcal{S}\mathrm{m}/k))$. Then the following are equivalent*

(1) f is a stable weak equivalence.

(2) For all subgroups $H \leq G$, the morphism $\Phi^H(f)$ is a stable equivalence of non-equivariant spectra.

Proof. Assume that f is a stable equivalence. Let $\mathcal{P}_H$ be the family of all proper subgroups of H. When applying the left Quillen functor $\widetilde{E\mathcal{P}_H} \wedge -$ we still have a stable equivalence and by Proposition 4.3.3 for all subgroups H of G we thus have a non-equivariant stable equivalence

$$(\widetilde{E\mathcal{P}_H} \wedge f)^H : (\widetilde{E\mathcal{P}_H} \wedge X)^H \to (\widetilde{E\mathcal{P}_H} \wedge Y)^H$$

which implies by Lemma 4.3.10 that $\Phi^H(f)$ is a stable equivalence.

Conversely, assume that for all subgroups H of G the map $\Phi^H(f)$ on geometric fixed points is a stable equivalence. Now we proceed by induction on the order of G. For $|G| = 1$ there is nothing to show, since Φ^G is basically the identity then. So let G be non-trivial and assume the claim to be true for all proper subgroups of G. So $res^G_H f$ is an equivariant stable equivalence for all proper subgroups H of G and by Proposition 4.3.3 this implies that for these subgroups also f^H is an non-equivariant stable equivalence. We are going to show that f^G is a stable equivalence as well. Smashing f with the norm sequence (4.11) for $E\mathcal{P}$ we obtain a diagram

$$\begin{tikzcd} E\mathcal{P} \wedge X \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, "f"] & \widetilde{E\mathcal{P}} \wedge X \arrow[d] \\ E\mathcal{P} \wedge Y \arrow[r] & Y \arrow[r] & \widetilde{E\mathcal{P}} \wedge Y \end{tikzcd}$$

where $E\mathcal{P}_+ \wedge f$ is a stable equivalence by an argument completely analogous to the proof of Lemma 4.3.9. We may apply $(-)^G$ to the whole diagram above and using Lemma 4.3.10 we find that f^G is surrounded by stable equivalences in the diagram

$$\begin{tikzcd} (E\mathcal{P} \wedge X)^G \arrow[r] \arrow[d, "\sim"] & X^G \arrow[r] \arrow[d, "f^G"] & (\widetilde{E\mathcal{P}} \wedge X)^G \arrow[d, "\sim"] \\ (E\mathcal{P} \wedge Y)^G \arrow[r] & Y^G \arrow[r] & (\widetilde{E\mathcal{P}} \wedge Y)^G \end{tikzcd}$$

with rows cofiber sequences. Therefore, f^G is a stable equivalence and we conclude again by using Proposition 4.3.3 that f itself is stable equivalence. □

4.4. **Generalized Motivic Tate Cohomology.** This section has inherited its name from a book of nearly the same name by Greenlees and May [GM95]. In fact, its content is in parts inspired by Part 1 of that book, where Greenlees and May develop the general theory of the technical core of [Gre88b]. Given a G-spectrum E one may associate to it an E-Borel spectrum $c(E)$, a co-Borel spectrum $f(E)$ and a Tate spectrum $t(E)$ and study their interplay by a so-called *norm sequence*

$$f(E) \to c(E) \to t(E).$$

As discussed in the introduction this leads to a convenient framework for an investigation of the Segal conjecture and its generalizations.

We review the construction of the norm sequence directly in our motivic setting. Let E be a G-equivariant motivic spectrum. The morphism $(EG \to *)_+$ was already used to build the cofiber sequence (4.11) and we make use of it again to obtain an induced morphism

$$\epsilon : E = F(S^0, E) \to F(EG_+, E)$$

of equivariant motivic spectra. Next, we smash the morphism ϵ with the cofiber sequence (4.11) and this provides us with what is called a Tate diagram in [HKO11b, 3.3]; the diagram

(4.13)
$$\begin{tikzcd} EG_+ \wedge E \arrow[r] \arrow[d, "\sim"] & E \arrow[r] \arrow[d, "\epsilon"] & \widetilde{EG} \wedge E \arrow[d] \\ EG_+ \wedge F(EG_+, E) \arrow[r] & F(EG_+, E) \arrow[r] & \widetilde{EG} \wedge F(EG_+, E) \end{tikzcd}$$

with rows cofiber sequences and the bottom row known as the norm sequence. The following nomenclature is standard in topology (cf. [GM95, p. 178]) and will be adopted here:

$EG_+ \wedge E$ is stably equivalent to $EG_+ \wedge F(EG_+, E)$ by Lemma 4.3.9

and is called the *co-Borel spectrum* or *free spectrum* of E and is therefore commonly denoted by $f(E)$.

$F(EG_+, E)$ is known as the *Borel spectrum* associated to E.

In modern terminology it is denoted by $c(E)$ and also known as the *geometric completion* of E, but we stick to the more classical $b(E)$ since we will mostly consider it for representing ordinary motivic Borel cohomology.

$\widetilde{EG} \wedge F(EG_+, E)$ is the *Tate spectrum* associated to E and denoted by $t(E)$.

At this point we may comment a little bit on the aforementioned relation to the Atiyah-Segal completion theorem and the Segal conjecture. In a modern and generalized formulation of these statements it is the morphism ϵ from (4.13) which is shown to be a completion at the augmentation ideal I of the Burnside ring [GM92], i.e. the induced morphism

$$\epsilon_* : E_I^\wedge \to F(EG_+, E)$$

known as the *completion conjecture map* is a stable equivalence, when E is the equivariant push forward of complex topological K-theory in case of the Atiyah-Segal completion theorem or when E is the sphere spectrum in case of the Segal conjecture. Motivically there has been a first progress towards an algebraic K-theory analogue of the Atiyah-Segal theorem by the - more or less ad-hoc - results of Knizel-Neshitov [KN11] and Krishna [Kri12].

Next, we implement a notation of equivariant motivic cell spectra in analogy to [DI05].

Definition 4.4.1. A spectrum X in $\mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathrm{sPre}_{\cdot}(G\mathcal{S}\mathrm{m}/k), \mathbb{T}_G \wedge -)$ is an equivariant cell spectrum if it is the homotopy colimit of a diagram $D : I \to \mathcal{S}\mathrm{p}^{\mathbb{N}}(G\mathcal{S}\mathrm{m}/k)$, such that for all $i \in I$ the object $D(i)$ is an element of the set $\{G/H_+ \wedge S^{p,q} \mid p, q \in \mathbb{Z}, H \leq G\}$. Moreover, X is called cellularly G-free if it is $\{G_+ \wedge S^{p,q} \mid p, q \in \mathbb{Z}\}$-cellular in the sense above.

Lemma 4.4.2. *Let X be a cellularly G-free spectrum and let E be any G-spectrum. Then the Tate cohomology associated to E vanishes on X, i.e.*

$$t(E)^{p,q}(X) = 0 \text{ for all } p, q \in \mathbb{Z}.$$

Proof. Any map $X \to t(E)$ is build from morphisms $G_+ \wedge S^{p,q} \to t(E)$. These correspond by Quillen adjunction to non-equivariant morphisms $S^{p,q} \to t(E)$ which are null-homotopic by Lemma 4.3.8. □

Definition 4.4.3. A spectrum X in $\mathcal{S}\mathrm{p}^{\mathbb{N}}(G\mathcal{S}\mathrm{m}/k)$ is called G-free if the morphism

$$EG_+ \wedge X \xrightarrow{0_+ \wedge \mathrm{id}} S^0 \wedge X \cong X$$

is an equivariant stable equivalence.

Lemma 4.4.4. *Any cellularly G-free spectrum is G-free.*

Proof. By Proposition 4.3.11 the geometric fixed point functors detect stable equivariant equivalences and by Lemma 4.3.7 Φ^H is compatible with smash products. Therefore, the statement follows from non-equivariant equivalence $(EG_+)^e \simeq S^0$ of simplicial presheaves. □

Lemma 4.4.5. *Let X be a free G-space or G-spectrum and let E be any G-spectrum. Then the Tate homology associated to E vanishes on X, i.e.*

$$t(E)_{p,q}(X) = 0 \text{ for all } p, q \in \mathbb{Z}.$$

Proof. As X is a free G-spectrum, by definition we know that

$$X \wedge EG_+ \wedge E \to X \wedge E$$

is an equivariant stable equivalence and hence the homology of the co-Borel and the Borel theory associated to E coincide and the Tate homology vanishes on X. □

5. Motivic Borel Cohomology

In the theory of transformation groups it has become an essential tool to study a G-object X by studying the cohomology of the associated action groupoid. This idea was introduced in equivariant topology by Borel in [Bor60] where he first made 'a systematic use of the twisted product $X \times_G EG$ of X with a universal bundle for G'. This twisted product became later known as the *Borel construction* or the *homotopy orbit space.* Nowadays, the Borel construction is often just attributed to the maxim of cofibrant replacement before applying a left Quillen functor to get 'the correct homotopy type'. However, Borel's idea was based on the insight that the space $X \times_G EG$ comes along with a fibration to the classifying space of G with fiber X and with a projection to the orbit space X/G with fibers being the classifying spaces of the isotropy subgroups of G. Thus the cohomology of $X \times_G EG$ is rich in equivariant information, combining the cohomology of $X, X/G, X^G, BG$, and the $B(G_x)$'s.

5.1. **A Digression on Classifying Spaces.** Any discussion of Borel cohomology clearly requires a discussion of classifying spaces and the universal bundle. In this section we will define the classifying space of a simplicial sheaf of groups as a two sided bar construction and compare this with the étale classifying space, a geometric model and Totaro's finite dimensional approximations of the principal G-bundle $EG \to BG$ [Tot99]. Our discussion is highly influenced by the corresponding discussions in [MV99, Tot99, EG98]

We start with the construction of that classifying space of simplicial sheaves which is topologically convenient in the sense that it is just a slight generalization of a topological model for the classifying space (cf. [May75, §7]).

Definition 5.1.1 (two-sided bar construction)**.** Let G be a simplicial sheaf of groups and let X, Y be simplicial sheaves with a right, resp. left, G-action. Define the simplicial sheaf $B(X, G, Y)$ to be given in simplicial degree n by

$$B(X,G,Y)_n : \mathcal{S}\mathrm{m}/k^{\mathrm{op}} \to \mathrm{s}\mathcal{S}\mathrm{et}, U \mapsto X(U)_n \times (G(U)_n)^{\times n} \times Y(U)_n.$$

The i-th face map is defined to be

$$d_n^i : B(X,G,Y)_n \to B(X,G,Y)_{n-1},$$

$$(x, g_1, \ldots, g_n, y) \mapsto \begin{cases} (x \cdot g_1, g_2, \ldots, g_n, y) & i = 0 \\ (x, g_1, \ldots, g_i \cdot g_{i+1}, \ldots, g_n, y) & 0 < i < n \\ (x, g_1, \ldots, g_{n-1}, g_n y) & i = n. \end{cases}$$

The bar-construction is functorial in all three variables. One defines

$$E(G) := B(*, G, G) \text{ and } B(G) := B(*, G, *).$$

Lemma 5.1.2. *Let G be a finite abstract group. Then $E(G)$ is non-equivariantly locally weak equivalent to* $\mathrm{Spec}(k)$.

