ON MOTIVIC SPHERICAL BUNDLES

FLORIAN STRUNK

Dissertation
zur
Erlangung des Doktorgrades (Dr. rer. nat.)
des Fachbereichs Mathematik/Informatik
an der Universität Osnabrück

vorgelegt von
Florian Strunk
aus Lemgo

Institut für Mathematik
Universität Osnabrück

Dezember 2012

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Introduction

This thesis deals with a motivic version of the J-homomorphism from algebraic topology. The classical J-homomorphism was introduced under a different name in 1942 by Whitehead [Whi42] in order to study homotopy groups of the spheres. It can be constructed as follows: An element of the special orthogonal group $SO(m)$ is a rotation $\mathbb{R}^m \to \mathbb{R}^m$ and yields by one-point compactification a homotopy equivalence $S^m \to S^m$ fixing the point at infinity. One gets a map $SO(m) \to \Omega^m S^m$ which can be adjusted to a pointed one by composing with minus the identity $\Omega^m S^m \to \Omega^m S^m$. This composition induces a morphism $J_{n,m} : \pi_n(SO(m)) \to \pi_{n+m}(S^m)$ on homotopy groups (alternatively, one could use the Hopf construction which associates a map $H(\mu) : X * Y \to \Sigma Z$ to $\mu : X \times Y \to Z$). The construction of $J_{n,m}$ behaves well (up to sign) with an increase of m and one gets a morphism $J_n : \pi_n(SO) \to \pi_n^s$ into the n-th stable stem and a morphism $J_n : \pi_n(O) \to \pi_n^s$ for $n \geq 1$. Precomposition with $\pi_n(U) \to \pi_n(O)$ induced by the inclusions $U(n) \hookrightarrow O(2n)$ yields the complex J-homomorphism.

Another viewpoint on J_n is taken by Atiyah in [Ati61b]: Fix a connected finite CW-complex X. To a vector bundle E on X, one can associate a spherical bundle $S(E) \to X$ by removing the zero section from E. While E and X are homotopy equivalent, the total space $S(E)$ may contain interesting homotopical information about the bundle. Define the monoid $J(X)$ by considering spherical bundles $S(E)$ up to homotopy equivalence over X and up to the addition of trivial bundles. This abelian monoid is in fact a group since for every E there is a D such that $E \oplus D$ is a trivial bundle over X [Ati61b, Lemma 1.2]. Hence, there is a natural epimorphism $J_X : \tilde{KO}^0(X) \twoheadrightarrow J(X)$ from the reduced real K-theory of X. For the special case of $n \geq 1$ and $X = S^{n+1}$ with the canonical basepoint, this is related to the morphism J_n from above by the diagram

$$\begin{array}{ccc}
\pi_n(O) & \xrightarrow{\quad J_n \quad} & \pi_n^s \\
\| & & \\
\langle S^n, O\rangle & & \\
\| & & \\
\langle S^n, \Omega BO\rangle & & \uparrow \\
\| & & \\
\langle S^{n+1}, BO\rangle & & \\
\| & & \\
\tilde{KO}^0(S^{n+1}) & \xrightarrow{J_{S^{n+1}}} \!\!\!\!\rightarrow & J(S^{n+1})
\end{array}$$

and $J(S^{n+1})$ is the image of J_n in the n-th stable stem [Ati61b, Proposition 1.4], induced by the clutching construction: A spherical bundle $S(E) \to S^{n+1}$ with fiber S^m may be pulled back to the upper and to the lower hemisphere $D_{\pm}^{n+1}$ where it trivializes. Hence, there is a homotopy equivalence $S^n \times S^m \to S^n \times S^m$ over the intersection S^n of the two hemispheres and by adjunction a morphism $S^n \to \Omega^m S^m$.

We may consider J_X as a morphism $J_X : \tilde{KO}^0(X) \to \tilde{\mathcal{S}}\mathrm{ph}(X)$ into the group of (fiber homotopy classes of) stable spherical fibrations on X with the above $J(X)$ as its image: The join $E *_X D$ over X of two spherical fibrations $E \to X$ and $D \to X$ (fibrations whose fiber is homotopy equivalent to a sphere) is again a spherical fibration [Rud98, Proposition IV.1.43]. The group $\tilde{\mathcal{S}}\mathrm{ph}(X)$ is the group completion of the monoid constituted by the equivalence classes of the relation declaring two spherical fibrations as equivalent if they are fiber homotopy equivalent after joining with trivial spherical fibrations of arbitrary dimension. The subspace $F(m)$ of $\Omega^m S^m$ consisting of (basepoint preserving) homotopy equivalences of S^m is a topological monoid under composition and the classifying space BF of the colimit F has the property that $\tilde{\mathcal{S}}\mathrm{ph}(X) = [X, BF]$ [Sta63].

The Bott periodicity theorem [Bot59] computes all the groups $\pi_n(O)$ and $\pi_n(U)$, whereas the stable homotopy groups of spheres π_n^s are hard to describe and not all known up to now.

n	0	1	2	3	4	5	6	7	
$\pi_n(O)$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\circlearrowleft$
$\pi_n(U)$	0	$\mathbb{Z}$	$\circlearrowleft$						
π_n^s	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/240\ldots$	

Adams was able to show in a series of papers [Ada63, Ada65a, Ada65b, Ada66] that, assuming the Adams conjecture, the image of J is always a direct summand, that J is injective for the $\mathbb{Z}/2$-terms (exept for $n = 0$) and that for the $\mathbb{Z}$-terms (where $n = 4s-1$), the image of J is $\mathbb{Z}/m(2s)$ where $m(2s)$ is the denominator of the quotient $B_s/4s$ of the sth Bernoulli number B_s.

The Adams conjecture states that for each integer k and each vector bundle E over a finite CW-complex X there exists a natural number t (dependent on k and E) such that

$$J(k^t(\Psi^k - 1)E) = 0.$$

The Adams conjecture was first proven by Quillen [Qui71]. Later, other proofs were given by Sullivan [Sul74], Friedlander [Fri73] and Becker-Gottlieb [BG75]. An unpublished note of Brown [Bro73] modifies the latter proof avoiding the use of the property that BF is an infinite loop space. In [Ada63], Adams himself proved the conjecture for line bundles and two dimensional bundles $E \to X$. The statement of the Adams conjecture above is linear in E and true for trivial bundles since the Adams operations Ψ^k are the identity on them. Hence, a method of proof would be to reduce the question successively from bundles with structure group $O(m)$ to bundles with a smaller structure group until we reach $O(1)$ or $O(2)$ where it holds true by Adam's result. The steps in Becker-Gottlieb's and Brown's proofs are

$$O(m) \underset{k=\lceil \frac{m}{2} \rceil}{\rightsquigarrow} O(2k) \rightsquigarrow \Sigma_k \wr O(2) = O(2)^k \rtimes \Sigma_k \rightsquigarrow O(2)$$

where we will ignore the first easy step. A principal $O(2k)$-bundle $\hat{E} \to X \cong \hat{E}/O(2k)$ associated to a vector bundle $E \to X$ of rank $2k$ induces morphisms

$$\hat{E}/(O(2) \times (\Sigma_{k-1} \wr O(2))) \xrightarrow{g} \hat{E}/(\Sigma_k \wr O(2)) \xrightarrow{f} \hat{E}/O(2k)$$

where f is a smooth fiber bundle whose fiber $O(2k)/(\Sigma_k \wr O(2))$ is connected and has invertible Euler characteristic and where g is an k-sheeted covering space since the group $O(2) \times (\Sigma_{k-1} \wr O(2))$ has index k in $\Sigma_k \wr O(2)$ [Ebe]. Now one argues 'from both sides': If $J(f^*E) = 0$ then $J(E) = 0$ and if $J(K) = 0$ then $J(g_!K) = 0$ where $g_!$ is the geometric transfer [Ati61a]. Finally, one shows that there is a bundle K whose structure group may be reduced to $O(2)$ such that $g_!K \cong f^*E$. The crucial step in the proof which uses that BF is an infinite loop space or Brown's trick is the first step involving f.

Motivic homotopy theory (also called $\mathbb{A}^1$-homotopy theory) is a homotopy theory of algebraic schemes. Although it is constructed by rather general categorical arguments [MV99], remarkable results like the Milnor conjecture were proven within this setting [Voe97]. Many objects and methods from algebraic topology have their motivic analogues like for instance algebraic K-theory takes the place of topological K-theory. Nevertheless, many new and interesting difficulties and phenomena arise. For example, there are two different types of objects which should be considered as spheres: A simplicial and an algebraic one, which yield the projective line $\mathbb{P}^1$ when they are smashed together. The behaviour of the algebraic sphere with respect to standard homotopical constructions seems to be a reason why there is not yet a Recognition theorem for $\mathbb{P}^1$-loop spaces like the one of Boardman-Vogt [BV68, Theorem A,B] in algebraic topology.

The organization of this text is as follows. In the first section we recall the construction of the motivic homotopy category and provide with Proposition 1.1.22 a slightly modified version of Morel-Voevodsky's fibrant replacement functor for one of its models. Moreover, some properties of homotopy sheaves are discussed. Section 1.3 contains a motivic version of (the second) Mather's cube theorem and some consequences for motivic fiber sequences inducing exact sequences of $\mathbb{A}^1$-homotopy sheaves are outlined. In the second section, we focus on the stable motivic homotopy category. A consequence of Morel's Motivic Hurewicz theorem which is of importance for our main result is established in Proposition 2.2.5. Afterwards, we make some observations on the motivic connectivity of morphisms and use Morel's results to prove a Blakers-Massey theorem 2.3.8. The third section begins with recalling the terms of principal bundles and fiber bundles and describes the procedure of extension and reduction of their structure group. Next, we outline how results of Morel, Moser, Rezk and Wendt are used to get motivic fiber sequences from certain bundles. In section 3.3, a motivic J-homomorphism $J : \tilde{K}^0(X) \to \tilde{\mathcal{S}}\mathrm{ph}(X)$ is constructed. Section 4 deals with vanishing results for the motivic J-homomorphism. We first establish a relation between the vanishing of the J-homomorphism and Thom classes. A general method to obtain a transfer morphism due to Ayoub, Hu, May, Röndigs-Østvær and others is discussed in section 4.2. Then we are ready to prove our main result 4.3.5 which is a motivic version of the above-mentioned trick of Brown:

Theorem. Let S be the spectrum of a perfect field, $F \to Y \xrightarrow{f} X$ a smooth Nisnevich fiber bundle with an $\mathbb{A}^1$-connected base X and an $\mathbb{A}^1$-connected fiber F with a dualizable suspension spectrum and invertible Euler characteristic $\chi(F)$. Let $E \to X$ be a vector bundle over X. Then $J(f^*E) = 0$ implies $J(E) = 0$.

Section 4.4 begins with a Corollary 4.4.2 of a result of Ayoub and Röndigs showing that for a line bundle E over a projective curve X over a perfect field, $J(E \otimes E)$ is zero. Afterwards, we discuss how a reduction process for the vanishing of the J-homomorphism could be realized using the established results. We examine a candidate for a fiber bundle $F \to Y \to X$ with fiber GL_n/NT where NT is the subgroup of the general linear group consisting of the monomial matrices. In particular for a base field of characteristic zero, the suspension spectrum of the smooth scheme GL_n/NT is dualizable.

1. Unstable motivic homotopy theory

Remark 1.0.1. Unless no other indication is made, definitions from algebraic geometry are taken from [Liu02]. A *base scheme* is a noetherian scheme of finite Krull dimension. Throughout this text, let S denote a base scheme, k a field and $\mathcal{S}\mathrm{m}_S$ the category of smooth schemes of finite type over S. The category $\mathcal{S}\mathrm{m}_{Spec(k)}$ is abbreviated by $\mathcal{S}\mathrm{m}_k$. An object $X \to S$ of $\mathcal{S}\mathrm{m}_S$ is often written as X. Every $S \in \mathcal{S}\mathrm{m}_k$ may serve as a base scheme. Due to the finiteness condition, $\mathcal{S}\mathrm{m}_S$ is an essentially small category. Smoothness for an object X of $\mathcal{S}\mathrm{m}_S$ may be characterized by the property, that Zariski-locally on X, the structure morphism $X \to S$ factorizes as an étale morphism $U \to \mathbb{A}^n_S$ followed by the projection [Liu02, Corollary 6.2.11].

Basic notations on category theory are taken from [Bor94a] and [Bor94b].

We use [MM92, Section III.2] as a reference for Grothendieck topologies. When no confusion is possible, we speak of a basis Cov of a Grothendieck topology as of a Grothendieck topology τ and mean the topology generated by the basis. This is an abuse of notation since different bases may generate the same Grothendieck topology. Whenever we speak of a covering of an object in a Grothendieck topology, we mean a covering for this object in an arbitrary basis for this topology.

The category of $\mathcal{S}$et-valued presheaves on $\mathcal{S}\mathrm{m}_S$ is denoted by Pre or by Pre(S), if the role of the base scheme has to be emphasized. The symbol Shv or Shv(S) stands for the category of $\mathcal{S}$et-valued sheaves on $\mathcal{S}\mathrm{m}_S$ with respect to a Grothendieck topology. The category sPre or sPre(S) of *simplicial presheaves* or *(motivic) spaces* is the category of simplicial objects in Pre. Simplicial sheaves form the category sShv or sShv(S).

Remark 1.0.2. Although the most-widespread way to define a scheme is to give a pair $(X, \mathcal{O}_X)$ consisting of a particular topological space X and a particular sheaf of rings $\mathcal{O}_X$ on the open sets of X, it may be intuitively more convenient to think of a scheme over Spec(k) in the motivic setting as of a particular functor from the category of finitely generated k-algebras to $\mathcal{S}$et. This equivalent interpretation is called the *functor of points approach*. It is explained extensively in [DG70] or in the first chapter of [Jan87].