Proof. $E(G)$ is the Čech nerve associated to the Nisnevich covering $G \to \mathrm{Spec}(k)$, so that $E(G)$ is contractible by Example 2.3.3. □

The universal principal G-bundle is the map $E(G) \to B(G)$, induced by mapping G to the terminal object in the third variable. There is a free G-action on $E(G)$ given in simplicial degree n by

$$(g, (g_0, \ldots, g_n)) \mapsto (g_0, \ldots, g_{n-1}, g \cdot g_n).$$

Lemma 5.1.3. *For the diagonal action on $E(G) \times X$ there is an isomorphism*

$$E(G) \times_G X \cong B(*, G, X).$$

Proof. We may neglect the choice of a section of the simplicial sheafs involved. The orbit space on the left hand side is the quotient of $B(*, G, G) \times X$, where tuples (g_i, X) and (h_i, x') are identified if and only if there exists a $g \in G$ such that

$$(\forall i = 0, \ldots, n-1 : g_i = h_i) \wedge (g \cdot g_n = h_n) \wedge (gx = x')$$

which is equivalent to

$$(\forall i = 0, \ldots, n-1 : g_i = h_i) \wedge (g_n^{-1} x = h_n^{-1} x').$$

Thus, the map

$$B(*, G, G) \times_G X \to B(*, G, X), [(g_0, \ldots, g_n, x)] \mapsto (g_0, \ldots, g_{n-1}, g_n^{-1} x)$$

is well-defined and seen to be an isomorphism by straightforward computation. □

It is worthwhile to note that this classifying space has some expected homotopical meaning: For a simplicial sheaf of groups G, a principal G-bundle is a G-equivariant morphism $E \to X$ of simplicial sheaves, such that E has a free G-action, X has a trivial G-action and the adjoint morphism $E/G \to X$ is an isomorphism. Let $P(X, G)$ denote the set of isomorphism classes of principal G-bundles over X. Morel and Voevodsky prove [MV99, Proposition 4.1.15] that for G of simplicial dimension zero there is a natural bijection

$$P(X, G) \to \mathrm{Hom}_{\mathcal{H}_s}(X, BG).$$

To impose étale descent on the classifying space BG (5.1.1) the so-called étale classifying space is introduced.

Definition 5.1.4. The identity functor $\mathrm{id}_{\mathcal{S}\mathrm{m}/k}$ induces a morphism of sites $\pi : \mathcal{S}\mathrm{m}/k_{et} \to \mathcal{S}\mathrm{m}/k_{Nis}$ and hence induces an adjunction

$$\pi^* : \mathcal{H}_s(k) \rightleftarrows \mathcal{H}_s(\mathrm{sShv}(\mathcal{S}\mathrm{m}/k_{et})) : R\pi_*$$

between the homotopy categories of the local model structures (see [MV99, Proposition 2.1.47]). Here $R\pi_*$ is the total right derived of the direct image functor π_*. Let G be a simplicial sheaf of groups on $\mathcal{S}\mathrm{m}/k_{Nis}$. The étale classifying space $B_{et}G$ is defined as $R\pi_*\pi^*BG$.

The unit $\eta : \mathrm{id} \to R\pi_*\pi^*$ gives a canonical morphism $BG \to B_{et}G$ in $\mathcal{H}_s(k)$. Since the classifying spaces classify principal bundles $P(-, G) = H^1(-, G)$ one obstruction for η_{BG} to be an isomorphism is the coincidence of the first Nisnevich and étale cohomology groups

$$H^1_{Nis}(U, G) = [U, BG] \xrightarrow{\eta_{BG*}} [U, B_{et}G] = H^1_{et}(U, G).$$

More precisely, we have the following lemma.

Lemma 5.1.5. *The canonical morphism $BG \to B_{et}G$ is an isomorphism in $\mathcal{H}_s(k)$ if and only if G is sheaf in the étale topology and $H^1_{Nis}(U, G) = H^1_{et}(U, G)$ for all $U \in \mathcal{S}\mathrm{m}/k$.*

Proof. This is [MV99, Lemma 4.1.18]. □

Unfortunately these classifying space are not exactly representable by schemes, as for example it is shown in Proposition 5.4.3 that the classifying spaces tend to have unbounded motivic cohomology in a way that conflicts with the vanishing theorem [MVW06, Theorem 3.6] which says that

$$H^{p,q}(X, G) = 0 \qquad \text{for } p > q + \dim X.$$

This kind of algebraic inaccessibility of EG and BG may have been the reason why it took so long for Borel cohomology to enter into algebraic geometry (cf. [Ful07, Lecture 1, Remark 1.4]). However, the following finite dimensional approximations by Totaro [Tot99] make classifying spaces accessible for algebraic geometry.

Lemma 5.1.6. *Let G be a finite group, $n \geq q$. There exists a representation V of G over k with an open G-invariant subset U_n such that G acts freely on U_n and complement of U_n has codimension $\geq n$ and such that $U_n \to U_n/G$ exists in $\mathcal{S}\mathrm{m}/k$.*

Proof. Assume $G \neq 0$. The standard action of G on the regular representation $\mathbb{A}[G]$ is free on an open subset U_1, such that the complement of U_1 has a non-zero codimension. Let U_n be the product of the n copies of U_1 in $\mathbb{A}[G]^n$. Thus G acts freely on U_n and the complement $V_n := \mathbb{A}[G]^n \setminus U_n$ has a codimension

$$\mathrm{codim}_{\mathbb{A}[G]^n}(V_n) = n \cdot \mathrm{codim}_{\mathbb{A}[G]}(V_1) \geq n.$$

The quotient $\varphi : \mathbb{A}^{nG} \to \mathbb{A}^{nG}/G$ exists in $\mathcal{S}\mathrm{m}/k$ by [MV99, §4.4.2] and so the restriction to the (image of the) G-free open subscheme $U_n \to U_n/G$ also exists in $\mathcal{S}\mathrm{m}/k$. □

Definition 5.1.7. Including $\mathbb{A}^{nG} \xrightarrow{\cong} \mathbb{A}^{nG} \times 0 \subset \mathbb{A}^{(n+1)G}$ the above proof gives closed embeddings $U_n \to U_{n+1}$ and $U_n/G \to U_{n+1}/G$. Define $E_{gm}G$ and $B_{gm}G$ to be the colimits of the corresponding sequences of simplicial étale sheaves, i.e.

$$E_{gm}G = \operatorname{colim}(U_1 \to U_2 \to \dots U_n \to \dots) \in \mathrm{sShv}(\mathcal{S}\mathrm{m}/k) \text{ and}$$
$$B_{gm}G = \operatorname{colim}(U_1/G \to U_2/G \to \dots U_n/G \to \dots) \in \mathrm{sShv}(\mathcal{S}\mathrm{m}/k).$$

The simplicial presheaf $B_{gm}G$ is called the *geometric classifying space* of G. Following [MV99, Section 4.2] we have $E_{gm}G$ as a G-object in $\mathrm{sShv}(\mathcal{S}\mathrm{m}/k)$, but recall from (3.3) that there is a left adjoint external action functor

$$ext : \mathrm{sPre}(G\mathcal{S}\mathrm{m}/k) \to {}^{G}\,\mathrm{sPre}(\mathcal{S}\mathrm{m}/k).$$

This allows us to consider the diagram $U_1 \to U_2 \to \dots$ as a diagram of representables in $\mathrm{sPre}(G\mathcal{S}\mathrm{m}/k)$ and hence to consider $E_{gm}G$ as an object in $\mathrm{sPre}(G\mathcal{S}\mathrm{m}/k)$, such that by abuse of notation we have $E_{gm}G = ext(E_{gm}G)$.

The schemes U_i/G approximate the motivic cohomology of $B_{gm}G$ in the following sense:

Proposition 5.1.8. *The inclusions $U_i \to U_{i+1}$ induce isomorphisms*

$$H^{r,s}(U_{i+1}/G, \mathbb{Z}/l) \xrightarrow{\cong} H^{r,s}(U_i/G, \mathbb{Z}/l)$$

in motivic cohomology of weight $s < i$.

Proof. This follows directly from [Voe03, Proposition 6.1]. □

The geometric classifying spaces depend on the choice of inclusions and representations only up to isomorphism in $\mathcal{H}_s(k)$ as we will see in the next proposition, which establishes a comparison between the étale and the geometric classifying spaces.

Proposition 5.1.9. *Let $G \in \mathcal{S}\mathrm{m}/k$ be a finite group over an infinite field. The map $B_{gm}G \to B_{et}G$ is a local weak equivalence.*

Proof. This follows from [MV99, Proposition 4.2.6.] since with our assumptions $(\mathbb{A}^{nG}, U_n, U_n \to U_{n+1})$ is an 'admissible gadget' with a 'nice action of G', which for finite G is automatic for infinite fields. □

Example 5.1.10. Together with Lemma 5.1.5 the above proposition leads to the following well known (cf. [VW99, Example 1.17]) comparison of geometric and simplicial classifying spaces:

$B_{gm}\mathbb{G}_m \simeq B\mathbb{G}_m$: This is the case $n = 1$ of a more general result [MV99, Proposition 3.7] and an implication of an instance of Hilbert's Theorem 90, telling that for any $X \in \mathcal{S}\mathrm{m}/k$ the canonical map

$$Pic(X) \xrightarrow{\cong} H^1_{et}(X, \mathbb{G}_m)$$

is an isomorphism [Mil80, Corollary 11.6].

$B_{gm}\mathbb{Z}/2 \not\simeq B\mathbb{Z}/2$: Assume that $char(k) \neq 2$. There is a local weak equivalence $\mathbb{Z}/2 \to \mu_2$, since the sheafifications are isomorphic, and thus an equivalence $B\mathbb{Z}/2 \simeq B\mu_2$. Furthermore, there is also an equivalence $B_{gm}\mathbb{Z}/2 \simeq B_{gm}\mu_2$. However, an obstruction for $B\mu_2 \simeq B_{gm}\mathbb{Z}/2$ is the vanishing of $H^1_{et}(K, \mu_2)$ for all finitely generated field extensions K of k [MV99, Lemma 4.1.18], but $H^1_{et}(K, \mu_2) \cong K^\times/K^{\times 2}$ is non-zero for $K = k(t)$.