Consider the Yoneda embedding

$$\begin{array}{rcl} y : k\text{-Alg}^{\mathrm{op}} & \to & \mathcal{F}\mathrm{un}(k\text{-Alg}, \mathcal{S}\mathrm{et}) \\ A & \mapsto & \hom_{k\text{-Alg}}(A, -) \end{array}$$

and call a representable yA an *affine scheme functor*. One wants to formulate the idea of a scheme functor as being glued out of affine scheme functors in a reasonable way. To do so, we have to choose along which morphisms affine schemes should be glued together. An *open immersion into an affine scheme functor* is a monomorphism $D(I) \rightarrowtail yA$ where $D(I)(B) = \{f : A \to B \mid \sum_{x \in I} Bf(x) = B\}$ for an ideal I of A. This notion is justified by identifying $D(I)$ evaluated at a field extension K of k with

$$\bigcup_{x \in I} \{f : A \to K \mid f(x) \neq 0\}$$

which is the complement in $yA(K)$ of the evaluation at K of the *closed immersion* $V(I) \rightarrowtail yA$ where $V(I)(B) = \{f : A \to B \mid \forall x \in I : f(x) = 0\}$. In particular for $x \in A \in k\text{-Alg}$, the the canonical map $A \to A_x$ defines an open immersion $yA_x \rightarrowtail yA$ since $D((x))(B) = \{f : A \to B \mid f(x) \in B^\times\} = yA_x(B)$.

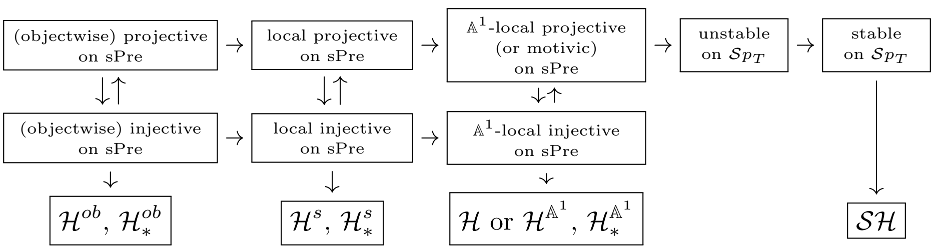
A set $\{U_i \to X\}$ of morphisms is a *covering* if $yX(K)$ is the union of the $yU_i(K)$ for all field extensions K of k. In particular for finitely many open immersions $yA_{x_i} \to yA$, this means that $\sum Ax_i = A$. These open coverings define a subcanonical topology on $k\text{-Alg}^{\mathrm{op}}$ called the *Zariski topology*.

In order to define the notion of a scheme functor, one defines a compatible notion of an *open immersion* into an arbitrary element X of $\mathcal{F}\mathrm{un}(k\text{-Alg}, \mathcal{S}\mathrm{et})$ as a monomorphism $i : U \rightarrowtail X$ such that for any $f : yA \to X$ there exists an ideal I of A with $f^{-1}i(U) = D(I)$. A *scheme functor* over $\mathrm{Spec}(k)$ may now be defined as a sheaf in the Zariski topology which admits an open covering by affine schemes. To a scheme $(X, \mathcal{O}_X)$ one associates a scheme functor $B \mapsto \hom_{\mathcal{S}\mathrm{ch}/k}(\mathrm{Spec}(B), X)$. Conversely, for a scheme functor F the scheme $(X, \mathcal{O}_X)$ is a left Kan extension. In particular, the structure sheaf $\mathcal{O}_X$ may be described as $\mathcal{O}_X(U) = \lim \mathcal{O}_{\mathrm{Spec}\,A}(j^{-1}(U)) = \lim \lim A_{f_i}$ where $j : \mathrm{Spec}\,A \to X$ is the canonical map into the colimit and $\{D((f_i))\}$ an open covering of $j^{-1}(U)$. This provides an equivalence of the two concepts [DG70, I.4.4].

Unfortunately, the definition of a scheme is too restrictive for homotopy theory and invariant theory since the category of schemes does for example not allow to build arbitrary quotients. Hence, one weakens the second of the above conditions on a functor to be a scheme. This leads to the notion of algebraic stacks and, if one gets rid of the second condition completely, to the whole category of Zariski-sheaves.

Although the functor of points approach may be used as a motivation for the upcoming categorical constructions, we still use [Liu02] as a reference for our notations.

Remark 1.0.3. The following diagram provides an overview on how the different model structures and their homotopy categories coming up in this text are designated where the star refers to the pointed model respectively.



1.1. The unstable motivic homotopy category

Remark 1.1.1. The full and faithful *Yoneda embedding* $y : \mathcal{S}\mathrm{m}_S \hookrightarrow \mathrm{Pre}$ defined by $yX = \hom_{\mathcal{S}\mathrm{m}_S}(-, X)$ commutes with products [MM92, Section I.1] and is a canonical way to embed $\mathcal{S}\mathrm{m}_S$ into a bicomplete category. This construction has the drawback of preserving almost no colimits which may be seen as follows: The embedding of the usual construction

$$\begin{array}{ccc} \mathbb{A}^1 \setminus \{0\} & \longrightarrow & \mathbb{A}^1 \\ \downarrow & & \downarrow \\ \mathbb{A}^1 & \longrightarrow & \mathbb{P}^1 \end{array}$$

of the projective line is a pushout diagram in Pre if and only if all presheaves X induce a pullback diagram

$$\begin{array}{ccc} X(\mathbb{P}^1) & \longrightarrow & X(\mathbb{A}^1) \\ \downarrow & & \downarrow \\ X(\mathbb{A}^1) & \longrightarrow & X(\mathbb{A}^1 \setminus \{0\}) \end{array}$$

of sets. For the special choice of $X = \mathrm{Pic}(-)$, one has $\mathrm{Pic}(\mathbb{A}^1) \cong 0$ but $\mathrm{Pic}(\mathbb{P}^1) \cong \mathbb{Z}$ (confer [HS09, Section 2]).

For a subcanonical Grothendieck topology, the Yoneda embedding factorizes over Shv and the diagram

$$\coprod yU_i \times_{yX} yU_j \rightrightarrows \coprod yU_i \twoheadrightarrow yX$$

is a coequalizer in the category of sheaves for every covering $\{U_i \to X\}$ in $\mathrm{Cov}(X)$. Hence, one may think of yX as being glued from the covering and the restriction of the image of the Yoneda embedding to Shv as a way of preserving at least some colimits.

Remark 1.1.2. The isomorphism $\mathrm{sPre} \cong \mathcal{F}\mathrm{un}(\mathcal{S}\mathrm{m}_S^{\mathrm{op}} \times \Delta^{\mathrm{op}}, \mathcal{S}\mathrm{et}) \cong \mathcal{F}\mathrm{un}(\mathcal{S}\mathrm{m}_S^{\mathrm{op}}, \mathrm{s}\mathcal{S}\mathrm{et})$ is used implicitly throughout the whole text. Composing the Yoneda embedding with the *discrete simplicial set functor* $\mathcal{S}\mathrm{et} \hookrightarrow \mathrm{s}\mathcal{S}\mathrm{et}$ provides an embedding of categories $\mathcal{S}\mathrm{m}_S \hookrightarrow \mathrm{sPre}$. Recall that both functors, the discrete embedding and the evaluation at zero preserve colimits and limits respectively. Considering a simplicial set as constant in the scheme direction yields an embedding $\mathrm{s}\mathcal{S}\mathrm{et} \hookrightarrow \mathrm{sPre}$. This embedding as well as the evaluation at the terminal scheme preserve colimits and limits. By abuse of notation, objects are identified by their image under these embeddings respectively.

Remark 1.1.3. For an object U of a category $\mathcal{C}$ let $\mathcal{C}/_U$ be the over category $\mathcal{C}{\downarrow}U$.

The category Pre is cartesian closed with respect to the product and internal hom

$$\underline{\mathrm{hom}}(X, Y) = \mathrm{hom}_{\mathrm{Pre}}(X \times y(-), Y) \cong \mathrm{hom}_{\mathrm{Pre}/_{y(-)}}(X \times y(-), Y \times y(-))$$

which is a sheaf if Y is a sheaf (considered as a presheaf) [MM92, Proposition III.6.1].

The category $\mathcal{S}\mathrm{m}_S/_U$ is a essentially small and often different from the category $\mathcal{S}\mathrm{m}_U$ unless for example U is étale over S [Gro67, Proposition 17.3.4]. Since there is an equivalence $\mathrm{Pre}/_{yU} \cong \mathrm{Pre}(\mathcal{S}\mathrm{m}_S/_U)$, this internal hom may be rewritten further as

$$\mathrm{hom}_{\mathcal{S}\mathrm{m}_S}(X \times -, Y) \cong \mathrm{hom}_{\mathcal{S}\mathrm{m}_S/_-}(X \times -, Y \times -) = \mathrm{hom}_{\mathcal{S}\mathrm{m}_S/_-}(X|_-, Y|_-)$$

for the special case that X and Y are from $\mathcal{S}\mathrm{m}_S$. The same holds true when Pre is replaced by Shv and the slice Grothendieck topologies are used.

The categories sPre and sShv are cartesian closed with internal hom

$$\underline{\mathrm{hom}}(X, Y) = \mathrm{hom}(X \times y(-) \times \Delta^{(\text{-})}, Y).$$

Remark 1.1.4 (confer Theorem A.1.17)**.** There exists the *(objectwise) projective* model structure on the category sPre as well as the *(objectwise) injective* model structure on sPre and a monoidal Quillen equivalence

$$\mathrm{sPre}_{proj} \rightleftarrows \mathrm{sPre}_{inj}.$$

Both model structures are monoidal, proper, combinatorial and simplicial. Their (equivalent) homotopy categories are denoted by $\mathcal{H}_{ob}$ or $\mathcal{H}_{ob}(S)$ with morphism sets $[-,-]_{ob}$. The tensor for the simplicial enrichment is the categorical product with an (embedded) simplicial set. The cofibrations for the injective structure are exactly the monomorphism since limits are calculated objectwise [Bor94a, Proposition 2.15.1]. A set of generating cofibrations $\mathcal{I}_{proj}$ for the projective structure is given by

$$U \times \partial\Delta^n \to U \times \Delta^n$$

for $U \in \mathcal{S}\mathrm{m}_S$ and $n \geq 0$ and a set of generating acyclic cofibrations $\mathcal{J}_{proj}$ is

$$U \times \Lambda_i^n \to U \times \Delta^n$$

for $n \geq 1$ and $0 \leq i \leq n$. We will refer to an objectwise projective fibrant simplicial presheaf in the following just as an objectwise fibrant simplicial presheaf.

Definition 1.1.5. The *Zariski topology* $\mathcal{Z}\mathrm{ar}$ on $\mathcal{S}\mathrm{m}_S$ consists of the collections

$$\mathcal{Z}\mathrm{ar}(X) = \{\{f_i : U_i \to X \mid \text{each } f_i \text{ is an open immersion, } \cup f_i(U_i) = X\}\}$$

of *Zariski coverings*. It is the topology generated by a *complete regular cd-structure* constituted by the *Zariski distinguished squares*

$$\begin{array}{ccc} W & \overset{j}{\rightarrowtail} & Y \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ U & \overset{i}{\rightarrowtail} & X \end{array}$$

which are pullback squares such that i and p are open immersions and $X = p(Y) \cup i(U)$ [Voe10, Theorem 2.2]. This defines the *(big) Zariski site* $(\mathcal{S}\mathrm{m}_S, \mathcal{Z}\mathrm{ar})$.

The *Nisnevich topology* $\mathcal{N}\mathrm{is}$ on $\mathcal{S}\mathrm{m}_S$ consists of the collections

$$\begin{aligned} \mathcal{N}\mathrm{is}(X) = \{\{f_i : U_i \to X \mid &\text{ each } f_i \text{ is étale, } \cup f_i(U_i) = X, \\ &\text{for all } x \in X \text{ exists some } u \in f_i^{-1}(x), \text{ such that } k(x) \cong k(u)\}\} \end{aligned}$$

of *Nisnevich coverings*. It is the topology generated by a complete regular cd-structure constituted by the *Nisnevich distinguished squares*

$$\begin{array}{ccc} W & \overset{j}{\rightarrowtail} & Y \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ U & \overset{i}{\rightarrowtail} & X \end{array}$$

which are pullback squares such that i is an open immersion, p is an étale morphism and $p^{-1}((X \setminus i(U))_{red}) \xrightarrow{\cong} (X \setminus i(U))_{red}$ [Voe10, Proposition 2.17.]. This defines the *(big) Nisnevich site* $(\mathcal{S}\mathrm{m}_S, \mathcal{N}\mathrm{is})$.

The *étale topology* $\acute{\mathcal{E}}\mathrm{t}$ on $\mathcal{S}\mathrm{m}_S$ consists of the collections

$$\acute{\mathcal{E}}\mathrm{t}(X) = \{\{f_i : U_i \to X \mid \text{each } f_i \text{ is is étale, } \cup f_i(U_i) = X\}\}$$

and the *fppf topology* $\mathcal{F}\mathrm{ppf}$ on $\mathcal{S}\mathrm{m}_S$ of

$$\begin{aligned} \mathcal{F}\mathrm{ppf}(X) = \{\{f_i : U_i \to X \mid &\text{ each } f_i \text{ is flat and locally of finite presentation,} \\ &\cup f_i(U_i) = X\}\}. \end{aligned}$$

Remark 1.1.6. An étale covering $\{U_i \to X\}$ is a Nisnevich covering if for each point $\operatorname{Spec} k(x) \to X$, there is an index i and a lift in the diagram

$$\begin{array}{ccc} & & U_i \\ & \nearrow & \downarrow \\ \operatorname{Spec} k(x) & \longrightarrow & X. \end{array}$$

Let S be the spectrum of a field k of characteristic different from two and a a nonzero element of k. The open immersion $\mathbb{A}^1 \setminus \{a\} \hookrightarrow \mathbb{A}^1$ together with the morphism $\mathbb{A}^1 \setminus \{0\} \to \mathbb{A}^1$ where $z \mapsto z^2$ are always an étale covering of $\mathbb{A}^1$ but a Nisnevich covering if and only if a is a square. In the latter case, the two morphisms do not constitute a Nisnevich distinguished square by pullback. To form a Nisnevich distinguished square, one could remove one of the two roots b of a, id est $\mathbb{A}^1 \setminus \{a\} \hookrightarrow \mathbb{A}^1 \leftarrow \mathbb{A}^1 \setminus \{0, b\}$ [MVW06, Example 12.1].