Recall from Lemma 3.3.8 that the trivial G-action functor $\mathcal{S}\mathrm{m}/k \to G\mathcal{S}\mathrm{m}/k$ Kan-extends to a left Quillen functor $tr : \mathrm{sPre}(\mathcal{S}\mathrm{m}/k) \to \mathrm{sPre}(G\mathcal{S}\mathrm{m}/k)$, which we will also denote by tr_g in the following for distinction and to emphasize its geometric nature. Further, in (3.3) we have constructed an externalization adjunction

$$ext : \mathrm{sPre}(G\mathcal{S}\mathrm{m}/k) \rightleftarrows {}^G\,\mathrm{sPre}(\mathcal{S}\mathrm{m}/k) : int$$

which composes with the external quotient adjunction

$$-/G : {}^G\,\mathrm{sPre}(\mathcal{S}\mathrm{m}/k) \rightleftarrows \mathrm{sPre}(\mathcal{S}\mathrm{m}/k) : tr_e$$

to give a non-commutative diagram

$$\begin{tikzcd}
 & \mathrm{sPre}(\mathcal{S}\mathrm{m}/k) \arrow[dl, shift left, "tr_g"] \arrow[dr, shift left, leftarrow, "tr_e"] & \\
\mathrm{sPre}(G\mathcal{S}\mathrm{m}/k) \arrow[ur, shift left] \arrow[rr, shift left, "ext"] & & {}^G\,\mathrm{sPre}(\mathcal{S}\mathrm{m}/k) \arrow[ll, shift left, "int"] \arrow[ul, shift left, leftarrow, "-/G"]
\end{tikzcd}$$

There is a difference between tr_g and $= int \circ tr_e$. Suppose they are equal, then for all $X \in \mathrm{sPre}(\mathcal{S}\mathrm{m}/\mathbb{R})$ there are isomorphisms

$$\mathrm{Hom}_{\mathrm{sPre}(G)}(\mathbb{C}_{gal}, tr_g X) \cong \mathrm{Hom}_{\mathrm{sPre}(G)}(\mathbb{C}_{gal}, int \circ tr_e X),$$

but the left hand side is isomorphic to $\mathrm{Hom}_{\mathrm{sPre}}(\mathbb{R}, X)$ and the right hand side is isomorphic to $\mathrm{Hom}_{\mathrm{sPre}}(ext(\mathbb{C}_{gal})/G, X)$ by the adjunctions. Since we now have

$$\mathrm{Hom}_{\mathrm{sPre}}(\mathbb{R}, X) \cong \mathrm{Hom}_{\mathrm{sPre}}(ext(\mathbb{C}_{gal})/G, X)$$

for all $X \in \mathrm{sPre}(\mathcal{S}\mathrm{m}/\mathbb{R})$, it would follow that the simplicial presheaf represented by $\mathrm{Spec}(\mathbb{R})$ is isomorphic to the quotient $ext(\mathbb{C}_{gal})/G$ which is defined by the push out square

$$\begin{tikzcd}
ext(\mathbb{C}_{gal}) \times \widetilde{G} \arrow[r, "\alpha_*"] \arrow[d, "pr"'] & ext(\mathbb{C}_{gal}) \arrow[d] \\
ext(\mathbb{C}_{gal}) \arrow[r] & ext(\mathbb{C}_{gal})/G.
\end{tikzcd}$$

Using $\mathrm{Hom}(\widetilde{\mathrm{Spec}(\mathbb{R})}, ext(\mathbb{C}_{gal})) = \emptyset$, we see that evaluating at the point $\mathrm{Spec}(\mathbb{R})$ gives $ext(\mathbb{C}_{gal})/G(\mathbb{R}) = \emptyset$, while $\widetilde{\mathrm{Spec}(\mathbb{R})}(\mathbb{R}) = *$. Moreover, the lemma below implies that tr_g is not even weakly equivalent to $int \circ tr_e$.

Lemma 5.1.11. *Let X in $G\mathcal{S}\mathrm{m}/k$ such that the categorical quotient X/G exists as a scheme and let F be in $\mathrm{sPre}(\mathcal{S}\mathrm{m}/k)$. Then we have $tr_g(F)(X) \cong F(X/G)$.*

Proof. As tr_g is a left Kan extension, we have

$$tr_g(F)(X) \cong \mathrm{s}\mathcal{S}\mathrm{et}_{\mathrm{sPre}(G\mathcal{S}\mathrm{m}/k)}\left(\widetilde{X}, tr_g(\operatorname*{colim}_i \widetilde{(U_i, [n_i])})\right),$$

where F in $\mathrm{sPre}(G\mathcal{S}\mathrm{m}/k) \cong \mathrm{Pre}(G\mathcal{S}\mathrm{m}/k \times \Delta)$ is written as a colimit of representables, so that

$$\begin{aligned}
&\cong \mathrm{s}\mathcal{S}\mathrm{et}_{\mathrm{sPre}(G\mathcal{S}\mathrm{m}/k)}\left(\widetilde{X}, \operatorname*{colim}_i \widetilde{(tr_g(U_i), [n_i])}\right)\\
&\cong \operatorname*{colim}_i \mathrm{Hom}_{G\mathcal{S}\mathrm{m}/k}\left(X, tr_g(U_i)\right) \times \Delta^{n_i}\\
&\cong \operatorname*{colim}_i \mathrm{Hom}_{\mathcal{S}\mathrm{m}/k}\left(X/G, U_i\right) \times \Delta^{n_i}\\
&\cong \mathrm{s}\mathcal{S}\mathrm{et}_{\mathrm{sPre}(\mathcal{S}\mathrm{m}/k)}\left(\widetilde{X/G}, \operatorname*{colim}_i \widetilde{(U_i, [n_i])}\right)\\
&\cong \mathrm{s}\mathcal{S}\mathrm{et}_{\mathrm{sPre}(\mathcal{S}\mathrm{m}/k)}\left(\widetilde{X/G}, F\right) \cong F(X/G)
\end{aligned}$$

□

Given an $X \in \mathrm{sPre}(G\mathcal{S}\mathrm{m}/k)$ we could investigate simplicial Borel cohomology, that is to study the motivic cohomology

$$H^{*,*}(ext(EG \times X)/G, A).$$

Instead we will focus on geometric Borel cohomology which we define as follows:

Definition 5.1.12. Let A be an abelian group. The motivic Borel cohomology of a space $X \in \mathrm{sPre}(G\mathcal{S}\mathrm{m}/k)$ with coefficients in A is given by

$$b^{p,q}(X, A) := H^{p,q}\left((EG \times X)//G, A\right) := [EG \times X, tr_g(K(p,q,A))]^G$$

where $K(p,q,A)$ denotes the corresponding Eilenberg-MacLane space for A coefficients.

5.2. A Representation of Motivic Borel Cohomology.

In Definition 5.1.12 we have defined mod p motivic Borel cohomology for a space $X \in \mathrm{sPre}_{\cdot}(G\mathcal{S}\mathrm{m}/k)$ as the group of homotopy classes of maps

$$b^{r,s}(X, \mathbb{Z}/p) := [EG_+ \wedge X, tr_g K(r,s,\mathbb{Z}/p)]^G.$$

For any ring R, the motivic cohomology groups with coefficients in R are representable by motivic Eilenberg-MacLane spaces $K(p,q,R)$ [Voe03]. A closer analysis of these spaces provides a canonical morphism $i : T \to K(2,1,R)$ and multiplication morphisms

$$\mu : K(p,q,R) \wedge K(p',q',R) \to K(p+p',q+q',R), \tag{5.1}$$

cf. [Voe03, (2.1) & Theorem 2.2], which assemble to give a motivic Eilenberg-MacLane spectrum HR defined by

$$HR_n = K(2n,n,R) \text{ and}$$
$$T \wedge HR_n \xrightarrow{\mu \circ i} HR_{n+1}.$$

Building on the representability of motivic cohomology, we may construct a spectrum that represents motivic Borel cohomology:

Proposition 5.2.1. *There is an equivariant motivic spectrum b representing* mod p *motivic Borel cohomology. Explicitly b is modeled by the equivariant spectrum $F(EG_+, H\mathbb{Z}/p)$.*

Proof. As we have already noticed above the Kan-extended trivial G-action functor

$$tr_g : \mathrm{sPre}_\cdot(\mathcal{S}\mathrm{m}/k) \to \mathrm{sPre}_\cdot(G\mathcal{S}\mathrm{m}/k)$$

prolongates canonically to a functor from non-equivariant spectra to naive G-spectra. Thus, we may directly consider maps into the trivial Eilenberg-MacLane spectrum $tr_g H\mathbb{Z}/p$.

We have

$$\begin{aligned}
b^{r,s}(X,\mathbb{Z}/p) &= [EG_+ \wedge X, tr_g(K(p,q,\mathbb{Z}/p))]^G \\
&\cong [EG_+ \wedge X, tr_g(\Omega^{r,s}K(0,0,\mathbb{Z}/p))]^G \\
&\cong [EG_+ \wedge X, ev_0(tr_g(\Omega^{r,s}H\mathbb{Z}/p))]^G \\
&\cong [\Sigma^\infty_T(EG_+ \wedge X), tr_g(\Omega^{r,s}H\mathbb{Z}/p)]^{naive} \\
&\cong [\Sigma^\infty_T(EG_+ \wedge X), \Omega^{r,s} tr_g H\mathbb{Z}/p]^{naive} \\
&\cong [S^{r,s} \wedge EG_+ \wedge X, tr_g H\mathbb{Z}/p]^{naive} \\
&\cong [X, F(EG_+, tr_g H\mathbb{Z}/p)]^{naive}_{r,s}
\end{aligned}$$

and in view of Proposition 4.2.6 and the fact that $i_*\Sigma^\infty_T X \cong \Sigma^\infty_{\mathbb{T}_G} X$ we may also represent motivic Borel cohomology by a genuine G-spectrum

$$\cong [X, i_* F(EG_+, tr_g H\mathbb{Z}/p)]^{genuine}_{r,s}$$

where we observe $i_*F(EG_+, X) \cong F(EG_+, i_*X)$ and recall from Definition (4.8) that $i_* \circ tr_g = (-)_{fixed}$, so that we end up with

$$\cong [X, F(EG_+, H\mathbb{Z}/p_{fixed})]^{genuine}_{r,s}.$$

□

For some arguments concerning the convergence of the motivic Borel cohomology Adams spectral sequence in Section 6.2 it will be useful to know the following.

Lemma 5.2.2. *The underlying non-equivariant spectrum $(i^*(H\mathbb{Z}/p_{fixed}))^e$ of the trivially equivariant Eilenberg-MacLane spectrum is stably equivalent to $H\mathbb{Z}/p$.*

Proof. By Lemma 4.2.3 the unit of the change of universe adjunction is a non-equivariant stable equivalence and so we conclude

$$H\mathbb{Z}/p \cong (tr_g H\mathbb{Z}/p)^e \xrightarrow{\sim} (i^* H\mathbb{Z}/p_{fixed})^e.$$

□

5.3. **Motivic co-Borel Theories.** This subsection continues and specializes the investigations from Section 4.4. We study the norm sequence

$$EG_+ \wedge F(EG_+, E) \to F(EG_+, E) \to \widetilde{EG} \wedge F(EG_+, E)$$

of (4.13) for the equivariant spectrum $E = H\mathbb{Z}/p$ from Section 5.2. For this reason we refine our abbreviations and set

$$\begin{aligned} c &:= f(H\mathbb{Z}/p) = EG_+ \wedge F(EG_+, H\mathbb{Z}/p), \\ b &:= b(H\mathbb{Z}/p) = F(EG_+, H\mathbb{Z}/p), \\ t &:= t(H\mathbb{Z}/p) = \widetilde{EG} \wedge F(EG_+, H\mathbb{Z}/p). \end{aligned}$$

We say that $c^{*,*}(X)$ and $b^{*,*}(X)$ are the ordinary motivic (co-) Borel cohomology groups of a G-space X. The ordinary motivic Tate cohomology groups $t^{*,*}(X)$ will not play any prominent role in our investigations outside of this subsection. The following results are in parallel to [Gre88b, Section 1] and will be used for computing the cohomology operations of motivic Borel cohomology in the next subsection.

Lemma 5.3.1. *The coefficient module $c_{*,*}$ of the motivic co-Borel spectrum c is isomorphic to $H_{*,*}(BG_+)$.*

Proof. We have

$$c_{*,*} = [S^{*,*}, c] = [S^{*,*}, EG_+ \wedge b]$$

and apply Lemma 4.3.9 which gives a G-equivalence

$$EG_+ \wedge b = EG_+ \wedge F(EG_+, H\mathbb{Z}) \xrightarrow{\sim} EG_+ \wedge H\mathbb{Z}$$

to obtain

$$c_{*,*} \cong [S^{*,*}, EG_+ \wedge H\mathbb{Z}].$$

Now, the equivariant homotopy classes from the push forward of the non-equivariant sphere are by Lemma 4.3.2 isomorphic to the non-equivariant homotopy classes of maps from the spheres to the (Lewis-May) fixed-points $(EG_+ \wedge H\mathbb{Z})^G$. Hence, we may use the Adams isomorphism [HKO11b, Theorem 8] and triviality of the G-action on $H\mathbb{Z}$ to finally compute

$$c_{*,*} \cong [S^{*,*}, (EG_+ \wedge H\mathbb{Z})^G] \cong [S^{*,*}, EG_+ \wedge_G H\mathbb{Z}] \cong [S^{*,*}, BG_+ \wedge H\mathbb{Z}] \cong H_{*,*}(BG).$$

□

In [HKO10] a cellular spectrum E is defined to be k-connective if $\pi_{m,n}E = 0$ for all $m, n \in \mathbb{Z}$ with $m - n < k$. Furthermore, E is called connective if it is 0-connective.

Lemma 5.3.2. *Motivic co-Borel cohomology with $\mathbb{Z}/l$ coefficients is represented by a connective spectrum.*

Proof. $B_{gm}G$ is consists of schemes U_j/G, which by [Tot99, Remark 1.4] are smooth quasi-projective schemes of some dimension d and so by [RØ08, Theorem 4.9] there is a duality isomorphism $H_{m,n}(U_j/G) \cong H^{2d-m,d-n}(U_j/G)$. Therefore, we have

$$\pi_{m,n}c \cong H_{m,n}(BG) \cong \operatorname*{colim}_j H_{m,n}(U_j/G) = 0 \text{ for } m < n,$$

by the vanishing theorem [MVW06, Theorem 3.6]. □

Proposition 5.3.3. *Let X be a G-equivariant motivic spectrum. From* (4.13) *we obtain a long exact sequence*

$$\ldots \to c^{*,*}(X) \to b^{*,*}(X) \to t^{*,*}(X) \to c^{*,*}(\Sigma^{1,0}X) \to \ldots$$

Corollary 5.3.4. *If X is cellularly G-free, then $c^{*,*}(X) \cong b^{*,*}(X)$.*

Proof. This is an immediate consequence of the above proposition and the vanishing of Tate theories in the case of a cellularly free G-action by Lemma 4.4.2. □

Corollary 5.3.5. *If X is non-equivariantly weakly equivalent to* Spec(k)*, then $c^{*,*}(X) \cong t^{*,*}(X)$.*

Proof. Since X is non-equivalently contractible Lemma 4.3.9 implies that $EG_+ \wedge X$ is equivariantly contractible and therefore Borel cohomology vanishes on X. □

Proposition 5.3.6. *We have isomorphisms*

$$c^{*,*}c \cong b^{*,*}c \cong b^{*,*}b.$$

Proof. The spectrum $c = EG_+ \wedge b$ is by Lemma 4.3.9 stably equivalent to $EG_+ \wedge H\mathbb{Z}/p_{tr}$, so we can see that c is a cellularly G-free spectrum. Thus, we may apply Corollary 5.3.4 to obtain an isomorphism

$$c^{*,*}c \xrightarrow{\cong} b^{*,*}c$$

The second isomorphism can be explained by smashing the cofiber sequence (4.11) with b to acquire the cofiber sequence

$$c = EG_+ \wedge b \to b \to \widetilde{EG} \wedge b.$$

Nonequivariantly $\widetilde{EG} \wedge b$ is still contractible, cf. Lemma 4.3.8, and therefore by Corollary 5.3.5 the Borel cohomology of c is isomorphic to that of b. □

5.4. **Stable Operations of Mod 2 Borel Cohomology.** This subsection starts out with a discussion of the algebraic structures on motivic Borel cohomology. For example, the ring structure on usual motivic cohomology gives a ring structure on Borel cohomology. Furthermore, there are actions of $H^{*,*}(BG)$ and the Steenrod algebra on motivic Borel cohomology. We recall Greenlees non-motivic computation of the stable operations and proceed to give an analogous result. Technically most important is that the graded algebra of stable operations is still bounded below, which is used in the next section to construct the spectral sequence. Computationally, the bigger picture of Theorem 5.4.11 is of course essential. From this subsection on, we restrict our attention to cyclic groups $\mathbb{Z}/l$ and assume that the base field contains a primitive l-th root of unity. This assumption enables us to use Voevodsky's computations concerning $B\mu_2$ below, while talking about $B\mathbb{Z}/2$.

The multiplication from (5.1) given by morphisms $\mu : K(p, q, R) \wedge K(p', q', R) \to K(p+p', q+q', R)$ on Eilenberg-MacLane spectra further induces a multiplicative structure on motivic cohomology:

Theorem 5.4.1. *Let R be a commutative ring. Motivic cohomology $H^{*,*}(-, R)$ has the structure of an associative and graded commutative R-algebra.*

Proof. [Voe03, Theorem 2.2.] □

Corollary 5.4.2. *Motivic Borel cohomology $b^{*,*}(-, R)$ has the structure of an associative and graded commutative R-algebra.*

By use of the above multiplicativity and the induced map

$$p^* : H^{*,*}(BG, R) \to H^{*,*}(EG_+ \wedge_G X, R)$$

we have $b^{*,*}(\mathrm{Spec}(k), R)$-module structure on $b^{*,*}(X, R)$. The base ring $b^{*,*}$ has been computed in various cases by Voevodsky:

Proposition 5.4.3 ([Voe03, Theorem 6.10]). *Let G be cyclic of order l and k be a field of characteristic zero containing a primitive l-th root of unity. There is an isomorphism of $H^{*,*}$-modules*

$$b^{*,*}(\mathrm{Spec}(k), \mathbb{Z}/l) \cong H^{*,*}(\mathrm{Spec}(k), \mathbb{Z}/l)[a, b]/\sim$$

where $|a| = (1, 1), |b| = (2, 1)$, and

$$\sim \text{ is given by the relation } \begin{cases} a^2 = 0, & \text{if } l \neq 2 \\ a^2 = \tau b + \rho a, & \text{if } l = 2. \end{cases}$$

This can be extended to give a computation of motivic Borel cohomology of spaces with a trivial G-action. For convenience we also state the case of a free G-action in the following lemma.

Lemma 5.4.4. *Let X be in $G\mathcal{S}\mathrm{m}/k$.*

(1) If X has a free G-action and the quotient scheme X/G exists, then the morphism
$$(EG_{gm} \times X)/G \to X/G$$
to the represented simplicial presheaf is an $\mathbb{A}^1$-local weak equivalence.

(2) If X has a trivial G-action, then the morphism
$$(EG_{gm} \times X)/G \to BG_{gm} \times X$$
is an $\mathbb{A}^1$-local weak equivalence.

Proof.

(1) Since X has a free G-action the projections $(W_i \times X)/G \to X/G$ are vector bundles [Voe03, Proof of Lemma 6.3].

(2) In this case the quotients $(W_i \times X)/G$ are isomorphic to $W_i/G \times X$.

□

For the case of a trivial G-action the above lemma gives a description of the Borel Cohomology of X.

Proposition 5.4.5. *Let G be cyclic of order l and let $X \in G\mathcal{S}\mathrm{m}/k$ have a trivial G-action and k be a field of characteristic zero. There is an isomorphism of $H^{*,*}(X, G)$-modules*

$$b^{*,*}(X, \mathbb{Z}/l) \cong H^{*,*}(X, \mathbb{Z}/l)[[a, b]]/\sim$$

where $|a| = (1, 1), |b| = (2, 1)$, and

$$\sim \text{ is given by the relation } \begin{cases} a^2 = 0, & \text{if } l \neq 2 \\ a^2 = \tau b + \rho a, & \text{if } l = 2. \end{cases}$$

Proof. This is [Voe03, Theorem 6.10] combined with the lemma directly above.

□

So far we have noticed the ring structure of Borel cohomology and the action of the coefficient ring. Another essential ingredient to the stable operations of Borel cohomology is given by the natural action of the motivic Steenrod algebra.

Definition 5.4.6. The motivic Steenrod algebra $A^{*,*} = A^{*,*}(k, \mathbb{Z}/l)$ is defined to be the subalgebra of the algebra of bistable cohomological operations on motivic cohomology with coefficients in $\mathbb{Z}/l$ generated by

- the Bockstein operation $\beta : H^{p,q}(-, \mathbb{Z}/l) \to H^{p+1,q}(-, \mathbb{Z}/l)$ (cf. [Voe03, Section 8]) and
- the power operations $P^i : H^{p,q}(-, \mathbb{Z}/l) \to H^{p+2i(l-1),q+i(l-1)}(-, \mathbb{Z}/l), i \geq 0$ (cf. [Voe03, Section 9]).

This is essentially Voevodsky's definition from [Voe03], where he explicitly postpones the discussion of the following question. Is the motivic Steenrod algebra equal to the algebra of *all* bistable cohomological operations and does it coincide with the endomorphisms of the corresponding Eilenberg-MacLane spectrum? A positive answer to this question in the case we are anyhow already restricted to, that is $char(k) = 0$ and $\mathbb{Z}/l$ coefficients, was published relatively recently in [Voe10].

We will need the following two propositions to deal with the stable operations of motivic Borel cohomology later.

Proposition 5.4.7. *The motivic Steenrod algebra is a free $H^{*,*}$-module. Moreover, the admissible monomials*

$$P^I = \beta^{e_0} P^{s_1} \beta^{e_1} \ldots P^{s_k} \beta^{e_k}$$

where $I = (e_0, s_1, e_1, \ldots s_l, e_k)$ with $e_i \in \{0, 1\}, 0 \leq s_i$ and $s_i \geq l \cdot s_{i+1} + e_i$, form a basis.

Proof. This is a combination of Lemma 11.1 and Corollary 11.5 of [Voe03]. □

Proposition 5.4.8 ([Voe10, Proposition 3.49])**.** *Let k be a field of characteristic zero. Then the motivic Steenrod algebra $A^{*,*}(k, \mathbb{Z}/l)$ is the algebra of all bistable cohomological operations on motivic cohomology with $\mathbb{Z}/l$ coefficients. There is an isomorphism*

$$A^{*,*}(k, \mathbb{Z}/l) \cong H\mathbb{Z}/l^{*,*} H\mathbb{Z}/l.$$

All the statements about algebraic structure and actions on Borel cohomology have analogs in classical topology. It is a computation of Greenlees [Gre88b, Theorem 2.7] that this already gives all the stable cohomology operations:

Theorem 5.4.9. *The algebra b^*b of stable cohomology operations on* mod p *Borel cohomology is given by the twisted tensor product*

$$b^*b \cong H^*(BG_+, \mathbb{Z}/p) \widetilde{\otimes} \mathcal{A}_p.$$

Remark 5.4.10. The above isomorphism is even an algebra isomorphism instead of only one of modules. Therefore the tensor product is decorated as $\widetilde{\otimes}$ to denote a twisted tensor product. The underlying module structure of the twisted tensor product is that of the usual tensor product, but Greenlees further investigated how coefficients of Borel cohomology and the Steenrod squares and powers interact multiplicatively by introducing Wu-classes [Gre88b, Lemma 2.2], which leads to the twist.

A sketch of a proof of 5.4.9. To compute b^*b is the same as to compute b^*c by a result analogous Proposition 5.3.6. Greenlees proceeds by proving that evaluation induces a G-equivalence $EG_+ \wedge F(EG_+, X) \to EG_+ \wedge X$ [Gre88b, Lemma 2.8] to have $b^*b \cong b^*(HR_{triv} \wedge EG_+)$. Then applying a Künneth formula there is a group isomorphism $b^*b \cong b^* \otimes A_p^*$. □

Now we are going to compute the stable operations on mod p motivic Borel cohomology in a way similar to the one just sketched. The goal is to give a proof of the following theorem.