Remark 1.1.7. A conservative set of points for the big Zariski site $\mathcal{S}\mathrm{m}_S$ is given by

$$\begin{aligned} X \mapsto \underset{(x\in V \xrightarrow{open} U)^{\mathrm{op}}}{\mathrm{colim}} X(V) &\cong \underset{(x\in \mathrm{Spec}(B)\to U)^{\mathrm{op}}}{\mathrm{colim}} \underset{\underset{(special)}{A}}{\mathrm{colim}} \hom_{\mathcal{S}\mathrm{m}_S}(\mathrm{Spec}(B), \mathrm{Spec}(A)) \\ &\cong \mathrm{colim}\ \underset{A}{\mathrm{colim}}\ \hom_{\mathrm{Alg}}(A, B) \\ &\cong \underset{A}{\mathrm{colim}}\ \hom_{\mathrm{Alg}}(A, \mathrm{colim}\ B) \\ &\cong \underset{A}{\mathrm{colim}}\ \hom_{\mathrm{Alg}}(A, \mathcal{O}_{U,x}) \\ &\cong \underset{A}{\mathrm{colim}}\ \hom_{\mathcal{S}\mathrm{ch}/_{\mathrm{S}}}(\mathrm{Spec}(\mathcal{O}_{U,x}), \mathrm{Spec}(A)) \\ &\approx X(\mathrm{Spec}(\mathcal{O}_{U,x})) \end{aligned}$$

for U (in a skeleton) of $\mathcal{S}\mathrm{m}_S$ and $x \in U$. There is no reason for the scheme $\mathrm{Spec}(\mathcal{O}_{U,x})$ to be of finite type over S. Hence, the notion $X(\mathrm{Spec}(\mathcal{O}_{U,x}))$ for an $X : \mathcal{S}\mathrm{m}_S^{\mathrm{op}} \to \mathrm{s}\mathcal{S}\mathrm{et}$ is just an intuitive abbreviation for the colimit above.

A conservative set of points for the big Nisnevich site $\mathcal{S}\mathrm{m}_S$ is analogously given by

$$X \mapsto \underset{(x\in V \xrightarrow{nis} U)^{\mathrm{op}}}{\mathrm{colim}} X(V) \cong \underset{A}{\mathrm{colim}}\ \hom_{\mathrm{Alg}}(A, \mathcal{O}^h_{U,x}) \approx X(\mathrm{Spec}(\mathcal{O}^h_{U,x}))$$

for U (in a skeleton) of $\mathcal{S}\mathrm{m}_S$ and and $x \in U$, where $\mathcal{O}^h_{U,x}$ is the henselization of the Zariski stalk $\mathcal{O}_{U,x}$ which is again just an abbreviation of the actual colimit (confer [Gro67, Corollaire 18.6.10] and [Nis87, 1.9]). It is referred to this set of points as to the *canonical set of points*.

For a not-necessarily smooth scheme X over S, the functor

$$\hom_{\mathcal{S}\mathrm{ch}/_S}(-, X) : \mathcal{S}\mathrm{m}_S^{\mathrm{op}} \to \mathcal{S}\mathrm{et}.$$

is a natural candidate for viewing X as an element of Pre. With the help of the concrete description of points from above, one can show for example that the non-smooth coordinate cross $\mathrm{Spec}\, k[x,y]/(xy)$ is isomorphic to the usual pushout $\mathbb{A}^1 \leftarrow * \to \mathbb{A}^1$ in the category of Zariski-sheaves on $\mathcal{S}\mathrm{m}_S$. This would not be true when sheaves on $\mathcal{S}\mathrm{ch}/_S$ were considered instead.

Let K be a field and $\mathrm{Spec}(K) \to S$ a morphism of $\mathcal{S}\mathrm{m}_S$. Since $\mathcal{O}^h_{\mathrm{Spec}(K),(0)} \cong K$, the functor evaluating an object of $\mathrm{Shv}(S)$ at the spectrum of such a field K is a point of the big Nisnevich site. This can be seen without citing [Nis87, 1.9] as follows: It suffices to show that evaluating at $\mathrm{Spec}(K)$ is a point of the small Nisnevich site $Nis(\mathrm{Spec}(K))$, this is it commutes with colimits and finite limits. Evaluating at any U commutes with limits so we have to examine colimits. Every object of this site is of the form $\coprod_{i=1}^n \mathrm{Spec}(L_i) \to \mathrm{Spec}(K)$, where each field L_i is a finite separable field extension of K and it is a Nisnevich morphism if and only if at least one of the L_i equals K. A presheaf F on this site defines a sheaf if and only if $F(\emptyset) \cong *$ and $F(L \sqcup L') \cong F(L) \times F(L')$. From this, one observes that evaluating at $\mathrm{Spec}(K)$ commutes with colimits.

Remark 1.1.8. Unless no other indication is made, sheaves are always considered with respect to the Nisnevich topology in the following. A reason for using the Nisnevich topology in motivic homotopy theory is that it shares a good property with the Zariski and another good one with the étale topology:

- Similar to the Zariski topology, algebraic K-theory has descent properties with respect to the Nisnevich topology [TT90, Theorem 10.8].
- Similar to the étale topology, an object from $\mathcal{S}\mathrm{m}_k$ of Krull dimension n looks Nisnevich locally like affine space, this is for a finite separable field extension

L of k and a point $x : \mathrm{Spec}\,(L) \to X$ there is an isomorphism

$$X \big/ (X \setminus \{x\}) \cong \mathbb{A}^n \big/ (\mathbb{A}^n \setminus \{0\})$$

of Nisnevich sheaves.

Definition 1.1.9. A morphism $f : X \to Y$ of simplicial presheaves is called

- *local weak equivalence* if for all canonical points x (equivalently for all points) $x^* f$ is a weak equivalence of simplicial sets,
- *local projective fibration (respectively local injective fibration)* if it has the right lifting property with respect to projective cofibrations (respectively injective cofibrations, id est monomorphisms) which are also simplicial weak equivalences.

In the context of a morphism of simplicial sheaves, these properties refer to the underlying morphism of simplicial presheaves.

Theorem 1.1.10 (Blander, Jardine, Joyal)**.** There is a monoidal, proper, combinatorial and simplicial model structure on sPre (respectively on sShv) called the *local injective model structure* with weak equivalences the local weak equivalences, cofibrations the monomorphisms and fibrations the local injective fibrations. In the case of simplicial presheaves, it is the Bousfield localization of the injective model structure at a certain set consisting of hypercovers (confer [DHI04]).

Moreover, there is a monoidal, proper, combinatorial and simplicial model structure on sPre (respectively on sShv) called the *local projective model structure* with weak equivalences the local weak equivalences, cofibrations the projective cofibrations and fibrations the local projective fibrations. In the case of simplicial presheaves, it is the Bousfield localization of the projective model structure at the morphisms $P(Q) \to X$ where $P(Q)$ is a model for the homotopy pushout of the upper half $U \leftarrow W \to Y$ of a Nisnevich distinguished square Q (confer [Bla01]) and where we implicitly first choose a skeleton of the category $\mathcal{S}\mathrm{m}_S$.

There is a diagram of monoidal Quillen equivalences

$$\begin{array}{ccc} \mathrm{sPre}_{lproj} & \underset{i}{\overset{a}{\rightleftarrows}} & \mathrm{sShv}_{lproj} \\ \Big\downarrow\Big\uparrow \mathrm{id} & & \Big\downarrow\Big\uparrow \mathrm{id} \\ \mathrm{sPre}_{linj} & \underset{i}{\overset{a}{\rightleftarrows}} & \mathrm{sShv}_{linj} \end{array}$$

where a denotes the sheafification functor. The associated *local homotopy category* is denoted by $\mathcal{H}^s$ or $\mathcal{H}^s(S)$ with morphism sets $[-,-]_s$.

Proof. The local injective model structure on simplicial presheaves is examined by Jardine in [Jar87] and on simplicial sheaves by Joyal in [Joy84]. The existence of the local projective structures on simplicial presheaves and simplicial sheaves is proven in [Bla01]. The statements about the Bousfield localizations follow from [DHI04, Theorem 6.2], [DHI04, Proposition 6.7] and [Bla01, Lemma 4.3]. The monoidality is addressed for instance in [Toë09] or [Hor06, Theorem 1.9]. A detecting Quillen adjunction whose unit consists of local weak equivalences is a Quillen equivalence and [DHI04, Proposition A.2] or the discussion below shows this for $\mathrm{id} \to ia$. The sheafification a is strong monoidal and Lemma A.2.5 implies the monoidality of the adjunction. □

Remark 1.1.11. Due to [Jar11, Lemma 2.26], injective cofibrations are exactly stalkwise cofibrations, id est morphisms which induce cofibrations on all canonical points (equivalently on all points). The analogue is not true for *stalkwise fibrations* (or sometimes called *local fibrations* which conflicts with our notation), which are defined to be fibrations on all canonical points (equivalently on all points). All local projective fibrations (and therefore all local injective fibrations) are stalkwise fibrations [Jar11, Example 6.23].

Remark 1.1.12. The fibrant objects of the local projective model structure on simplicial presheaves may be identified exactly as the *flasque* simplicial presheaves, which are defined to be the the objectwise projective fibrant simplicial presheaves F such that $F(\emptyset) \cong *$ and for every Nisnevich distinguished square

$$\begin{array}{ccc} W & \longrightarrow & Y \\ \downarrow & & \downarrow \\ U & \longrightarrow & X \end{array}$$

the diagram

$$\begin{array}{ccc} F(X) & \longrightarrow & F(U) \\ \downarrow & & \downarrow \\ F(Y) & \longrightarrow & F(W) \end{array}$$

is a homotopy pullback of simplicial sets [Bla01, Lemma 4.1]. In particular, every Nisnevich distinguished square is a homotopy pushout square (id est a homotopy colimit diagram in the sense of Remark A.1.21) in the local model structures.

Remark 1.1.13. An important class of local weak equivalences are induced by Čech complexes. The *Čech complex* of a morphism $f : U \to X$ of presheaves is a simplicial presheaf $\check{C}(U)$ (or $\check{C}(f)$) defined by

$$\check{C}_n(U) = U \times_X \ldots \times_X U \qquad (n + 1\text{-times})$$

with structure maps given by the diagonals and projections together with the canonical factorization $U \xrightarrow{i_f} \check{C}(U) \xrightarrow{p_f} X$ of f. The morphism p_f is always a stalkwise fibration but if f is a stalkwise epimorphism, p_f is also a local weak equivalence which is implied by the corresponding statement for simplicial sets on the Nisnevich-points [MV99, Lemma 2.1.15].

In particular, if $\{U_i \to X\}$ is a covering and $f : \coprod yU_i \to yX$, this evaluates as

$$\check{C}_n(f) \cong \coprod_{i_0,\ldots,i_n} yU_{i_0} \times_{yX} \ldots \times_{yX} yU_{i_n}.$$

It is helpful to note that $U \times_X U \cong U$ if $U \to X$ is a monomorphism and one writes shortly

$$\begin{array}{rcl} \check{C}(f) & \cong & \Big(\ldots \;\substack{\rightarrow\\\rightarrow\\\rightarrow}\; \coprod_{i_0,i_1} U_{i_0 i_1} \;\rightrightarrows\; \coprod_{i_0} U_{i_0} \Big) \\ & \overset{ob}{\sim} & \operatorname{hocolim} \Big(\ldots \;\substack{\rightarrow\\\rightarrow\\\rightarrow}\; \coprod_{i_0,i_1} U_{i_0 i_1} \;\rightrightarrows\; \coprod_{i_0} U_{i_0} \Big) \end{array}$$

for the simplicial presheaf associated to the covering where for the last objectwise weak equivalence, the Čech complex is viewed as a bisimplicial presheaf since every level can be considered as a discrete simplicial presheaf. As the U_i are simplicially discrete, this object is objectwise weak equivalent to the (simplicially discrete) simplicial presheaf

$$\pi_0(\check{\mathrm{C}}(f)) = \operatorname{colim}\left(\coprod_{i_0,i_1} U_{i_0 i_1} \rightrightarrows \coprod_{i_0} U_{i_0}\right)$$

by [Dug98, Lemma 3.3.5]. For a covering, f is a stalkwise epimorphism and there is a local weak equivalence $\check{\mathrm{C}}(f) \xrightarrow{\sim} X$ as mentioned before. Therefore X may be considered as a sheaf up to local weak equivalence (confer also [Dug98, Lemma 4.1.6]).

Recall that an objectwiese projective fibrant simplicial presheaf F is said to *satisfy descent* with respect to $f : U \to X$ if the canonical morphism

$$F(X) \to \operatorname{holim} F(\check{\mathrm{C}}(f))$$

is a weak equivalence of simplicial sets and it *satisfy descent with respect to a topology* if it satisfies descent for all the coverings of the topology in the above sense. Hence a simplicially discrete F satisfy descent with respect to a topology if and only if F is a sheaf for that topology. All local injective fibrant objects F satisfy descent for the chosen topology defining the local structure [DHI04, Theorem 1.1.]. As mentioned before, the converse to this statement is not true, even for objectwise injective fibrant objects [DHI04, Example A.10].

Remark 1.1.14. A fibrant simplicial set viewed as a simplicial presheaf is a fibrant object in the local model structures. A simplicially constant simplicial presheaf is fibrant in the local models if and only if it is a sheaf [Jar11, Lemma 5.11]. In particular, the objects from $\mathcal{S}\mathrm{m}_S$ are fibrant. According to Remark 1.1.1, the presheaf $\mathrm{Pic}(-)$ is not a Nisnevich sheaf and therefore not fibrant in the local models. The objects from $\mathcal{S}\mathrm{m}_S$ and the simplicial sets are cofibrant for the local model structures as can be observed by considering the generating cofibratons of Remark 1.1.4.

Since $- \times \Delta^1$ induces cylinder objects for the local injective model, it follows from the cofibrancy and fibrancy of the simplicially constant simplicial presheaves $\hom(-, U)$ that there is an embedding $\mathcal{S}\mathrm{m}_S \hookrightarrow \mathcal{H}^s$.

Definition 1.1.15. A simplicial presheaf X is called $\mathbb{A}^1$*-local* if the induced map

$$[Y, X]_s \xrightarrow{pr^*} [Y \times \mathbb{A}^1, X]_s$$

is a bijection for all simplicial presheaves Y.

An object X of $\mathcal{S}\mathrm{m}_S$ is called $\mathbb{A}^1$*-rigid* if the induced map

$$X(U) = \hom_{\mathcal{S}\mathrm{m}_S}(U, X) \xrightarrow{pr^*} \hom_{\mathcal{S}\mathrm{m}_S}(U \times \mathbb{A}^1, X) = X(U \times \mathbb{A}^1)$$

is a bijection for all objects U of $\mathcal{S}\mathrm{m}_S$.

Lemma 1.1.16. Choose either the local injective or the local projective model. If X is fibrant in this model, then X is $\mathbb{A}^1$*-local* if and only if it is $\{U \times \mathbb{A}^1 \to U\}$-local in the sense of Definition A.3.7. The latter condition writes out as

$$\mathrm{s}\mathcal{S}\mathrm{et}(U, X) \xrightarrow{pr^*} \mathrm{s}\mathcal{S}\mathrm{et}(U \times \mathbb{A}^1, X)$$

being a weak equivalence for all objects U of $\mathcal{S}\mathrm{m}_S$.

An object of $\mathcal{S}\mathrm{m}_S$ is $\mathbb{A}^1$-rigid if and only if it is $\mathbb{A}^1$-local.

Proof. We consider the injective situation as the projective is analogous. Choose a skeleton of the category $\mathcal{S}\mathrm{m}_S$ without changing the notation. Let X be a local injective fibrant simplicial presheaf. To be $\{U \times \mathbb{A}^1 \to U\}$-local is by an enriched version of the Yoneda lemma equivalent to $X(U) \to X(U \times \mathbb{A}^1)$ being a weak equivalence for all objects U of $\mathcal{S}\mathrm{m}_S$. This implies the last statement of the lemma. One has

$$\begin{aligned}
[U \times A, X]_s &\cong \pi_0 \mathrm{s}\mathcal{S}\mathrm{et}(U \times A, X) \\
&\cong [\Delta^0, \mathrm{s}\mathcal{S}\mathrm{et}(U \times A, X)]_{\mathrm{s}\mathcal{S}\mathrm{et}} \\
&\cong [\Delta^0, \underline{\mathrm{hom}}(U \times A, X)(S)]_{\mathrm{s}\mathcal{S}\mathrm{et}} \\
&\cong [\Delta^0, \underline{\mathrm{hom}}(A, \underline{\mathrm{hom}}(U, X))(S)]_{\mathrm{s}\mathcal{S}\mathrm{et}} \\
&\cong [\Delta^0, \underline{\mathrm{hom}}(A, \mathrm{s}\mathcal{S}\mathrm{et}(U, X))(S)]_{\mathrm{s}\mathcal{S}\mathrm{et}} \\
&\cong [A, \mathrm{s}\mathcal{S}\mathrm{et}(U, X)]_{\mathrm{s}\mathcal{S}\mathrm{et}}
\end{aligned}$$

by the monoidality for all simplicial sets A and objects U of $\mathcal{S}\mathrm{m}_S$. Hence, the Yoneda lemma shows that $\mathbb{A}^1$-locality implies $\{U \times \mathbb{A}^1 \to U\}$-locality of X. The converse direction is a consequence of lemma A.3.10 since Theorem A.3.8 allows a Bousfield localization at the set $\{U \times \mathbb{A}^1 \to U\}$. □

Remark 1.1.17. Choose either the local injective or the local projective model. A fibrant X in this model is $\{U \times \mathbb{A}^1 \to U\}$-local in the sense of Definition A.3.7 if and only if

$$\mathrm{s}\mathcal{S}\mathrm{et}(U, X) \xrightarrow{pr^*} \mathrm{s}\mathcal{S}\mathrm{et}(U \times \mathbb{A}^1, X)$$

is a weak equivalence for all connected objects U of $\mathcal{S}\mathrm{m}_S$. Each object U of $\mathcal{S}\mathrm{m}_S$ is a finite coproduct $\coprod_i U_i$ of connected schemes U_i in the category $\mathcal{S}\mathrm{m}_S$. Hence, the statement follows from $yU \sqcup yV \sim y(U \sqcup V)$ being local weakly equivalent, the fact that $\mathrm{s}\mathcal{S}\mathrm{et}(-, X)$ is a Quillen functor by Definition A.3.1, the equation

$$\mathrm{s}\mathcal{S}\mathrm{et}(U \sqcup V, X) \cong \mathrm{s}\mathcal{S}\mathrm{et}(U, X) \times \mathrm{s}\mathcal{S}\mathrm{et}(V, X)$$

and the fact that taking π_0 commutes with products.

Remark 1.1.18. The projective n-space $\mathbb{P}^n$ is not $\mathbb{A}^1$-rigid as there are non-constant morphisms $\mathbb{A}^1 \to \mathbb{P}^n$.

The $\mathbb{A}^1$-rigidity of $\mathbb{G}_m = S \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(\mathbb{Z})[x, y]/(xy - 1)$ however may be seen as follows: As $\mathbb{G}_m$ is a Zariski-sheaf, so is $\underline{\mathrm{hom}}_{\mathrm{Pre}}(X, \mathbb{G}_m)$ for every presheaf X (confer Remark 1.1.3). Hence, we may argue Zariski locally and therefore affine. The assertion follows from the fact, that $x \mapsto 0$ maps the units of $A[x]$ isomorphically to the units of A.

Theorem 1.1.19 (Morel, Voevodsky)**.** Choose a skeleton of $\mathcal{S}\mathrm{m}_S$ without changing the notation. The Bousfield localization of the local injective model structure on sPre (respectively on sShv) at the set $\{U \times \mathbb{A}^1 \to U\}$ is monoidal, proper, combinatorial and simplicial. It is called the $\mathbb{A}^1$-*local injective* model structure and weak equivalences, cofibrations and fibrations are denoted likewise.

The Bousfield localization of the local projective model structure on sPre (respectively on sShv) at the set $\{U \times \mathbb{A}^1 \to U\}$ is monoidal, proper, combinatorial and simplicial. It is called the $\mathbb{A}^1$-*local projective* or *motivic model structure* and weak equivalences, cofibrations and fibrations are denoted likewise.

There is a diagram of monoidal Quillen equivalences

$$\begin{array}{ccc}
\mathrm{sPre}_{\mathbb{A}^1 proj} & \underset{i}{\overset{a}{\rightleftarrows}} & \mathrm{sShv}_{\mathbb{A}^1 proj} \\
\Big\downarrow\Big\uparrow \mathrm{id} & & \Big\downarrow\Big\uparrow \mathrm{id} \\
\mathrm{sPre}_{\mathbb{A}^1 linj} & \underset{i}{\overset{a}{\rightleftarrows}} & \mathrm{sShv}_{\mathbb{A}^1 inj}.
\end{array}$$

The associated $\mathbb{A}^1$-*local homotopy category* (or *motivic homotopy category*) is denoted by $\mathcal{H}^{\mathbb{A}^1}$ or $\mathcal{H}^{\mathbb{A}^1}(S)$ with morphism sets $[-,-]_{\mathbb{A}^1}$ or just by $\mathcal{H}$ or $\mathcal{H}(S)$ with morphism sets $[-,-]$, if no confusion is possible.

Proof. The existence, the left-properness, combinatoriality and simpliciality of the two Bousfield localizations follow from Theorem A.3.8 respectively. The right-properness is a non-trivial issue and was proven for simplicial sheaves in the injective setting in [MV99, Theorem 2.2.7] and for simplicial presheaves in the injective setting in [Jar11, Corollary 6.21]. The projective setting is treated in [Bla01, Theorem 3.2] (The reference to [MV99] in the context of presheaves can be bypassed citing [Jar00a, Theorem A.5]). The monoidality is a consequence of the monoidality of the local structure and Remark A.3.9 whose assumptions hold by the existence of a strong monoidal (with respect to the product) fibrant replacement functor $R^{\mathbb{A}^1}$ as discussed in Remark 1.1.21 below. The horizontal adjunctions of the diagram are monoidal by Lemma A.2.5 and Quillen adjunctions by [Hir03, Theorem 3.3.20]. □

Remark 1.1.20. As a consequence of Remark 1.1.14, Remark 1.1.18 and Lemma A.3.10, the projective n-space $\mathbb{P}^n$ is not motivically fibrant, whereas $\mathbb{G}_m$ is $\mathbb{A}^1$-local injective fibrant.

The functor $- \times \mathbb{A}^1$ induces cylinder objects for the $\mathbb{A}^1$-local injective model.

Remark 1.1.21. In the discussion around [MV99, Lemma 3.2.6], Morel and Voevodsky construct a fibrant replacement functor $R^{\mathbb{A}^1}$ for the $\mathbb{A}^1$-local injective model structure on sShv given by

$$X \mapsto R^s \circ (R^s \circ \mathcal{S}\mathrm{ing})^{|\mathbb{N}|} \circ R^s$$

where R^s denotes some local injective fibrant replacement functor and $\mathcal{S}$ing the *Suslin-Voevodsky construction* or the *singular functor*, id est the functor

$$\begin{array}{rcl} \mathcal{S}\mathrm{ing} : \mathrm{sPre} & \to & \mathrm{sPre} \\ X & \mapsto & (U,[n]) \mapsto X(U \times \Delta^n)_n \end{array}$$

restricted to simplicial sheaves where the cosimplicial object $\Delta^{(\text{-})} : \Delta \to \mathcal{S}\mathrm{m}_S$ is given by

$$\Delta^n = S \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(\mathbb{Z})[x_0,\dots,x_n]/(1-\textstyle\sum x_i),$$

which is non-canonically isomorphic to $\mathbb{A}^n$ as explained after [MV99, Lemma 3.2.5]. According to [MV99, Theorem 2.1.66], one may choose a local injective fibrant replacement functor commuting with finite limits. Together with the fact that filtered colimits of sets commute with finite limits (confer [Bor94a, Theorem 2.13.4]) and together with the property of the singular functor commuting with all limits, there exists a strong monoidal $\mathbb{A}^1$-local injective fibrant replacement functor.

Proposition 1.1.22. There is an $\mathbb{A}^1$-local injective fibrant replacement functor $R^{\mathbb{A}^1}$ on sPre given by

$$X \mapsto R^s \circ (R^s_p \circ \mathcal{S}\mathrm{ing})^{|\mathbb{N}|}$$

where R^s_p denotes some local projective fibrant replacement functor and where R^s denotes some local injective fibrant replacement functor.

Proof. The method of the proof follows [MV99, Lemma 3.21]. It is to show that $R^{\mathbb{A}^1}(X)$ is $\mathbb{A}^1$-local injective fibrant for a simplicial presheaf X. As $R^{\mathbb{A}^1}(X)$ is local injective fibrant, it suffices by Lemma 1.1.16 to show, that $R^{\mathbb{A}^1}(X)$ is $\mathbb{A}^1$-local but since $\mathbb{A}^1$-locality is invariant under local weak equivalence, we may as well prove that $R'(X) = (R_p^s \circ \mathcal{S}\mathrm{ing})^{|\mathbb{N}|}(X)$ is $\mathbb{A}^1$-local. The simplicial presheaf $R'(X)$ is local projective fibrant as there is a set $\mathcal{J}'$ of generating acyclic cofibrations detecting the fibrant objects (confer Remark A.3.8) such that the domains and codomains of these cofibrations are finitely presentable (confer [DRØ03b, Lemma 2.15.]). Therefore, we have to show that

$$R'(X)(U) \cong \mathrm{s}\mathcal{S}\mathrm{et}(U, R'(X)) \xrightarrow{pr^*} \mathrm{s}\mathcal{S}\mathrm{et}(U \times \mathbb{A}^1, R'(X)) \cong R'(X)(U \times \mathbb{A}^1)$$

is a weak equivalence of simplicial sets for every U of $\mathcal{S}\mathrm{m}_S$ by Lemma 1.1.16. Pick an U of $\mathcal{S}\mathrm{m}_S$. Since the morphism pr^* is a section, it suffices to show that for all basepoints $x : \Delta^0 \to R'(X)(U)$ the map

$$\pi_i(R'(X)(U), x) \xrightarrow{(pr^*)_*} \pi_i(R'(X)(U \times \mathbb{A}^1), pr^*(x))$$

is a surjection for all $i \geq 0$. Let x be such a basepoint, $i \geq 0$ an integer and

$$[lb] \in \pi_i(R'(X)(U \times \mathbb{A}^1), pr^*(x))$$

an arbitrary element. As $R'(X)(U \times \mathbb{A}^1)$ is a fibrant simplicial set, the equivalence class $[lb]$ is represented by a morphism $lb : \partial\Delta^{i+1} \to R'(X)(U \times \mathbb{A}^1)$ under Δ^0 or equivalently by considering the adjoint maps of simplicial presheaves, it is represented by a morphism $lb : U \times \mathbb{A}^1 \times \partial\Delta^{i+1} \to R'(X)$. This morphism is under $U \times \mathbb{A}^1$ or more precisely, the composition $U \times \mathbb{A}^1 \times \Delta^0 \xrightarrow{\mathrm{id} \times *} U \times \mathbb{A}^1 \times \partial\Delta^{i+1} \xrightarrow{lb} R'(X)$ equals $U \times \mathbb{A}^1 \times \Delta^0 \xrightarrow{pr} U \xrightarrow{x} R'(X)$. Since lb has a finitely presentable domain, it factorizes through some object ${}^nR'(X) = (R_p^s \circ \mathcal{S}\mathrm{ing})^n(X)$ and by possibly enlarging n, we assume that the basepoint factorizes as well. Therefore, we obtain a morphism $b : U \times \mathbb{A}^1 \times \partial\Delta^{i+1} \to {}^nR'(X)$.