Theorem 5.4.11. *There is an isomorphism*

$$b^{*,*}b = H^{*,*}(BG_+, \mathbb{Z}_p) \otimes \mathcal{A}_p$$

of $H^{*,*}$-modules between the stable cohomology operations of mod p *motivic Borel cohomology and its coefficients tensored with the* mod p *motivic Steenrod algebra.*

To be able to apply a Künneth formula we introduce the extended translation of Joshua's work on a Künneth theorem [Jos01] to motivic homotopy theory given by Dugger and Isaksen [DI05, Section 8].

Definition 5.4.12. The class of *linear motivic spectra* is the smallest class of motivic spectra that consists of all the spheres $S^{p,q}$, is closed under stable weak equivalence, and satisfies the following two conditions:

(1) If two of the objects in a homotopy cofiber sequence are linear, then so is the third.
(2) Let $\xi : E \to X$ be a vector bundle in $\mathcal{S}\mathrm{m}/k$. The suspension spectrum $\Sigma^\infty Th(\xi)$ of the Thom space of ξ is linear if and only if $\Sigma^\infty X_+$ is linear.

Example 5.4.13. All the projective spaces $\mathbb{P}^n$ give linear suspension spectra. We see immediately that $\Sigma^\infty \mathbb{P}^1 \simeq S^{2,1}$ is linear. By induction and use of the homotopy cofiber sequence

$$\mathbb{P}^{n-1} \to \mathbb{P}^n \to S^{2n,n}$$

[MV99, Cor. 3.2.18] all other projective spaces give linear motivic spectra as well.

We state a version of Dugger and Isaksen's Künneth theorem [DI05, Theorem 8.12].

Theorem 5.4.14. *Let E, X, Y be motivic spectra, such that E is a motivic ring spectrum that 'satisfies Thom isomorphism' and X is a linear motivic spectrum. Then there is a strongly convergent Künneth spectral sequence*

$$\left(\mathrm{Tor}_{p_1}^{E^{*,*}}(E^{*,*}(X), E^{*,*}(Y))\right)^{p_2,q} \Rightarrow E^{p_2-p_1,q}(X \wedge Y).$$

Remark 5.4.15.

(1) That a motivic ring spectrum E satisfies Thom isomorphism means that for every vector bundle ξ of some rank n over a smooth scheme X, there is a cohomology class $u \in E^{2n,n}(Th(\xi))$ (a 'Thom class') such that

$$E^{p,q}(X) \to E^{p+2n,q+n}(Th(\xi)), \qquad v \mapsto \xi^*(v) \cup u$$

is an isomorphism.

(2) As remarked in loc. cit. the Eilenberg-MacLane spectrum $H\mathbb{Z}$ is known to satisfy Thom isomorphism [Voe03, Proposition 4.3]. By work of Panin, Pimenov and Röndigs [PPR08] such an orientation is equivalent to a morphism $MGL \to H\mathbb{Z}$, which implies a Thom isomorphism for $H\mathbb{Z}/p$ by composition with the induced morphism $H\mathbb{Z} \to H\mathbb{Z}/p$.

(3) Also remarked in loc. cit. is that the above spectral sequence collapses to give a Künneth isomorphisms

$$E^{*,*}(X) \otimes_{E^{*,*}} E^{*,*}(Y) \cong E^{*,*}(X \wedge Y)$$

if the higher Tor's vanish, e.g. if $E^{*,*}(X)$ is a free $E^{*,*}$-module.

Lemma 5.4.16. *Let $G = \mathbb{Z}/l$. Then $BG_{gm} = \mathrm{colim}_n W_n$ (as in Definition 5.1.7) and W_n is isomorphic to $E(\lambda^{\otimes l})^\times$ the complement of the zero section of the l-th power of the canonical line bundle over $\mathbb{P}^{n-1}$.*

Proof. This is essentially [Voe03, Lemma 6.3], respectively taken from the proof of the Lemma there. □

Corollary 5.4.17. *Let $G = \mathbb{Z}/l$. Then $\Sigma^\infty BG_{gm_+}$ is a sequential colimit of linear motivic spectra.*

Proof. We show that the spectra $\Sigma^\infty E(\lambda^{\otimes l})^\times_+$ are linear. By the closedness property of linear spectra and the homotopy cofiber sequence

$$E(\lambda^{\otimes l})^\times \to E(\lambda^{\otimes l}) \to Th(\lambda^{\otimes l})$$

it is equivalent to observe that $E(\lambda^{\otimes l})$ and $Th(\lambda^{\otimes l})$ give linear suspension spectra. Vector bundles are $\mathbb{A}^1$-local weak equivalences, so $E(\lambda^{\otimes l})$ is weakly equivalent to $\mathbb{P}^{n-1}$ and hence gives a linear spectrum by Example 5.4.13. The linearity of $\Sigma^\infty \mathbb{P}^{n-1}_+$ also implies the linearity of the Thom space $Th(\lambda^{\otimes l})$. □

Proof of Theorem 5.4.11. Starting with the isomorphism from Proposition 5.3.6 we compute

$$\begin{aligned} b^{*,*}b &\cong b^{*,*}c \\ &\cong [EG_+ \wedge H\mathbb{Z}/p, F(EG_+, H\mathbb{Z}/p)]^G \\ &\cong [EG_+ \wedge EG_+ \wedge H\mathbb{Z}/p, H\mathbb{Z}/p]^G \end{aligned}$$

by Lemma 4.3.9 this is

$$\cong [EG_+ \wedge H\mathbb{Z}/p, H\mathbb{Z}/p]^G$$

which is now the situation of Proposition 4.2.6, so we may continue with

$$\begin{aligned} &\cong [EG_+ \wedge H\mathbb{Z}/p_{tr}, H\mathbb{Z}/p_{tr}]^{naive} \\ &\cong \pi_0 \mathrm{s}\mathcal{S}\mathrm{et}\left(\operatorname*{colim}_{n\geq 0} \widetilde{U_{n}}_+ \wedge \operatorname*{colim}_i tr_g(\widetilde{X_i}), tr_g(H\mathbb{Z}/p^f)\right) \end{aligned}$$

since $tr_g(H\mathbb{Z}/p)$ is still stably fibrant, compute $tr_g(X_n)^H \xrightarrow{\sim} \Omega_T(tr_g X_{n+1})^H$ for a stably fibrant X,

$$\cong \pi_0 \lim_{n\geq 0} \lim_i \mathrm{s}\mathcal{S}\mathrm{et}\left(\widetilde{U_n \times X_{i,tr}}/(\widetilde{U_n} \times x_{i,o}), tr_g(H\mathbb{Z}/p^f)\right)$$

where the quotients of U_n times something G-trivial exist and which is therefore by Lemma 5.1.11 the same as

$$\begin{aligned} &\cong \pi_0 \lim_{n\geq 0} \lim_i \mathrm{s}\mathcal{S}\mathrm{et}\left(\widetilde{U_n/G}_+ \wedge \widetilde{X_i}, H\mathbb{Z}/p^f\right) \\ &\cong [BG_+ \wedge H\mathbb{Z}/p, H\mathbb{Z}/p] \end{aligned}$$

having arrived at this point we may switch to a motivic cohomology notation for convenience

$$\cong H^{*,*}(BG_+ \wedge H\mathbb{Z}/p, \mathbb{Z}/p)$$

where we now may use the corollary above to express BG as a sequential colimit of linear motivic spectra

$$\cong H^{*,*}(\operatorname*{colim}_n (W_n \wedge H\mathbb{Z}/p), \mathbb{Z}/p)$$

and the corresponding $\lim^1$ terms vanish by a Mittag-Leffler argument, because of Proposition 5.1.8 and the fact of Proposition 5.4.7 that $\mathcal{A}_p$ is a free $H^{*,*}$-module, so that we have

$$\cong \lim_n H^{*,*}(W_n \wedge H\mathbb{Z}/p, \mathbb{Z}/p),$$

which by the Künneth theorem 5.4.14 may be written as

$$\cong \lim_n \left(H^{*,*}(W_n, \mathbb{Z}/p) \otimes_{H^{*,*}} A_p \right).$$

Again, we use that A_p is a free and hence flat $H^{*,*}$-module by 5.4.7, so tensoring commutes with inverse limits

$$\cong \left(\lim_n H^{*,*}(W_n, \mathbb{Z}/p) \right) \otimes_{H^{*,*}} A_p$$

and finally the $\lim^1$-argument (cf. [Voe03, Corollary 6.2]) from above allows us to compute

$$\cong H^{*,*}(BG_+, \mathbb{Z}/p) \otimes_{H^{*,*}} A_p = b^{*,*} \otimes A_p.$$

□

6. The Borel Cohomology Adams Spectral Sequence

In this section we construct an Adams spectral sequence based on motivic Borel cohomology. Adams spectral sequences have proven to be a powerful tool in stable homotopy theory. They serve the purpose of extracting homotopical information from homological algebra, sort of inverse to the Hurewicz map

$$[X, Y] \to \mathrm{Hom}_{E^{*,*}E}(E^{*,*}Y, E^{*,*}X), \quad f \mapsto f^*.$$

In non-equivariant motivic homotopy theory the techniques of applying Adams spectral sequences have been studied from an early point on, cf. [Mor99], and remained to be of interest since then [DI09, HKO10, HKO11a]. Recall from the preamble of Subsection 5.4 that we assume a base field of characteristic zero, which contains a primitive l-th root of unity.

6.1. Construction.

The construction of an E-cohomology Adams spectral sequences is usually done by finding a 'geometric projective resolution' of the cohomology E^*Y of some space Y. We prepare this construction with a technical lemma, which gives some first restrictions on the space Y.

Lemma 6.1.1. *Let $(p_i, q_i)_{i \in I}$ be a sequence of integers such that for every i there are only finitely many $j \in I$ with $p_i > p_j$ and $q_i > q_j$. Then*

(1) the canonical algebraic map

$$\bigoplus_i c^{*,*}(S^{p_i,q_i} c) \to \prod_i c^{*,*}(S^{p_i,q_i} c)$$

is an isomorphism and

(2) *the canonical geometric map*

$$\nu : \bigvee_i S^{p_i,q_i} c \to \prod_i S^{p_i,q_i} c$$

is a stable equivalence.

Proof.

(1) We look at the bidegree (p,q) of the domain and codomain of the map and compute that

$$\left(\prod_i c^{*,*}(S^{p_i,q_i}c)\right)^{(p,q)} = \prod_i c^{p,q}(S^{p_i,q_i}c) = \prod_i c^{p-p_i,q-q_i}c$$

We have seen in Proposition 5.3.6 that $c^{*,*}c \cong b^{*,*}b$. Hence, from Theorem 5.4.11 it follows that $c^{*,*}c$ is bounded below, since both factors of the graded tensor product, the motivic Steenrod algebra and the coefficients $b^{*,*}$, are bounded below. The first by Proposition 5.4.7 and the latter by Proposition 5.4.3. So the condition on the set $(p_i,q_i)_i$ of bidegree ensures that this product consists only of finitely many non-zero factors and therefore the canonical morphism from the direct sum to the direct product is an isomorphism.

(2) To show that the canonical geometric morphism is an equivariant stable weak equivalence, we use the characterization of Proposition 4.3.11. For a subgroup $e < H \leq G$, the map $\Phi^H(\nu)$ on geometric H-fixed points is trivial since the action of G on c is free and hence it suffices to show that ν is a stable weak equivalence when forgetting the group action. Recall that by Lemma 4.3.9 and Lemma 5.2.2 there is a stable weak equivalence $c \to EG_+ \wedge H\mathbb{Z}/l$, which means that non-equivariantly c is equivalent to the Eilenberg-MacLane spectrum, via the composition

$$c \xrightarrow{ev} EG_+ \wedge H\mathbb{Z}/l \xrightarrow{EG_+ \to S^0} H\mathbb{Z}/l.$$

So we obtain a square

$$\begin{tikzcd}
\bigoplus_i \pi_{p-p_i,q-q_i}(c) \arrow[r, "\Phi^e(\nu)_*"] \arrow[d, "\cong"'] & \prod_i \pi_{p-p_i,q-q_i}(c) \arrow[d, "\cong"] \\
\bigoplus_i \pi_{p-p_i,q-q_i}(H\mathbb{Z}/l) \arrow[r] & \prod_i \pi_{p-p_i,q-q_i}(H\mathbb{Z}/l)
\end{tikzcd}$$

of bigraded stable homotopy groups and by [HKO10, Lemma 6], saying that $H\mathbb{Z}/l$ is of finite type, the lower horizontal morphism is an isomorphism and hence also the upper horizontal one. We conclude with [Jar00, Lemma 3.7] that the $\pi_{*,*}$-isomorphism $\Phi^e(\nu)$ is a stable weak equivalence.

□

Adams resolution and Adams tower. As usual for Adams style spectral sequences we aim to geometrically realize a free resolution

$$\ldots \to P_2 \to P_1 \to P_0 \to b^{*,*}Y$$

of $c^{*,*}c$-modules. Let Y be a cellularly G-free spectrum such that $b^{*,*}Y$ is bounded below and finite in each degree and by Proposition 5.4.11 the same is true for $b^{*,*}b = c^{*,*}c$. Hence, we may insist on this property for an algebraic resolution as $P_\bullet$. Now we inductively produce spectra Q_i, such that setting $P_i = b^{*,*}Q_i$ gives a resolution as above.

Let $Y_0 := Y$. We construct a map which hits every generator α_j of $b^{*,*}Y_0 = c^{*,*}Y_0$ by mapping the identity element of a shifted copy of $c^{*,*}c$ to it. This way, we obtain a morphism

$$\prod_j c(S^{p_j,q_j}c) \xrightarrow{\cong} \bigoplus_j c(S^{p_j,q_j}c) \to b^{*,*}Y_0,$$

where the first map is the isomorphism given by Lemma 6.1.1. This composition corresponds by the Yoneda lemma to a class in $[Y_0, \wedge_j S^{p_j,q_j}c]$, which we can lift to a map $r_0 : Y_0 \to Q_0$, where Q_0 is defined to be a fibrant replacement of the wedge of suspensions of c. We focus on the indicated part of the fiber sequence for r_0:

$$\begin{tikzcd}
\cdots \arrow[r] & \Omega Q_0 \arrow[r] & Y_1 \arrow[d] & & \\
& & Y_0 \arrow[r, "r_0"] & Q_0 \arrow[r] & \Sigma Y_1 \arrow[r] & \cdots
\end{tikzcd}$$

Note that Y_1 is the homotopy fiber of a map of cellularly G-free spectra and therefore it is cellularly G-free itself (cf. [DI05, Lemma 2.2]). Furthermore, the induced long exact sequence in motivic Borel cohomology guarantees that $b^{*,*}Y_1$ is also bounded below. Inductively proceeding as above gives a sequence of fiber sequences in the stable equivariant motivic model category, which we line up as usual to build an Adams tower of Y:

(6.1)
$$\begin{tikzcd} \vdots \arrow[r] & \vdots \arrow[r] \arrow[d] & \vdots \\ \Omega Q_1 \arrow[r] & Y_2 \arrow[r, "r_2"] \arrow[d] & Q_2 \\ \Omega Q_0 \arrow[r] & Y_1 \arrow[r, "r_1"] \arrow[d] & Q_1 \\ & Y_0 \arrow[r, "r_0"] & Q_0. \end{tikzcd}$$

As we work in a stable model category we can change to an Adams resolution without losing any information. Define $Y^i := \Sigma^i Y_i$ and $Q^i := \Sigma^i Q_i$ and line up the (co-) fiber sequences as:

(6.2)
$$\begin{tikzcd} Y \arrow[rr, "r_0"] & & Q^0 \arrow[rr, dashed] \arrow[dr] & & Q^1 \arrow[rr, dashed] \arrow[dr] & & \cdots \\ & & & Y_1 \arrow[ur, "r_1"'] & & Y_2 \arrow[ur, "r_2"'] & \end{tikzcd}$$

Here the dotted arrows are defined as the composition of the solid ones. It is now easy to see that the top line gives a free resolution in Borel cohomology.

Before we roll up the resolution to an exact couple, we make sure that a morphism of spectra induces up to homotopy a morphism of towers.

Lemma 6.1.2. *Let $f : Y' \to Y$ be a given map of equivariant motivic spectra and let $\{Q_i', Y_i'\}_i$ be a c-Adams tower of Y'. Then there exist homotopy classes $\varphi_i \in [Y_i', Y_i]$ such that the diagram*

$$\begin{tikzcd} Y' \arrow[d, "f"] & Y_1' \arrow[l] \arrow[d, "\varphi_1"] & Y_2' \arrow[l] \arrow[d, "\varphi_2"] & \cdots \arrow[l] \\ Y & Y_1 \arrow[l] & Y_2 \arrow[l] & \cdots \arrow[l] \end{tikzcd}$$

in the homotopy category commutes.

Proof. Let $\varphi_0 := f$. Suppose we are already given φ_i and all squares left from φ_i commute. The map $r_i : Y_i \to Q_i$ is equivalent to a family of cohomology classes $\{x_\alpha\}_\alpha \in c^{*,*}Y_i$. Since the map $r_i' : Y_i' \to Q_i'$ is taken to a surjective map under $c^{*,*}$ there exists a family $\{y_\alpha\}_\alpha \in c^{*,*}Q_i'$ such that $(r_i')^*(y_\alpha) = \varphi_i^*(x_\alpha)$ for all α. Define $\psi_{i+1} : Q_i' \to Q_i$ to be the homotopy class corresponding to the family

$\{y_\alpha\}_\alpha$ and define $\varphi_i : Y'_{i+1} \to Y_{i+1}$ to be the class induced by ψ_i and φ_i on the homotopy fibers. □

The exact couple and its spectral sequence. The next step in our construction of the spectral sequence is to recollect the spectra of (6.1) in an exact couple of tri-graded abelian groups. For this, we apply $[S^{p,q} \wedge X, -]$ for some equivariant spectrum X and sum up groups in a manner indicated by the following diagram.

(6.3)
$$\begin{tikzcd}
{[S^{p,q} \wedge X, Y_s]} \arrow[rr, "i"] & & {[S^{p,q} \wedge X, Y_{s-1}]} \arrow[dl, "j"] \\
{[\Omega S^{p,q} \wedge X, Y_s]} & {[S^{p,q} \wedge X, Q_{s-1}]} \arrow[l, "k"] &
\end{tikzcd}$$

That is, we define

$$D = \bigoplus_{p,q,s} [S^{p-s,q} \wedge X, Y_s]$$

and

$$E = \bigoplus_{p,q,s} [S^{p-s,q} \wedge X, Q_s]$$

and obtain an exact couple with differentials $i : D \to D, j : D \to E, k : E \to D$ of respective tri-degrees $(-1,-1,0), (0,0,0)$, and $(1,0,0)$. This exact couple gives an spectral sequence with E_1-page defined by E and differential $d = j \circ k$ with degree $(1,0,0)$, i.e.

$$d_1 : E_1^{s,p,q} \to E_1^{s+1,p,q}.$$

Theorem 6.1.3. *Let X, Y be equivariant motivic spectra, such that Y is cellularly G-free and $b^{*,*}Y$ is bounded below and finite in each degree. There is a tri-graded spectral sequence with*

$$E_2^{s,p,q} = \mathrm{Ext}^{s,(p,q)}_{c^{*,*}c}\,(c^{*,*}Y, c^{*,*}X)$$

and with target group $[X, Y]^G_*$.

Proof. Most of what the theorem states has been constructed and explained above. Still to show is just the computation of the E_2-page. First, we rewrite the E_1-page as

$$\begin{aligned} E_1^{s,p,q} &\cong [S^{p,q} \wedge X, S^{s,0} Q_s] \\ &\cong [S^{p,q} \wedge X, S^{s,0} \wedge \bigvee_{j \in J_s} S^{(p_j,q_j)} c] \\ &\cong [S^{p,q} \wedge X, \prod_{j \in J_s} S^{s+p_j,q_j} Q_s]. \end{aligned}$$

Recall from the construction of Q_s that the index set J_s satisfies the hypothesis of Lemma 6.1.1, so clearly the shifted set of bidegrees $J_s + (s, 0)$ does as well and we continue with

$$\begin{aligned} [S^{p,q} \wedge X, \prod_{j \in J_s} S^{s+p_j,q_j} c] &\cong \prod_{j \in J_s} [S^{p,q} \wedge X, S^{s+p_j,q_j} c] \\ &\cong \prod_{j \in J_s} c^{s+p_j,q_j} (S^{p,q} \wedge X) \\ &\cong \prod_{j \in J_s} \mathrm{Hom}_{c^{*,*}c} \left(c^{*,*}(S^{s+p_j,q_j} c), c^{*,*}(S^{p,q} \wedge X) \right) \\ &\cong \mathrm{Hom}_{c^{*,*}c} \left(\bigoplus_{j \in J_s} c^{*,*}(S^{s+p_j,q_j} c), c^{*,*}(S^{p,q} \wedge X) \right) \\ \text{(again by Lemma 6.1.1)} &\cong \mathrm{Hom}_{c^{*,*}c} \left(\prod_{j \in J_s} c^{*,*}(S^{s+p_j,q_j} c), c^{*,*}(S^{p,q} \wedge X) \right) \\ &\cong \mathrm{Hom}_{c^{*,*}c} \left(c^{*,*}(\bigvee_{j \in J_s} S^{s+p_j,q_j} c), c^{*,*}(S^{p,q} \wedge X) \right) \\ &\cong \mathrm{Hom}_{c^{*,*}c} \left(c^{*,*}(S^{s,0} Q_s), c^{*,*}(S^{p,q} \wedge X) \right) \\ &= \mathrm{Hom}^{p,q}_{c^{*,*}c} \left(c^{*,*}(S^{s,0} Q_s), c^{*,*}(X) \right) \end{aligned}$$

Tracing the differential d_1 back to the Adams resolution (6.2) we see that d_1 is induced by the composition that gives the differentials ∂ of the projective resolution of Y after taking motivic Borel cohomology. By naturality in the above transformations we can thus rewrite the differential as

$$d_1 = \partial^* : \mathrm{Hom}^{p,q}_{c^{*,*}c} \left(c^{*,*}(S^{s,0} Q_s), c^{*,*}(X) \right) \to \mathrm{Hom}^{p,q}_{c^{*,*}c} \left(c^{*,*}(S^{s+1,0} Q_{s+1}), c^{*,*}(X) \right).$$

Since the E_2-page is the cohomology of this complex, we obtain exactly

$$E_2^{s,p,q} = \mathrm{Ext}^{s,(p,q)}_{c^{*,*}c} \left(c^{*,*} Y, c^{*,*} X \right).$$