Consider the diagram

$$\begin{array}{ccccc} & & U \times \mathbb{A}^1 \times \partial\Delta^{i+1} & \xrightarrow{b} & {}^nR'(X) \\ & \overset{c}{\nearrow} & \uparrow{\scriptstyle i_0 \times \mathrm{id}} & \underset{b'}{\nearrow} & \\ U \times \mathbb{A}^1 \times \partial\Delta^{i+1} & \xrightarrow{pr \times \mathrm{id}} & U \times \partial\Delta^{i+1} & & \end{array}$$

where c and b' are defined as the compositions respectively. As c is elementary $\mathbb{A}^1$-homotopic to the identity (confer [MV99, Section 2.3] for a definition of this term), the morphism b is elementary $\mathbb{A}^1$-homotopic to the composition $bc = b' \circ (pr \times \mathrm{id})$ and hence their adjoints

$$b, pr^*b' : \partial\Delta^{i+1} \to {}^nR'(X)(U \times \mathbb{A}^1)$$

are elementary $\mathbb{A}^1$-homotopic. Since [MV99, Proposition 2.3.4] holds for simplicial presheaves as well, applying the functor $\mathcal{S}\mathrm{ing}$ shows that there is an elementary simplicial homotopy (id est a homotopy with respect to the cylinder object induced by $- \times \Delta^1$) between the two morphisms

$$\mathcal{S}\mathrm{ing}(b), \mathcal{S}\mathrm{ing}(pr^*b') : \partial\Delta^{i+1} = \mathcal{S}\mathrm{ing}(\partial\Delta^{i+1}) \to \mathcal{S}\mathrm{ing}({}^nR'(X)(U \times \mathbb{A}^1))$$

and hence also between these two morphisms composed with

$$\mathcal{S}\mathrm{ing}({}^nR'(X)(U \times \mathbb{A}^1)) \xrightarrow{r} {}^{n+1}R'(X)(U \times \mathbb{A}^1).$$

With the identifications above, following the basepoints and considering the canonical morphism $l' : {}^{n+1}R^{\mathbb{A}^1}(X) \to R'(X)$ into the colimit, this shows that

$$[l' r \mathcal{S}\mathrm{ing}(b')] \in \pi_i(R^{\mathbb{A}^1}(X)(U), x)$$

is a preimage of $[lb] = [l' r \mathcal{S}\mathrm{ing}(b)]$ under $(pr^*)_*$ which implies the result. □

Remark 1.1.23. As far as the author knows, it is not known whether $\mathcal{S}\mathrm{ing}(X)$ is $\mathbb{A}^1$-local for every object X from $\mathcal{S}\mathrm{m}_S$ although there are counterexamples for a simplicially discrete X [MV99, Example 3.2.7]. According to [Mor12, Theorem 7.2], $\mathcal{S}\mathrm{ing}(GL_n)$ with $n \neq 2$ is $\mathbb{A}^1$-local for example.

Remark 1.1.24. Following Remark A.4.2, we may equip the categories sPre and sShv with an initial object S and denote them by sPre_* or $\mathrm{sPre}(S)_*$ and Shv_* or $\mathrm{sShv}(S)_*$ respectively. One gets pointed, monoidal combinatorial and simplicial analogues of all the considered model structures by Lemma A.4.5 and Lemma A.4.7 which we call likewise as in the unpointed setting when no confusion is possible. The models are right proper since the functor which forgets the basepoint preserves limits and pushouts (confer Lemma A.4.3).

According to [Hir03, Theorem 3.3.20], one gets the pointed model structures also by performing a Bousfield localization at those sets obtained by applying the functor $(-)_*$ to the unpointed sets localized at. More precisely and similar for the injective situation, there is an objectwise projective monoidal, proper, combinatorial and simplicial model structure on $\mathrm{sPre}_* \cong \mathcal{F}\mathrm{un}(\mathcal{S}\mathrm{m}_S, \mathrm{s}\mathcal{S}\mathrm{et}_*)$ which can again be localized twice where we implicitly choose a skeleton of the category $\mathcal{S}\mathrm{m}_S$: For the first step and to obtain the local projective model structure, one localizes the objectwise projective model at the pointed versions $P(Q)_* \to X_*$ of the morphisms considered in Theorem 1.1.10. One should note, that applying a canonical point to a morphism of pointed simplicial presheaves yields a map of pointed simplicial sets. A second Bousfield localization at the set $\{U_* \wedge \mathbb{A}^1_* \to U_*\}$ provides the motivic model structure on sPre_* with homotopy category denoted by $\mathcal{H}_*$ or $\mathcal{H}_*^{\mathbb{A}^1}$. For a canonical point p^*, it holds

$$p^*(X \wedge Y) = p^*X \wedge p^*Y.$$

The statement [DRØ03b, Corollary 2.16] shows that the motivic model structure on sPre_* is weakly finitely generated and thus shares the properties of Lemma A.1.16 where a set $\mathcal{J}'$ of generating acyclic cofibrations with the necessary properties may be constructed by choosing a skeleton of $\mathcal{S}\mathrm{m}_S$ and considering Remark A.3.9.

Moreover, it is shown in [DRØ03b, Lemma 2.20], that smashing with an arbitrary simplicial presheaves preserves motivic weak equivalences.

1.2. Homotopy sheaves and fiber sequences

Definition 1.2.1. Let X be an element of sPre. One defines the *Nisnevich sheaf of local connected components* of X on $\mathcal{S}\mathrm{m}_S$ as

$$\pi_0^s(X) = a[-, X]_{ob} \cong a[-, X]_s$$

and the *Nisnevich sheaf of* $\mathbb{A}^1$*-connected components* of X on $\mathcal{S}\mathrm{m}_S$ as

$$\pi_0^{\mathbb{A}^1}(X) = a[-, X]_{\mathbb{A}^1}.$$

The simplicial presheaf X is called *locally connected* if $\pi_0^s(X) \cong *$ and X is called $\mathbb{A}^1$*-connected* if $\pi_0^{\mathbb{A}^1}(X) \cong *$.

Remark 1.2.2. The identification of the sheafified objectwise and the sheafified local homotopy classes follows from the natural isomorphism

$$\begin{aligned} \pi_0(p^*X) &\cong \pi_0(\operatorname{colim}_V X(V)) \\ &\cong \operatorname{colim} \pi_0(X(V)) \\ &\cong p^*\pi_0(X(-)) \\ &\cong p^*a\pi_0(X(-)) \\ &\cong p^*a[-,X]_{ob} \end{aligned}$$

for an objectwise fibrant X and a canonical point p applied to the local weak equivalence $X \to R^s X$. Moreover, if R^{ob} denotes the objectwise fibrant replacement functor obtained by applying Kan's Ex-functor R objectwise, one has

$$\pi_0(p^*R^{ob}X) \cong \pi_0(Rp^*X) \cong \pi_0(p^*X).$$

as R commutes with filtered colimits.

Remark 1.2.3. The Unstable $\mathbb{A}^1$-connectivity theorem 1.2.20 implies that a locally connected simplicial presheaf is also $\mathbb{A}^1$-connected. The projective line $\mathbb{P}^1$ is an example of a non-locally but $\mathbb{A}^1$-connected object (confer the diagram of Remark 1.1.1). The object $\mathbb{G}_m$ of $\mathcal{S}\mathrm{m}_S$ is not $\mathbb{A}^1$-connected. According to [AM11, Lemma 2.2.11, Corollary 2.4.4], a connected proper object of $\mathcal{S}\mathrm{m}_k$ is $\mathbb{A}^1$-connected if it has a finite Zariski covering by some $\mathbb{A}^n_k$ such that each intersection of those has a k-point (the last condition is obsolete if k has infinite many elements). Hence, each projective space $\mathbb{P}^n$ is $\mathbb{A}^1$-connected.

If k is a field of characteristic zero, then $\mathbb{A}^1$-connectedness of connected proper objects of $\mathcal{S}\mathrm{m}_k$ is invariant under k-birational equivalence [AM11, Corollary 2.4.6] where connected objects X and Y from $\mathcal{S}\mathrm{m}_k$ are called k-birationally equivariant, if there is a k-algebra isomorphism $k(X) \cong k(Y)$ of their function fields.

Definition 1.2.4. Let X be an element of sPre_*. For $n \geq 0$ one defines the *n-th local homotopy sheaf* of X as

$$\pi_n^s(X) = a\langle(-)_* \wedge S^n, X\rangle_{ob} \cong a\langle(-)_* \wedge S^n, X\rangle_s$$

and the *n-th $\mathbb{A}^1$-homotopy sheaf* as

$$\pi_n^{\mathbb{A}^1}(X) = a\langle(-)_* \wedge S^n, X\rangle_{\mathbb{A}^1}.$$

A pointed simplicial presheaf X is called *locally n-connected* if $\pi_i^s(X) \cong *$ for all $i \leq n$ and *$\mathbb{A}^1$-n-connected* if $\pi_i^{\mathbb{A}^1}(X) \cong *$ for all $i \leq n$. If the morphism $X \to S$ is a motivic weak equivalence, X is called *$\mathbb{A}^1$-contractible*.

Remark 1.2.5. For $n = 0$ and a pointed simplicial presheaf, this definition is consistent with Definition 1.2.1. It is not true that a morphism of pointed simplicial presheaves is a local (respectively motivic) weak equivalence, if it induces an isomorphism on all local (respectively $\mathbb{A}^1$-) homotopy sheaves, even if one takes all basepoints $S \to X$ into account. Instead, one has to consider all morphisms $U \to X$ for schemes U in $\mathcal{S}\mathrm{m}_S$ which may be interpreted as basepoints of $\mathcal{S}\mathrm{m}_U$. Due to the facts that homotopy groups of simplicial sets commute with filtered colimits, the canonical Nisnevich points are given as filtered colimits and points commute with colimits and finite limits, one may define local (respectively motivic) weak equivalences as those morphisms inducing an isomorphism on all local (respectively $\mathbb{A}^1$-) homotopy sheaves with respect to all $U \to X$ (confer Remark 1.2.2, [Jar96, Lemma 26]).

Remark 1.2.6. The $\mathbb{A}^1$-homotopy sheaves $\pi_n^{\mathbb{A}^1}(X) : \mathcal{S}\mathrm{m}_S^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ interpreted as simplicially discrete objects of sPre are fibrant in the local structures and one is tempted to ask, whether they are fibrant in the $\mathbb{A}^1$-local structures. This is precisely to ask, if they are $\mathbb{A}^1$-local by Lemma 1.1.16. Although the $\mathbb{A}^1$-homotopy presheaves are $\mathbb{A}^1$-local by definition, this is not at all clear for their sheafification. For $n \geq 1$ this was shown by Morel [Mor12] but the case $n = 0$ remains open [Mor12, Conjecture 12].

Remark 1.2.7. Along with the terms introduced in Section A.6, we may consider *local cofiber sequences* and *local fiber sequences* as well as *motivic cofiber sequences* and *motivic fiber sequences*

$$F \to E \to X.$$

Due to Lemma A.6.7, its dual and Remark A.1.19, we do not have to refer to the specific injective or projective model when dealing with cofiber sequences and fiber sequences. Moreover, we do not have to specify the action by Lemma A.6.7. One should recall the following facts with the help of Lemma A.3.10 and Lemma A.6.7.

- Every diagram $A' \to B' \to C'$ isomorphic in $\mathcal{H}_*^s$ to a local cofiber sequence $A \to B \to C$ is a local cofiber sequence of the same isomorphism class. The same holds for motivic cofiber sequences and $\mathcal{H}_*^{\mathbb{A}^1}$ instead of $\mathcal{H}_*^s$.
- If $A \to B \to C$ is a local cofiber sequence, then it is a motivic cofiber sequence.
- Every diagram $F' \to E' \to X'$ isomorphic in $\mathcal{H}_*^s$ to a local fiber sequence $F \to E \to X$ is a local fiber sequence of the same isomorphism class. The same holds for motivic fiber sequences and $\mathcal{H}_*^{\mathbb{A}^1}$ instead of $\mathcal{H}_*^s$.
- A local fiber sequence $F \to E \to X$ is a motivic fiber sequence if and only if $R^{\mathbb{A}^1}F \to R^{\mathbb{A}^1}E \to R^{\mathbb{A}^1}X$ is a local fiber sequence.
- A local fiber sequence $F \to E \to X$ is a motivic fiber sequence if and only if $R^{\mathbb{A}^1}\mathrm{hofib}^s(E \to X)$ is isomorphic in the pointed local homotopy category $\mathcal{H}_*^s$ to $\mathrm{hofib}^s(R^{\mathbb{A}^1}E \to R^{\mathbb{A}^1}X) \sim \mathrm{hofib}^{\mathbb{A}^1}(E \to X)$.

In the sequel, we will consider fiber sequences up to isomorphism without explicitly mentioning it.