□

6.2. **Convergence Issues.** The target $[X,Y]^G_*$ of the spectral sequence from Theorem 6.1.3 is filtered as

$$\ldots \subseteq F^r[X,Y]^G_* \subseteq \ldots \subseteq F^1[X,Y]^G_* \subseteq F^0[X,Y]^G_* = [X,Y]^G_*,$$

where $F^r[X,Y]^G_*$ is the image of the r-th power of the morphism i in the exact couple used to construct the spectral sequence, i.e.

$$F^r[X,Y]^G_{p,q} = \mathrm{Image}\left([S^{p,q}X, Y_r]^G \xrightarrow{(i^r)_*} [S^{p,q}X,Y]^G\right).$$

Lemma 6.2.1. *This filtration does not depend on the choice of the c-Adams tower over Y.*

Proof. Assume that $\{Y'_s \xrightarrow{j_s} Y'_{s-1}, Q'_s\}_{s>0}$ is a second c-Adams tower over $Y'_0 := Y$. The identity on Y together with Lemma 6.1.2 gives morphisms

$$(\varphi_r)_* : [S^{p,q}X, Y_r]^G \to [S^{p,q}X, Y'_r]^G \quad (\text{resp. } (\psi_r)_* : [S^{p,q}X, Y'_r]^G \to [S^{p,q}X, Y_r]^G),$$

such that $(i^r)_* = (j^r \circ \varphi_r)_*$ (resp. $(j^r)_* = (i^r \circ \psi_r)_*$). Thus, the two corresponding filtrations mutually contain each other and are therefore equal. □

The first step in our investigation of convergence properties is to check for conditional convergence [Boa99, Definition 5.10], so for the exact couple (6.3) with $D_s = [X, Y_s]^G_*$ we have to investigate the limit

$$D_\infty = \lim_s(\ldots \to D_{s+1} \xrightarrow{i_s} D_s \to \ldots \to D_0)$$

and the derived limit $RD_\infty = \lim^1_s\{D_s, i_s\}$. Under our assumptions on Y and G by now, there is no reason for these groups to vanish. We will discuss stronger assumptions on Y and G to change this below, but for the moment we continue without further restrictions and manipulate the exact couple instead.

To assure the needed vanishing let Z denote the homotopy limit $\mathrm{holim}_s Y_s$, and for each $s \geq 0$ let W_s be defined as the homotopy cofiber

$$Z \xrightarrow{pr_s} Y_s \to W_s$$

and consider the homotopy commutative diagrams

(6.4)
$$\begin{tikzcd}
Z \arrow[r, equal] \arrow[d] & Z \arrow[r] \arrow[d] & * \arrow[d] \\
Y_s \arrow[r] \arrow[d, "c_s"] \arrow[dr, phantom, "\spadesuit"] & Y_{s-1} \arrow[r] \arrow[d, "c_{s-1}"] \arrow[dr, phantom, "\clubsuit"] & Q_{s-1} \arrow[d, "\varphi_{s-1}", "\sim"'] \\
W_s \arrow[r, "!_s"] & W_{s-1} \arrow[r, "h_{s-1}"] & Q'_{s-1}
\end{tikzcd}$$

where all three columns and the upper two rows are cofiber sequences by definition, so by the Four Cofibrations Lemma [Doe98, Lemma 1.4] the lower row is a cofiber sequence as well.

We define a second exact couple

$$\begin{tikzcd} \widetilde{D} \arrow[rr, "\widetilde{i}"] & & \widetilde{D} \arrow[dl, "\widetilde{j}"] \\ & E \arrow[ul, "\widetilde{k}"] & \end{tikzcd}$$

where $\widetilde{D}$ is the direct sum over all the $\widetilde{D}^{s,p,q} := [S^{p-s,q}X, W_s]^G$ and

- $\widetilde{i}$ is induced by the maps $!_s$,
- $\widetilde{j}$ is given by $(\varphi_s)_*^{-1} \circ (h_s)_*$ and
- $\widetilde{k}$ is the composition of $(\varphi_s)_*$ and the obvious connecting map of long exact sequence induced by the cofiber sequence $W_s \to W_{s-1} \to Q_{s-1}$.

Let $F := \oplus_{s,p,q}[S^{p-s}X, Z]^G$ and consider also the long exact sequence induces by the left column $Z \to Y_s \to W_s$ of the diagram (6.4). Altogether, we may now put all these objects and morphisms in place to obtain the following so-called Rees system:

(6.5)
$$\begin{tikzcd} & D \arrow[rr, "i"] \arrow[dd, "\alpha"] & & D \arrow[dl, "j"] \arrow[dd, "\alpha"] & \\ F \arrow[ur, "\gamma"] & & E \arrow[ul, "k"] \arrow[dl, "\widetilde{k}"] & & F \arrow[ul, "\gamma"] \\ & \widetilde{D} \arrow[ul, "\beta"] \arrow[rr, "\widetilde{i}"] & & \widetilde{D} \arrow[ul, "\widetilde{j}"] \arrow[ur, "\beta"] & \end{tikzcd}$$

The conditions for a diagram of this shape to form a Rees system are $\alpha \circ \widetilde{i} = i \circ \alpha, \widetilde{j} \circ \alpha = j, \widetilde{k} = \alpha \circ k$ and that α, β, γ give a long exact sequence. The latter is the case here, because α, β and γ are induced by the left column cofiber sequence of (6.4). The identities $\widetilde{j} \circ \alpha = j, \widetilde{k} = \alpha \circ k$ holds because of the homotopy commutativity of the square ♣ in (6.4) and finally $\alpha \circ \widetilde{i} = i \circ \alpha$ is encoded in ♠.

Lemma 6.2.2. *The motivic Borel cohomology spectral sequence of Theorem 6.1.3 is conditionally convergent to* $[X, Y/(\mathrm{holim}_s Y_s)]^G_*$.

Proof. The Rees system above tells us by [McC85, Corollary 3.5] that the spectral sequences associated to the exact couples (D, E, i, j, k) and $(\widetilde{D}, E, \widetilde{i}, \widetilde{j}, \widetilde{k})$ are identical. By Boardman's Theorem 4.5 in [Boa99] we have $\widetilde{D}_\infty = 0 = R\widetilde{D}_\infty$ and may conclude the conditional convergence. □

We finish this subsection with a concluding remark about how the convergence result can probably be improved for calculational purposes.

Remark 6.2.3.

(1) If $c^{*,*}Y$ is finitely generated $c^{*,*}c$-module, then the same is true for a nice resolution $P_\bullet$. If moreover $c^{*,*}X$ is locally finite, then $\mathrm{Hom}_{c^{*,*}c}(c^{*,*}Y, c^{*,*}X)$ is finite and all the ext-groups on the second page are finite as well. Therefore a Mittag-Leffler argument guarantees a vanishing of the obstruction group and conditional convergence implies strong convergence then.

(2) In [Gre88b, Section 4] Greenlees is able to show that under the additional assumptions that G is a p-group and Y is p-local spectrum, the spectral sequence converges to $[X, Y]^G$, i.e. $\mathrm{holim}_s Y_s \simeq *$. Among the key ingredients in the proofs of that section are localization, a homotopy fixed point spectra sequence and co-Borel homology. The existence of a conditionally convergent motivic fixed point spectral sequence is claimed (based on a different equivariant setup) by Isaksen and Shkembi [IS11, Theorem 3.8].

References

[AGV72] Michael Artin, Alexandre Grothendieck, and Jean-Louis Verdier, *Théorie des topos et cohomologie étale des schémas. Tome 1: Théorie des topos*, vol. 269, Springer-Verlag, 1972.

[Aso] Aravind Asok, *$\mathbb{A}^1$-contractibility and topological contractibility.*

[Bel01] Jason P. Bell, *The Equivariant Grothendieck Groups of the Russell-Koras Threefolds*, Canadian Journal of Mathematics **53** (2001), no. 1, 3–32.

[Blu06] Andrew J. Blumberg, *Continuous functors as a model for the equivariant stable homotopy category*, Algebraic & Geometric Topology **6** (2006), 2257–2295.

[Boa99] John Michael Boardman, *Conditionally convergent spectral sequences*, Contemporary Mathematics **239** (1999), 49–84.

[Bor60] Armand Borel, *Seminar on transformation groups*, vol. 46, Princeton Univ Pr, 1960.

[CJ11a] Gunnar Carlsson and Roy Joshua, *Equivariant Motivic Homotopy Theory*, Preprint, 2011.

[CJ11b] ———, *Motivic Spanier-Whitehead Duality and Motivic Becker-Gottlieb Transfer*, 2011.

[DG70] Michel Demazure and Alexandre Grothendieck, *Schémas en groupes: séminaire de géométrie algébrique du Bois Marie, 1962/64, SGA 3*, Lecture Notes in Mathematics, vol. 151–153, Springer-Verlag, 1970.

[DHI04] Daniel Dugger, Sharon Hollander, and Daniel C. Isaksen, *Hypercovers and simplicial presheaves*, Math. Proc. Cambridge Philos. Soc. **136** (2004), no. 1, 9–51.

[DI05] Daniel Dugger and Daniel C. Isaksen, *Motivic cell structures*, Algebr. Geom. Topol **5** (2005), 615–652.

[DI09] ———, *The motivic Adams spectral sequence*, Arxiv preprint arXiv:0901.1632 (2009).

[Doe98] Jean-Paul Doeraene, *Homotopy pull backs, homotopy push outs and joins*, Bulletin of the Belgian Mathematical Society Simon Stevin **5** (1998), no. 1, 15–38.

[Dré00] Jean-Marc Drézet, *Luna's Slice Theorem and Applications.*

[DRØ03a] Bjørn Ian Dundas, Oliver Röndigs, and Paul Arne Østvær, *Enriched functors and stable homotopy theory*, Documenta Math **8** (2003), 409–488.

[DRØ03b] ———, *Motivic Functors*, Documenta Mathematica **8** (2003), 489–525.

[EG98] Dan Edidin and William Graham, *Equivariant intersection theory (With an appendix by Angelo Vistoli: The Chow ring of M2)*, Inventiones Mathematicae **131** (1998), no. 3, 595–634.

[Fog73] John Fogarty, *Fixed Point Schemes*, American Journal of Mathematics (1973), 35–51.

[Ful07] William Fulton, *Equivariant Cohomology in Algebraic Geometry*, 2007.

[GM92] John P. C. Greenlees and Jon Peter May, *Completions of G-spectra at ideals of the Burnside ring*, Adams Memorial Symposium on Algebraic Topology, vol. 2, 1992, pp. 145–178.

[GM95] ———, *Generalized Tate cohomology*, vol. 113, Memoirs of the American Mathematical Society, no. 543, Amer Mathematical Society, 1995.

[Gre88a] John P. C. Greenlees, *How blind is your favourite cohomology theory*, Exposition. Math **6** (1988), 193–208.

[Gre88b] ———, *Stable maps into free G-spaces*, Transactions of the American Mathematical Society **310** (1988), no. 1, 199–215.

[Gro67] Alexandre Grothendieck, *Eléments de géométrie algébrique (rédigés avec la collaboration de Jean Dieudonné): IV. Etude locale des schémas et des morphismes de* *schémas, Quatrième partie*, vol. 32, Publications Mathématiques de l'Institut des Hautes Études Scientifiques, 1967.

[Gro85] ———, *Récoltes et semailles*, Réflexions et témoignage sur un passé de mathématicien (1985).

[Gui] Bert Guillou, *Models for Equivariant Homotopy Theory*, November 2006, Available online at http://www.math.uiuc.edu/ bertg/EquivModels.pdf.