According to Lemma A.6.3 and the exactness of the sheafification functor a, one gets a long exact sequence of (pointed) local homotopy sheaves

$$\ldots \to \pi_2^s(X) \to \pi_1^s(F) \to \pi_1^s(E) \to \pi_1^s(X) \to \pi_0^s(F) \to \pi_0^s(E) \to \pi_0^s(X)$$

for a local fiber sequence $F \to E \to X$ and a long exact sequence of (pointed) $\mathbb{A}^1$-homotopy sheaves

$$\ldots \to \pi_2^{\mathbb{A}^1}(X) \to \pi_1^{\mathbb{A}^1}(F) \to \pi_1^{\mathbb{A}^1}(E) \to \pi_1^{\mathbb{A}^1}(X) \to \pi_0^{\mathbb{A}^1}(F) \to \pi_0^{\mathbb{A}^1}(E) \to \pi_0^{\mathbb{A}^1}(X)$$

for a motivic fiber sequence $F \to E \to X$.

Remark 1.2.8. A diagram of simplicial presheaves

$$\begin{array}{ccc} P & \longrightarrow & C \\ \downarrow & & \downarrow \\ B & \longrightarrow & A \end{array}$$

is a homotopy pullback in the local homotopical category if and only if it is a homotopy pullback on all canonical points p^*. This is a consequence of the fact that p^* sends local injective fibrations to fibrations of simplicial sets [Jar11, Corollary 4.14] and commutes with finite limits. As a consequence, a diagram $F \to E \to X$ of pointed simplicial presheaves is a local fiber sequence if and only if $p^*F \to p^*E \to p^*X$ is a fiber sequence of pointed simplicial sets for every canonical point p^*.

By the same argument, a diagram is a homotopy colimit diagram for the local structures if and only if it is a homotopy colimit diagram on all canonical points.

Definition 1.2.9 (Rezk, [Rez10])**.** A morphism $p : E \to X$ of simplicial presheaves is called *sharp* with respect to a model structure, if for each diagram

$$\begin{array}{ccccc} C & \xrightarrow{i} & D & \longrightarrow & E \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle p} \\ Z & \underset{\sim}{\xrightarrow{j}} & Y & \longrightarrow & X \end{array}$$

consisting of pullbacks and a weak equivalence j, the morphism i is a weak equivalence as well.

Remark 1.2.10. The class of sharp morphisms is closed under pullbacks. All model structures on simplicial presheaves considered in this text are right proper and hence every fibration is sharp with respect to the model [Rez98, Proposition 2.2]. Moreover and with respect to a right proper model, a morphism p is sharp if and only if each (categorical) pullback diagram

$$\begin{array}{ccc} F & \longrightarrow & E \\ \downarrow & & \downarrow{\scriptstyle p} \\ Y & \longrightarrow & X \end{array}$$

is a homotopy pullback diagram [Rez98, Proposition 2.7]. Since the definition of sharp morphisms refers to the homotopical category only (confer Remark A.1.19), we define the notion of *locally sharp* morphisms as the property of being sharp with respect to one of the local models and $\mathbb{A}^1$-*sharp* morphisms as being sharp with respect to one of the $\mathbb{A}^1$-local models.

Theorem 1.2.11 (Rezk, [Rez98, Theorem 5.1])**.** Stalkwise fibrations and projections out of finite products are locally sharp.

Remark 1.2.12. The previous Theorem 1.2.11 implies in particular, that any morphism between objects from $\mathcal{S}\mathrm{m}_S$ is locally sharp since every map between discrete simplicial sets is a Kan-fibration.

Lemma 1.2.13. Let

$$\begin{array}{ccc} F & \longrightarrow & E \\ \downarrow & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow{f} & X \end{array}$$

be a local homotopy pullback diagram of simplicial presheaves such that X is $\mathbb{A}^1$-local. Then the diagram is an $\mathbb{A}^1$-homotopy pullback diagram. In particular, a locally sharp map with an $\mathbb{A}^1$-local codomain is $\mathbb{A}^1$-sharp.

Proof. Before giving a general proof, we want to mention that for the special case of an $\mathbb{A}^1$-local Y, this can be seen by a more direct argument which is in a similar situation attributed to Denis-Charles Cisinski [Wen07, Section 4.3]: Choose for instance the

$\mathbb{A}^1$-local injective model. One factorizes f as a motivic weak equivalence $Y \to Y'$ followed by an $\mathbb{A}^1$-local injective fibration $Y' \to X$ and obtains the diagram

$$\begin{array}{ccccc} F & \xrightarrow{i} & E \times_X Y' & \longrightarrow & E \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow{\sim} & Y' & \twoheadrightarrow & X \end{array}$$

consisting of pullbacks. The right square is an $\mathbb{A}^1$-homotopy pullback which uses right properness of the $\mathbb{A}^1$-local injective structure. Hence, it suffices to show that i is a motivic weak equivalence. As X and Y are $\mathbb{A}^1$-local injective fibrant by Lemma 1.1.16, Y' is $\mathbb{A}^1$-local injective fibrant and the weak equivalence between Y and Y' is a local weak equivalence. Since the original diagram is a local homotopy pullback, i is a local weak equivalence and hence in particular a motivic weak equivalence.

For the general case, factorize p as a local weak equivalence $E \to E'$ followed by a local injective fibration $p' : E' \to X$. As the diagram of the lemma is a local homotopy pullback, there is a local weak equivalence $F \to Y \times_X E'$. Therefore and to keep the notation simple, we assume that p is a local injective fibration in the first place.

By [Jar00a, Lemma A.2.4], we can factorize f as a motivic weak equivalence $Y \to Y'$ followed by an $\mathbb{A}^1$-local injective fibration $Y' \to X$ with the special property that $Y \to Y'$ is a so called *elementary* $\mathbb{A}^1$*-cofibration*. The result then follows by [Jar00a, Lemma A.3]. □

Corollary 1.2.14. Let E and X be simplicial presheaves. The projection morphism $E \times X \to X$ is $\mathbb{A}^1$-sharp.

Proof. As sharp morphisms are stable under basechange, it suffices to show that $E \to *$ is $\mathbb{A}^1$-sharp. The projection $E \to *$ is locally sharp by Theorem 1.2.11 and hence the result follows from Lemma 1.2.13 since the terminal object $*$ is $\mathbb{A}^1$-local. □

Corollary 1.2.15 (Wendt, [Wen07, Proposition 4.4.1])**.** Let $F \to E \to X$ be a local fiber sequence. If X is $\mathbb{A}^1$-local, then it is a motivic fiber sequence.

Theorem 1.2.16 (Morel, [Mor12, Theorem 5.52])**.** Let S be the spectrum of a perfect field and $F \to E \to X$ a local fiber sequence. If X is locally 1-connected, then it is a motivic fiber sequence.

Remark 1.2.17. Let S be the spectrum of a field. For a simplicial presheaf X, there is a canonical natural morphism

$$X \to R^{ob}X \to \pi_0 R^{ob}X \to a\pi_0 R^{ob}X \cong \pi_0^s(X) \to a\pi_0 R^{\mathbb{A}^1} R^{ob} X \cong \pi_0^{\mathbb{A}^1}(X)$$

of simplicial presheaves. Since applying S is a point and there is a fibrant replacement functor R^{op} preserving the zero simplices and the latter map of the composition above is an epimorphism by Theorem 1.2.20, a basepoint $S \to \pi_0^{\mathbb{A}^1}(X)$ can be lifted to a basepoint $S \to X$. This implies that the property of having a basepoint $S \to X$ is invariant under motivic weak equivalence. In particular, every $\mathbb{A}^1$-connected simplicial presheaf X admits a basepoint $S \to X$.

If X is an $\mathbb{A}^1$-rigid object of $\mathcal{S}\mathrm{m}_S$, then the morphism $X \to \pi_0^{\mathbb{A}^1}(X)$ is an isomorphism of sheaves (and hence a local weak equivalence) but in general one may extract non-trivial $\mathbb{A}^1$-connected components from X, id est for a basepoint $S \to X$, there is an $\mathbb{A}^1$-connected pointed simplicial presheaf X' such that $\pi_n^{\mathbb{A}^1}(X') \cong \pi_n^{\mathbb{A}^1}(X)$ for all $n \geq 1$. To obtain X', consider the pullback square

$$\begin{array}{ccc} X' & \longrightarrow & R^{\mathbb{A}^1}X \\ \downarrow & & \downarrow f \\ * & \longrightarrow & \pi_0^{\mathbb{A}^1}(X). \end{array}$$

This is a local homotopy pullback square, as the two objects on the right-hand side are local projective fibrant by Remark 1.1.14 and f is a morphism into a simplicially discrete object, hence an objectwise fibration. Therefore, there is a local fiber sequence $X' \to R^{\mathbb{A}^1}X \to \pi_0^{\mathbb{A}^1}(X)$. As f is an isomorphism on π_0^s, one gets that X' is locally connected and by Theorem 1.2.20, it is $\mathbb{A}^1$-connected as well. Referring to the local long exact sequence of Remark 1.2.7, we would get an isomorphism of the higher $\mathbb{A}^1$-homotopy sheaves if X' is $\mathbb{A}^1$-local. The latter is true as X' is a retract of the $\mathbb{A}^1$-local object $R^{\mathbb{A}^1}X$.

The simplicial presheaf X' does not have to be scheme if X comes from $\mathcal{S}\mathrm{m}_S$. In contrast to topological spaces (of a reasonable type), a simplicial presheaf is usually not the coproduct of its $\mathbb{A}^1$-connected components.

Remark 1.2.18. Let A, B, X and Y be simplicial presheaves. We have

$$\begin{aligned} [A \sqcup B, X]_{\mathbb{A}^1} &\cong \pi_0 \mathrm{s}\mathcal{S}\mathrm{et}(A \sqcup B, R^{\mathbb{A}^1}X) \\ &\cong \pi_0(\mathrm{s}\mathcal{S}\mathrm{et}(A, R^{\mathbb{A}^1}X) \times \mathrm{s}\mathcal{S}\mathrm{et}(A, R^{\mathbb{A}^1}X)) \\ &\cong \pi_0 \mathrm{s}\mathcal{S}\mathrm{et}(A, R^{\mathbb{A}^1}X) \times \pi_0 \mathrm{s}\mathcal{S}\mathrm{et}(B, R^{\mathbb{A}^1}X) \\ &\cong [A, X]_{\mathbb{A}^1} \times [B, X]_{\mathbb{A}^1} \end{aligned}$$

and similarly, since there is an $\mathbb{A}^1$-local injective strong monoidal fibrant replacement functor by Remark 1.1.21, it holds

$$[A, X \times Y]_{\mathbb{A}^1} \cong [A, X]_{\mathbb{A}^1} \times [A, Y]_{\mathbb{A}^1}.$$

For pointed simplicial presheaves, one gets

$$\begin{aligned} \langle A \vee B, X \rangle_{\mathbb{A}^1} &\cong \langle A, X \rangle_{\mathbb{A}^1} \times \langle B, X \rangle_{\mathbb{A}^1}, \\ \langle A, X \times Y \rangle_{\mathbb{A}^1} &\cong \langle A, X \rangle_{\mathbb{A}^1} \times \langle A, Y \rangle_{\mathbb{A}^1}. \end{aligned}$$

Remark 1.2.19. For integers a and b with $a \geq b \geq 0$, one defines the *motivic sphere* $S^{a,b}$ as $S^{a,b} = S^{a-b} \wedge \mathbb{G}_m^b$. Due to the existence of the Zariski distinguished square

$$\begin{array}{ccc} \mathbb{A}^1 \setminus 0 & \longrightarrow & \mathbb{A}^1 \\ \downarrow & & \downarrow \\ \mathbb{A}^1 & \longrightarrow & \mathbb{P}^1 \end{array}$$

there is an isomorphism $S^{2,1} \cong \Sigma \mathbb{G}_m \cong \mathbb{P}^1$ in the pointed motivic homotopy category.

Theorem 1.2.20 (Morel, Unstable $\mathbb{A}^1$-connectivity theorem)**.** Let X be a simplicial presheaf. The canonical morphism $\pi_0^s X \to \pi_0^{\mathbb{A}^1} X$ is an epimorphism of sheaves. Moreover for $n \geq 1$, if S is the spectrum of a perfect field and if X is locally n-connected, then it is $\mathbb{A}^1$-n-connected.

Proof. The first statement is [MV99, Corollary 2.3.22] for simplicial sheaves and an argument is as follows: First, one notes that for a simplicial presheaf Y, there is a diagram

$$\begin{array}{ccc} Y_0 & = & (\mathcal{S}\mathrm{ing}Y)_0 \\ \downarrow & & \downarrow \\ \pi_0 Y & \xrightarrow{f} & \pi_0 \mathcal{S}\mathrm{ing}Y \end{array}$$

inducing an epimorphism $\pi_0 Y \to \pi_0 \mathcal{S}\mathrm{ing}Y$ of presheaves and hence an epimorphism $a\pi_0 Y \to a\pi_0 \mathcal{S}\mathrm{ing}Y$ of the associated sheafifications. According to Remark 1.1.21, a specific $\mathbb{A}^1$-local injective fibrant replacement functor is available and one calculates

$$\begin{aligned} p^*\pi_0^s X = \pi_0(p^*X) &\to \pi_0(p^*R^{\mathbb{A}^1}X) \\ &\cong \operatorname{colim}(\pi_0 p^*X \xrightarrow{\cong} \pi_0 p^* R^s X \xrightarrow{g} \pi_0 p^* \mathcal{S}\mathrm{ing} R^s X \to \dots) \\ &\cong p^*\pi_0^{\mathbb{A}^1} X \end{aligned}$$

for all canonical points p^* since π_0 commutes with filtered colimits and $X \to R^s X$ is a local weak equivalence. The assertion follows from the surjectivity of the morphisms g since to be an epimorphism may be tested on points [Jar11, Lemma 2.26].

The second assertion of the theorem is much harder to prove and is stated for a pointed simplicial sheaf X as [Mor12, Theorem 5.37]. The result for simplicial presheaves follows from the observation that the sheafification a does neither change the local nor the $\mathbb{A}^1$-homotopy sheaves. □

Corollary 1.2.21. Let S be the spectrum of a perfect field and let $a > b \geq 0$ be integers. Then, the motivic sphere $S^{a,b}$ is $\mathbb{A}^1$-$(a-b-1)$-connected.

Proof. The Corollary would follow from the Unstable $\mathbb{A}^1$-connectivity theorem 1.2.20 if $S^{a-b} \wedge \mathbb{G}_m^b$ would be locally $(a-b-1)$-connected. The observation that for all canonical points p^* one has

$$p^*(S^{a-b} \wedge \mathbb{G}_m^b) \cong p^* S^{a-b} \wedge p^* \mathbb{G}_m^b \cong S^{a-b} \wedge p^* \mathbb{G}_m^b$$

reduces the question to the topological problem of showing that the $(a-b)$-th suspension of a pointed simplicial set is $(a-b-1)$-connected. This is for example a consequence of the Hurewicz theorem in topology [GJ99, Theorem III.3.7]. □

Lemma 1.2.22. Let $f : E \to X$ be a morphism of pointed simplicial presheaves with an $\mathbb{A}^1$-connected target X. Then, the following conditions are equivalent:

(1) f is a motivic weak equivalence,
(2) $\pi_0^{\mathbb{A}^1}(E) \cong *$ and $\pi_n^{\mathbb{A}^1}(f) : \pi_n^{\mathbb{A}^1}(E) \to \pi_n^{\mathbb{A}^1}(X)$ are isomorphisms for all $n \geq 1$,
(3) the $\mathbb{A}^1$-homotopy fiber of f is $\mathbb{A}^1$-contractible.

Proof. The implications (1) ⇒ (2) and (1) ⇒ (3) are obvious.

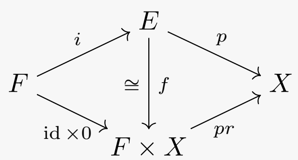
For implication (2) ⇒ (1), let p^* be an arbitrary canonical point. It is to show that $p^*R^{\mathbb{A}^1}f : p^*R^{\mathbb{A}^1}E \to p^*R^{\mathbb{A}^1}X$ is a weak equivalence of simplicial sets, id est $\pi_n(p^*R^{\mathbb{A}^1}E, x) \cong \pi_n(p^*R^{\mathbb{A}^1}X, p^*R^{\mathbb{A}^1}f(x))$ for all basepoints $x : \Delta^0 \to p^*R^{\mathbb{A}^1}E$. Since $\pi_0(p^*R^{\mathbb{A}^1}E)$ is trivial by assumption, this may be tested at just one arbitrary basepoint and we choose $\Delta^0 \cong p^*S \to p^*R^{\mathbb{A}^1}E$ induced by the basepoint of E in $\mathrm{sPre}(S)_*$. The isomorphism of the homotopy groups of simplicial sets is now exactly what is implied by condition (2) on the point p^*.

To prove (3) ⇒ (2), we may choose another model for $\mathrm{hofib}^{\mathbb{A}^1} \sim *$ as the point-set fiber of a fibration $f' : E' \to X'$ of motivically fibrant objects. Let p^* be a canonical point. By Remark 1.2.8, $p^*f' : p^*E' \to p^*X'$ is a fibration of simplicial sets with contractible fiber $p^*\mathrm{hofib}^{\mathbb{A}^1}$. Hence, from the triviality of $\pi_0(p^*X')$ it follows the triviality of $\pi_0(p^*E')$ and E is $\mathbb{A}^1$-connected. Since all homotopy fibers of p^*f' at different basepoints of p^*X' are equivalent by [Hir03, Proposition 13.4.7], condition (2) is implied by the exactness of the sequence of $\mathbb{A}^1$-homotopy sheaves. □

Lemma 1.2.23. Let X be a pointed simplicial presheaf. Then X is $\mathbb{A}^1$-connected if and only if it path connected in the sense of [CS06, Definition 5.1], this is for any morphism $f : E \to X$ of pointed simplicial presheaves it holds, that f is a motivic weak equivalence if and only if its $\mathbb{A}^1$-homotopy fiber $\mathrm{hofib}^{\mathbb{A}^1}(f)$ is $\mathbb{A}^1$-contractible.

Proof. Lemma 1.2.22 implies that an $\mathbb{A}^1$-connected simplicial presheaf is path connected. For the converse, let X be a path connected but not $\mathbb{A}^1$-connected simplicial presheaf which we assume to be $\mathbb{A}^1$-local injective fibrant. Now the same construction as in Remark 1.2.17 yields a morphism $X' \to X$ with trivial $\mathbb{A}^1$-homotoy fiber. Since X' is $\mathbb{A}^1$-connected, this cannot be a weak equivalence. □

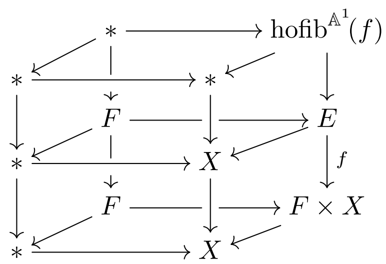
Proposition 1.2.24. Let X be a pointed $\mathbb{A}^1$-connected simplicial presheaf. Let $p : E \to X$ be a morphism in sPre_* with an $\mathbb{A}^1$-connected homotopy fiber F. The associated fiber sequence $F \to E \to X$ in $\mathcal{H}_*$ is trivial (confer Definition A.6.1), id est there is an isomorphism f making the diagram



commutative, if and only if there exists a morphism $g : E \to F$ in $\mathcal{H}_*$ such that the composition $F \xrightarrow{i} E \xrightarrow{g} F$ is an isomorphism.

Proof. If f exists, the composition $g : E \xrightarrow{f} F \times X \xrightarrow{pr} F$ with i is an isomorphism by the definition of id ×0 and by f being a morphism under F.

For the other direction, define f as the map into the product $F \times X$ in the category $\mathcal{H}_*$ induced by $g : E \to F$ and $p : E \to X$. Considering the cube



where the two lower horizontal squares are homotopy pullbacks and all vertical triples are homotopy fiber sequences, one concludes that the upper horizontal square is a homotopy pullback as well [Doe98, Lemma 1.4]. This means that $* \to \mathrm{hofib}^{\mathbb{A}^1} \to *$ is a motivic fiber sequence and it follows $\mathrm{hofib}^{\mathbb{A}^1}(f) \sim *$ by Lemma 1.2.22 after choosing a model for the fiber sequence. By the same Lemma 1.2.22 it is implied that f is a weak equivalence since F and X are $\mathbb{A}^1$-connected simplicial presheaves and so is $F \times X$. □

Remark 1.2.25. Theorem A.2.7 furnishes the pointed motivic homotopy category $\mathcal{H}_*$ with a closed symmetric monoidal structure with involved functors $- \wedge^{\mathcal{L}} -$ and $\underline{\text{hom}}_*^{\mathcal{R}}(-,-)$. Since every object is cofibrant in the pointed $\mathbb{A}^1$-local injective model, we will by abuse of notation often just write $- \wedge -$ instead of $- \wedge^{\mathcal{L}} -$.

For integers a, b with $a \geq b \geq 0$, we set $\Sigma^{a,b} : \text{sPre}_* \to \text{sPre}_*$ as the functor $- \wedge S^{a,b}$ and moreover $\Omega^{a,b} : \text{sPre}_* \to \text{sPre}_*$ as the functor $\underline{\text{hom}}(S^{a,b}, R^{\mathbb{A}^1}(-))$. This notion behaves well with the simplicial suspension and loop functors where $b = 0$.

Definition 1.2.26. Let X be an element of $\text{sPre}(S)_*$. For $a \geq b \geq 0$ one defines the *a-th $\mathbb{A}^1$-homotopy sheaf with weight b* of X as

$$\pi_{a,b}^{\mathbb{A}^1}(X) = a\langle(-)_* \wedge S^{a,b}, X\rangle_{\mathbb{A}^1}$$

and the *a-th local homotopy sheaf with weight b* of X analogously.

Theorem 1.2.27 (Morel, Algebraic loop space theorem)**.** Let S be the spectrum of a perfect field and X a pointed simplicial presheaf. If X is $\mathbb{A}^1$-connected, then $\Omega^{1,1}X$ is $\mathbb{A}^1$-connected.

Proof. For a pointed simplicial sheaf X, this is [Mor12, Theorem 5.12] and it was already observed that local weak equivalences like sheafification do not affect simplicial or $\mathbb{A}^1$-homotopy groups. □

Corollary 1.2.28. Let S be the spectrum of a perfect field, $a \geq b \geq 0$ integers and X a pointed simplicial presheaf. If $\pi_{a-b,0}^{\mathbb{A}^1}(X) \cong *$, then $\pi_{a,b}^{\mathbb{A}^1}(X) \cong *$.

Proof. One calculates

$$\begin{array}{rclcl} \pi_{a-b,0}^{\mathbb{A}^1}(X) & \cong & a\langle(-)_* \wedge S^{a-b,0}, X\rangle_{\mathbb{A}^1} & \cong & 0 \\ & \cong & a\langle(-)_* \wedge S^{0,0}, \Omega^{a-b,0}X\rangle_{\mathbb{A}^1} & & \\ & \Rightarrow & a\langle(-)_* \wedge S^{0,0}, \Omega^{b,b}\Omega^{a-b,0}X\rangle_{\mathbb{A}^1} & \cong & 0 \\ & \cong & a\langle(-)_* \wedge S^{a,b}, X\rangle_{\mathbb{A}^1} & & \\ & \cong & \pi_{a,b}^{\mathbb{A}^1}(X). & & \end{array}$$

□

Proposition 1.2.29. Let S be the spectrum of a perfect field and X a pointed simplicial presheaf. If $\pi_0^{\mathbb{A}^1}(X)(S) \cong *$, then $\pi_0^{\mathbb{A}^1}(\Omega^{1,1}X)(S) = *$.

Proof. It is given that $\pi_0^{\mathbb{A}^1}(X)(S) = a\langle S_*, X\rangle_{\mathbb{A}^1} \cong \langle S_*, X\rangle_{\mathbb{A}^1} \cong *$ and we have to show that $\pi_0^{\mathbb{A}^1}(\Omega^{1,1}X)(S) \cong \langle S_*, \Omega^{1,1}X\rangle_{\mathbb{A}^1} \cong \langle \mathbb{G}_m, X\rangle_{\mathbb{A}^1}$ is trivial. Since the dimension of $\mathbb{G}_m$ is one, there is an isomorphism

$$\langle \mathbb{G}_{m+}, X\rangle \cong H^1_{\mathcal{N}\text{is}}(\mathbb{G}_m; \pi_1^{\mathbb{A}^1}(X))$$

by [Mor12, Lemma 5.12] and $H^1_{\mathcal{N}\text{is}}(\mathbb{G}_m; \pi_1^{\mathbb{A}^1}(X))$ is trivial by [Mor12, Lemma 5.14] as $\pi_1^{\mathbb{A}^1}(X)$ is a strongly $\mathbb{A}^1$-invariant sheaf of groups by [Mor12, Theorem 9]. □

Corollary 1.2.30. Let S be the spectrum of a perfect field, $a \geq b \geq 0$ integers and X a pointed simplicial presheaf. If $\pi_{a-b,0}^{\mathbb{A}^1}(X)(S) \cong *$, then $\pi_{a,b}^{\mathbb{A}^1}(X)(S) \cong *$.

1.3. Mather's cube theorem and some implications

Remark 1.3.1. A *topos* is a category equivalent to a category $\mathrm{Shv}(\mathcal{C})$ of sheaves on a Grothendieck site $\mathcal{C}$. The associated sheaf functor $a : \mathrm{Pre}(\mathcal{C}) \to \mathrm{Shv}(\mathcal{C})$ commutes with colimits and finite limits [MM92, Theorem III.5.1]. For a morphism $f : Y \to X$ and a diagram $D : \mathcal{I} \to \mathrm{Shv}(\mathcal{C})$ over X, there is an isomorphism

$$\operatorname*{colim}_{i\in\mathcal{I}}(Y \times_X D(i)) \cong Y \times_X \operatorname*{colim}_{i\in\mathcal{I}} D(i)$$

since this property holds in $\mathcal{S}$et, colimits and limits in $\mathrm{Pre}(\mathcal{C})$ are computed objectwise [Bor94a, Proposition 2.15.1] and the sheafification functor has the above-mentioned properties. This property is sometimes denoted by saying that *colimits are stable under base change*. Considering the two pullback diagrams

$$\begin{array}{ccccc} E(i) & \longrightarrow & E & \longrightarrow & Y \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle f} \\ D(i) & \longrightarrow & \mathrm{colim} D & \longrightarrow & X \end{array}$$

one notes, that colimits are stable under base change, if and only if for every diagram D, every morphism $g : E \to \mathrm{colim} D$ and pullback diagram

$$\begin{array}{ccc} E(i) & \xrightarrow{k_i} & E \\ \downarrow & & \downarrow{\scriptstyle g} \\ D(i) & \xrightarrow{j_i} & \mathrm{colim} D \end{array}$$

the arrows k_i (together with all the arrows $E(i) \to E(i')$) constitute a colimit diagram, this is E is a colimit of the diagram $E(-)$ and the k_i are the canonical maps.

Definition 1.3.2 ([Rez98], [Rez10])**.** A model category $\mathcal{C}$ has *homotopy descent* if the following two conditions are fulfilled.

(**P1**) For any morphism $f : E \to D$ of diagrams $D, E : \mathcal{I} \to \mathcal{C}$ such that every square

$$\begin{array}{ccc} E(i) & \longrightarrow & \mathrm{colim} E \\ {\scriptstyle f_i}\downarrow & & \downarrow \\ D(i) & \longrightarrow & \mathrm{colim} D \end{array}$$

is a homotopy pullback and D is a homotopy colimit diagram, it follows that E is a homotopy colimit diagram.