[Har97] Robin Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics, Springer, 1997.

[Hir03] Philip Steven Hirschhorn, *Model Categories and Their Localizations*, Mathematical Surveys and Monographs, American Mathematical Society, 2003.

[HKO] Po Hu, Igor Kriz, and Kyle Ormsby, *Equivariant and Real Motivic Stable Homotopy Theory*, Preprint, February 12, 2010, K-theory Preprint Archives.

[HKO10] ———, *Convergence of the motivic Adams spectral sequence*, Journal of K-theory: K-theory and its Applications to Algebra, Geometry, and Topology **1** (2010), no. 1, 1–24.

[HKO11a] ———, *Remarks on motivic homotopy theory over algebraically closed fields*, Journal of K-theory: K-theory and its Applications to Algebra, Geometry, and Topology (2011), no. 7, 55–89.

[HKO11b] ———, *The homotopy limit problem for Hermitian K-theory, equivariant motivic homotopy theory and motivic Real cobordism*, Advances in Mathematics **228** (2011), no. 1, 434–480.

[Hov99] Mark Hovey, *Model Categories*, Mathematical Surveys and Monographs, vol. 63, American Mathematical Society, 1999.

[Hov01] ———, *Spectra and symmetric spectra in general model categories*, Journal of Pure and Applied Algebra **165** (2001), no. 1, 63–127.

[HS11] Philip Herrmann and Florian Strunk, *Enriched simplicial presheaves and the motivic homotopy category*, Journal of Pure and Applied Algebra **215** (2011), no. 7, 1663–1668.

[Hu03] Po Hu, *S-modules in the category of schemes*, American Mathematical Society, 2003.

[IM76] John Isbell and Barry Mitchell, *Exact colimits and fixed points*, Transactions of the American Mathematical Society (1976), 289–298.

[IS11] Daniel C. Isaksen and Armira Shkembi, *Motivic connective K-theories and the cohomology of $A(1)$*, Journal of K-theory: K-theory and its Applications to Algebra, Geometry, and Topology **7** (2011), no. 03, 619–661.

[Isa05] Daniel C. Isaksen, *Flasque model structures for simplicial presheaves*, K-Theory **36** (2005), no. 3-4, 371–395 (2006).

[Jar87] John Frederick Jardine, *Simplicial presheaves*, Journal of Pure and Applied Algebra **47** (1987), no. 1, 35–87.

[Jar96] ———, *Boolean localization, in practice*, Doc. Math **1** (1996), no. 13, 245–275.

[Jar00] ———, *Motivic Symmetric Spectra*, Documenta Mathematica **5** (2000), 445–552.

[Jar10] ———, *Local Homotopy Theory*, Springer-Verlag, 2010.

[Jos01] Roy Joshua, *Algebraic K-theory and higher Chow groups of linear varieties*, Mathematical Proceedings of the Cambridge Philosophical Society, vol. 130, Cambridge Univ Press, 2001, pp. 37–60.

[KN11] Alisa Knizel and Alexander Neshitov, *Algebraic analogue of Atiyah's theorem*, Arxiv preprint arXiv:1111.4685 (2011).

[KØ10] Amalendu Krishna and Paul Arne Østvær, *Nisnevich Descent for Deligne-Mumford Stacks*, Arxiv preprint arXiv:1005.0968 (2010).

[Koh11] Martin Kohls, *A user friendly proof of Nagata's characterization of linearly reductive groups in positive characteristics*, Linear and Multilinear Algebra **59** (2011), no. 3, 271–278.

[Kri12] Amalendu Krishna, *Completion theorem for equivariant K-theory*, arXiv:1201.5766, 2012.

[Lan02] Serge Lang, *Algebra*, Graduate Texts in Mathematics, vol. 211, Springer-Verlag, New York,, 2002.

[Lic73] Stephen Lichtenbaum, *Values of zeta-functions, étale cohomology, and algebraic K-theory*, Classical Algebraic K-Theory, and Connections with Arithmetic (1973), 487–501.

[LMS86] Lemoine G. Lewis, Jon Peter May, and Mark Steinberger, *Equivariant stable homotopy theory*, Lecture notes in mathematics ; 1213, Springer, 1986.

[Lun73] Domingo Luna, *Slices étales*, Mémoires de la Société Mathématique de France **33** (1973), 81–105.

[Lyd98] Manos Lydakis, *Simplical functors and stable homotopy theory*, 1998.

[Mac71] Saunders Mac Lane, *Categories for the Working Mathematician*, Graduate Texts in Mathematics, vol. 5, Springer, 1971.

[Man04] Michael A. Mandell, *Equivariant symmetric spectra*, Homotopy theory: relations with algebraic geometry, group cohomology, and algebraic K-theory: an international Conference on Algebraic Topology, March 24-28, 2002, Northwestern University, vol. 346, Amer Mathematical Society, 2004, pp. 399–452.

[May75] Jon Peter May, *Classifying spaces and fibrations*, vol. 155, American Mathematical Society, 1975.

[McC85] John McCleary, *User's guide to spectral sequences*, Mathematics lecture series, vol. 12, Publish or Perish, 1985.

[Mer05] Alexander S. Merkurjev, *Equivariant K-theory*, Handbook of K-theory, vol. 2, ch. Part V.2, pp. 925–954, Springer-Verlag, 2005.

[MFK94] David Mumford, John Fogarty, and Frances Clare Kirwan, *Geometric invariant theory*, vol. 34, Springer, 1994.

[Mil80] James Stuart Milne, *Étale Cohomology*, Princeton Mathematical Series, vol. 33, Princeton University Press, 1980.

[Mil11] ———, *Class Field Theory (v4.01)*, 2011, Available online at www.jmilne.org/math/, pp. 279+viii.

[Mit97] Stephen A. Mitchell, *Hypercohomology spectra and Thomason's descent theorem*, Algebraic K-theory (Toronto, ON, 1996) **16** (1997), 221–277.

[ML96] Leonid Makar-Limanov, *On the hypersurface $x+x2y+z2+t3=0$ in $\mathbb{C}^4$ or a $\mathbb{C}^4$-like threefold which is not $\mathbb{C}^3$*, Israel Journal of Mathematics **96** (1996), no. 2, 419–429.

[MM92] Saunders Mac Lane and Ieke Moerdijk, *Sheaves in Geometry and Logic*, Universitext, Springer, 1992.

[Mor99] Fabien Morel, *Suite spectrale d'Adams et invariants cohomologiques des formes quadratiques*, Comptes Rendus de l'Académie des Sciences-Series I-Mathematics **328** (1999), no. 11, 963–968.

[MRS01] Mark Mahowald, Douglas C. Ravenel, and Paul Shick, *The Thomified Eilenberg-Moore spectral sequence*, Progress in Mathematics **196** (2001), 249–262.

[MV99] Fabien Morel and Vladimir Voevodsky, *$\mathbb{A}^1$-homotopy theory of schemes*, Publications Mathématiques de l'Institut des Hautes Études Scientifiques **90** (1999), 45–143.

[MVW06] Carlo Mazza, Vladimir Voevodsky, and Charles A. Weibel, *Lecture notes on motivic cohomology*, Clay Mathematics Monographs, vol. 2, American Mathematical Society, Providence, RI, 2006.

[Nis89] Yevsey A. Nisnevich, *The completely decomposed topology on schemes and associated descent spectral sequences in algebraic K-theory, Algebraic*, Algebraic K-theory: connections with geometry and topology. (John Frederick Jardine and Victor P. Snaith, eds.), Proceedings of the NATO Advanced Study Institute held in Lake Louise, Alberta, December 7–11, 1987. NATO Advanced Science Institutes Series C: Mathematical and Physical Sciences, no. 279, Dordrecht: Kluwer Academic Publishers Group, 1989, pp. 241–342.

[Øst03] Paul Arne Østvær, *Étale descent for real number fields*, Topology **42** (2003), no. 1, 197–225.

[PPR08] Ivan A. Panin, Konstantin Pimenov, and Oliver Röndigs, *A Universality Theorem for Voevodsky's Algebraic Cobordism Spectrum*, Homology, Homotopy and Applications **10** (2008), no. 2, 211–226.

[Qui73] Daniel Gray Quillen, *Higher algebraic K-theory: I*, Higher K-theories (1973), 85–147.

[Rag10] Kári Ragnarsson, *Completion of G-spectra and stable maps between classifying spaces*, Arxiv preprint arXiv:1001.0771 (2010).

[RØ08] Oliver Röndigs and Paul Arne Østvær, *Modules over motivic cohomology*, Advances in Mathematics **219** (2008), no. 2, 689–727.

[RW99] John Rognes and Charles A. Weibel, *Étale descent for two-primary algebraic K-theory of totally imaginary number fields*, K-theory **16** (1999), no. 2, 101–104.

[Sch11] Stefan Schwede, *Lectures on Equivariant Stable Homotopy Theory*, December 2011.

[Ser01] Christian Serpé, *Eine triangulierte Kategorie äquivarianter Motive*, Ph.D. thesis, Westf. Wilhelms-Universität Münster, 2001.

[Ser10] ———, *Descent properties of equivariant K-theory*, Arxiv preprint arXiv:1002.2565 (2010).

[Ste10] Marc Stephan, *Elmendorf's Theorem for Cofibrantly Generated Model Categories*, Master's thesis, ETH Zurich, 2010.

[Tho85] R.W. Thomason, *Algebraic K-theory and étale cohomology*, Ann. Sci. École Norm. Sup.(4) **18** (1985), no. 3, 437–552.

[Tho87] ———, *Algebraic K-theory of group scheme actions*, Algebraic Topology and Algebraic K-theory (Ann. Math. Stud. 113) Princeton (1987), 539–563.

[Tot99] Burt Totaro, *The Chow ring of a classifying space*, Algebraic K-theory: AMS-IMS-SIAM Joint Summer Research Conference on Algebraic K-Theory, July 13-24, 1997, University of Washington, Seattle, vol. 67, Amer Mathematical Society, 1999, p. 249.

[Vis04] Angelo Vistoli, *Notes on Grothendieck topologies, fibered categories and descent theory*, Arxiv preprint math/0412512 (2004).

[Voe00] Vladimir Voevodsky, *Homotopy theory of simplicial sheaves in completely decomposable topologies*, K-theory Preprint Archives, September 2000.

[Voe01] ———, *Lectures on motivic cohomology 2000/2001 (written by Pierre Deligne)*, Preprint, November **19** (2001).

[Voe03] ———, *Reduced power operations in motivic cohomology*, Publications Mathématiques de L'IHÉS **98** (2003), no. 1, 1–57.

[Voe08] ———, *On Motivic Cohomology with $\mathbb{Z}/l$ coefficients*, arXiv:0805.4430 (2008).

[Voe10] ———, *Motivic Eilenberg-MacLane spaces*, Publications mathématiques de l'IHES **112** (2010), no. 1, 1–99.

[VW99] Vladimir Voevodsky and Charles A. Weibel, *Voevodsky's Seattle lectures: K-theory and motivic cohomology*, Proceedings of Symposia in Pure Mathematics, vol. 67, Providence, RI; American Mathematical Society; 1998, 1999, pp. 283–304.

[Wei49] A. Weil, *Numbers of solutions of equations in finite fields*, Bull. Amer. Math. Soc **55** (1949), no. 5, 497–508.

[Wri76] C.K. Wright, *A Flat Group Scheme G for which X^G is not Representable*, Journal of the London Mathematical Society **2** (1976), no. 2, 319.