(**P2**) For any morphism $f : E \to D$ of diagrams such that every square

$$\begin{array}{ccc} E(i) & \xrightarrow{E(i\to j)} & E(j) \\ {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle f_j} \\ D(i) & \xrightarrow{D(i\to j)} & D(j) \end{array}$$

is a homotopy pullback and D and E are homotopy colimit diagrams, it follows that every square as in (P1) is a homotopy pullback.

Remark 1.3.3. As discussed in Remark A.1.19, the property of having homotopy descent is rather a property of the homotopical category associated to a model category. Therefore, we will use the term *homotopy descent* as a property of a homotopical category obtained by a model category as well.

Remark 1.3.4. Property (P2) is not true for ordinary colimits and base change as noted in [Rez98, Remark 3.8]. In relation to Remark 1.3.1, one may phrase property (P1) by saying that *homotopy colimits are stable under homotopy base change.*

Theorem 1.3.5 (Rezk, [Rez98, Theorem 1.4])**.** The local homotopical category has homotopy descent.

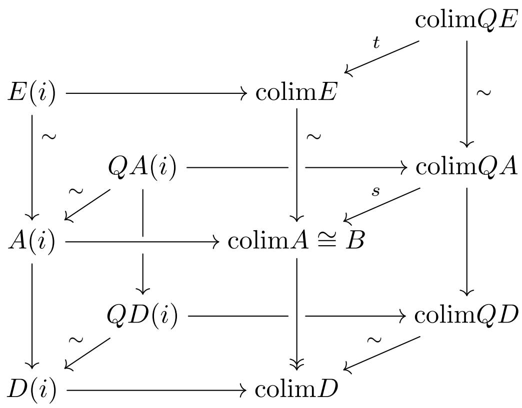
Remark 1.3.6. Unfortunately, the motivic homotopical category does not have homotopy descent [SØ10, Remark 3.5].

Theorem 1.3.7. The motivic homotopical category $\mathcal{H}$ satisfies the first property (P1) of Definition 1.3.2 of homotopy descent.

Proof. This proof follows the lines of the first part of the the proof [Rez98, Theorem 1.4]. Choose the $\mathbb{A}^1$-injective model structure as a model for the motivic homotopical category. Take a homotopy pullback square as considered in (P1) and factorize the right vertical map as a weak equivalence followed by a fibration

$$\mathrm{colim}E \xrightarrow{\sim} B \twoheadrightarrow \mathrm{colim}D.$$

We assume that D is a homotopy colimit diagram. It is to show that E is a homotopy colimit diagram. Let $A : \mathcal{I} \to \mathcal{C}$ be the diagram defined by pulling back the map $D(i) \to \mathrm{colim}D$ along $B \to \mathrm{colim}D$ and consider the following cube where Q is a cofibrant replacement functor for the diagram category with the projective structure (confer Remark A.3.11).



It is to show that t is a weak equivalence. The isomorphism $\mathrm{colim}A \cong B$ follows from the observation in Remark 1.3.1. The maps $E(i) \to A(i)$ are weak equivalences by the homotopy invariance of homotopy pullbacks and the fact that all squares of the front face are homotopy pullbacks. The morphism $\mathrm{colim}QD \to \mathrm{colim}D$ is a weak equivalence because D is a homotopy colimit diagram. The map $\mathrm{colim}QE \to \mathrm{colim}QA$ is a weak equivalence since it is the homotopy colimit of weak equivalences. Therefore, it suffices to show that s is a weak equivalence which would follow if we could show that the lower square of the right face is a categorical pullback, id est

$$\mathrm{colim}QA \cong B \times_{\mathrm{colim}D} \mathrm{colim}QD.$$

By Remark 1.3.1, this is implied by

$$QA(i) \cong B \times_{\mathrm{colim} D} QD(i).$$

being an isomorphism for every $i \in \mathcal{I}$ and since the lower front face of the above diagram is already a pullback, it suffices to show that the lower square of the left face is a categorical pullback.

To do so, we look at a specific model of a cofibrant replacement functor Q of the projective diagram model category. Considering Remark A.3.11, one calculates

$$\begin{aligned}
\operatorname*{hocolim}_{i\in\mathcal{I}} D \quad &\sim \quad \int^{n\in\Delta} \left(\coprod_{i_0 \to \ldots \to i_n} QD(i_0) \right) \times \Delta^n \\
&\cong \int^{n\in\Delta} \operatorname*{colim}_{i\in\mathcal{I}} \coprod_{i_0 \to \ldots \to i_n \to i} D(i_0) \times \Delta^n \\
&\cong \operatorname*{colim}_{i\in\mathcal{I}} \int^{n\in\Delta} \coprod_{i_0 \to \ldots \to i_n \to i} D(i_0) \times \Delta^n \\
&\cong \operatorname*{colim}_{i\in\mathcal{I}} \left(\left(\coprod_{i_0 \to \ldots \to i_{(-)} \to i} D(i_0)_{(-)} \right) \circ \mathit{diag} \right) \\
&\overset{def}{\cong} \operatorname*{colim}_{i\in\mathcal{I}} (QD)(i).
\end{aligned}$$

The natural augmentation map $QD \to D$ is the diagonal of a morphism of bisimplicial sets given in bidegree (m,n) on components of the coproduct by

$$D(i_0)_m \to \ldots \to D(i_n)_m \to D(i)_m = \underset{n}{const}\, D(i)_m$$

for a string of morphisms $i_0 \to \ldots \to i_n \to i$. Since the diagonal functor *diag* commutes with limits [GJ99, Chapter IV] and the diagram

$$\begin{array}{ccc}
\coprod_{i_0 \to \ldots \to i_n \to i} A(i_0)_m & \longrightarrow & \underset{n}{const}\, A(i)_m \\
\downarrow & & \downarrow \\
\coprod_{i_0 \to \ldots \to i_n \to i} D(i_0)_m & \longrightarrow & \underset{n}{const}\, D(i)_m
\end{array}$$

is a pullback for each $n, m \geq 0$, then so is the diagram

$$\begin{array}{ccc}
QA(i) & \longrightarrow & A(i) \\
\downarrow & & \downarrow \\
QD(i) & \longrightarrow & D(i)
\end{array}$$

for every $i \in \mathcal{I}$ and the result follows. □

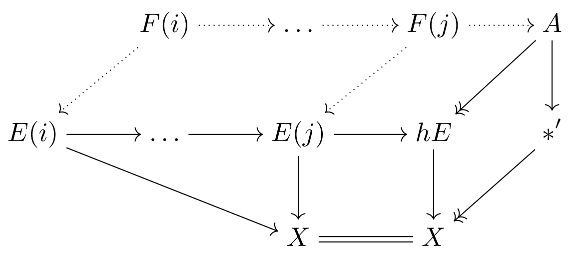
Theorem 1.3.8 (Puppe's Theorem)**.** Let $E : \mathcal{I} \to \mathrm{sPre}$ be a diagram over a pointed simplicial presheaf X. Then, there is an isomorphism

$$\operatorname*{hocolim}_{i\in\mathcal{I}} \mathrm{hofib}^{\mathbb{A}^1}(E(i) \to X) \xrightarrow{\cong} \mathrm{hofib}^{\mathbb{A}^1}(\operatorname*{hocolim}_{i\in\mathcal{I}} E(i) \to X).$$

in the motivic homotopy category.

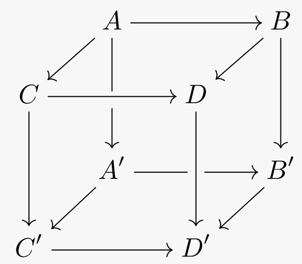
Proof. We recall the proof of [Wen07, Proposition 3.1.16] in order to show that only property (P1) of Definition 1.3.2 is needed. Choose the $\mathbb{A}^1$-injective model structure as a model for the motivic homotopical category.

Let $E : \mathcal{I} \to \mathrm{sPre}$ be a diagram over X with homotopy colimit hE and consider the solid diagram



where $*'$ is obtained by a factorization $* \xrightarrow{\sim} *' \twoheadrightarrow X$ and where A is the pullback of $hE \to X$ along $*' \twoheadrightarrow X$. By the right properness, this pullback is also a homotopy pullback and A is a model for the homotopy fiber of $hE \to X$. Let $F : \mathcal{I} \to \mathcal{C}$ be the diagram defined by the pullbacks of $E(i) \to hE$ along $A \to hE$ which are also homotopy pullbacks due to the right properness and hence $F(i)$ is a model for the homotopy fiber of $E(i) \to X$. Consider the diagrams $E(i) \to hE$ with hE as the terminal object and $F(i) \to A$ with A as the terminal object. Since $\mathrm{colim} E \cong hE$, we have $\mathrm{colim} F \cong A$ by Remark 1.3.1 and property (P1) implies $A \sim \mathrm{hocolim} F$. ☐

Theorem 1.3.9 (Mather's cube theorem)**.** Consider the cube



in the category of simplicial presheaves where the bottom face is a homotopy pushout and all the side faces are homotopy pullbacks with respect to the motivic homotopical structure. Then the top face is a homotopy pushout.

Proof. This is a special case of Theorem 1.3.7. ☐

Remark 1.3.10. Some consequences of Mather's cube theorem 1.3.9 can be found in [Doe98]. The authors pose the validity of the Cube theorem as an axiom. Hence, all the maps of simplicial presheaves with respect to a motivic model structure are *cube maps* in their notation.

Remark 1.3.11. For an object X of $\mathcal{S}\mathrm{m}_X$, let $\mathcal{S}\mathrm{m}/X$ denote the over category $\mathcal{S}\mathrm{m}_S \downarrow X$. This category is usually not the same as $\mathcal{S}\mathrm{m}_X$ as the objects of $\mathcal{S}\mathrm{m}/X$ need not to be smooth over X (if however X is étale over S, one has $\mathcal{S}\mathrm{m}/X \cong \mathcal{S}\mathrm{m}_X$ [Gro67, Proposition 17.3.4]). Since $\mathcal{S}\mathrm{m}_S$ is essentially small, $\mathcal{S}\mathrm{m}/X$ is an essentially small category as well. There is an equivalence of categories

$$\begin{aligned}
\mathrm{sPre}(\mathcal{S}\mathrm{m}/X) &\cong \mathcal{F}\mathrm{un}(\mathcal{S}\mathrm{m}/X^{\mathrm{op}} \times \Delta^{\mathrm{op}}, \mathcal{S}\mathrm{et}) \\
&\cong \mathcal{F}\mathrm{un}(\Delta^{\mathrm{op}}, \mathcal{F}\mathrm{un}(\mathcal{S}\mathrm{m}/X^{\mathrm{op}}, \mathcal{S}\mathrm{et})) \\
&\cong \mathcal{F}\mathrm{un}(\Delta^{\mathrm{op}}, \mathcal{F}\mathrm{un}(\mathcal{S}\mathrm{m}_S^{\mathrm{op}}, \mathcal{S}\mathrm{et})/X) \\
&\cong \mathcal{F}\mathrm{un}(\Delta^{\mathrm{op}}, \mathcal{F}\mathrm{un}(\mathcal{S}\mathrm{m}_S^{\mathrm{op}}, \mathcal{S}\mathrm{et}))/X \\
&\cong \mathrm{sPre}/X
\end{aligned}$$

omitting the notion for the Yoneda embedding and the discrete simplicial object functor. Hence, the (objectwise) projective model structure on sPre($\mathcal{S}\mathrm{m}/X$) can be identified with the over model structure of Theorem A.3.12 with respect to the (objectwise) projective model sPre. The equivalence $\mathrm{Pre}(\mathcal{S}\mathrm{m}/X) \cong \mathrm{Pre}(\mathcal{S}\mathrm{m}_S)/X$ in the third step is given as follows: An object $E \to X$ in $\mathrm{Pre}(\mathcal{S}\mathrm{m}_S)/X$ is mapped to the presheaf sending the object $u : U \to X$ of $\mathcal{S}\mathrm{m}/X$ to the set $E(u)$ of lifts in the diagram

$$\begin{array}{ccc} & & E \\ & {\overset{l}{\nearrow}} & \big\downarrow{\scriptstyle p} \\ U & \xrightarrow{\;u\;} & X \end{array}$$

again omitting the notion for the Yoneda embedding. Conversely, an object $E(-)$ of $\mathrm{Pre}(\mathcal{S}\mathrm{m}/X)$ is mapped to the presheaf sending the object U of $\mathcal{S}\mathrm{m}_S$ to $E(U)$ and

$$E(U) = \coprod_{u:U\to X} E(u) \to \coprod_{u:U\to X} * = X(U).$$

This equivalence can be promoted to an equivalence $\mathrm{Shv}(\mathcal{S}\mathrm{m}/X) \cong \mathrm{Shv}(\mathcal{S}\mathrm{m}_S)/X$ of topoi, where $\mathcal{S}\mathrm{m}/X$ carries the obvious Grothendieck topology which is generated by the over basis [MM92, Exercise III.8.(b)]. Hence, since a model structure is determined by its cofibrations and weak equivalences, the local projective model structure on sPre($\mathcal{S}\mathrm{m}/X$) can be identified with the over model structure with respect to the local projective model structure on sPre (confer [Bla01, Theorem 1.6]). The author considers it as unlikely, that for every X in $\mathcal{S}\mathrm{m}_S$ the model structure on sPre($\mathcal{S}\mathrm{m}/X$) obtained by a Bousfield localization of the above mentioned local model at the $U \times \mathbb{A}^1 \to U$ can be identified with the over model structure on sPre$/X$ with respect to the motivic model structure on sPre.

In what follows, we will use of the over model structure on sPre$/X$ with respect to the $\mathbb{A}^1$-local injective or the motivic model structure on sPre only.

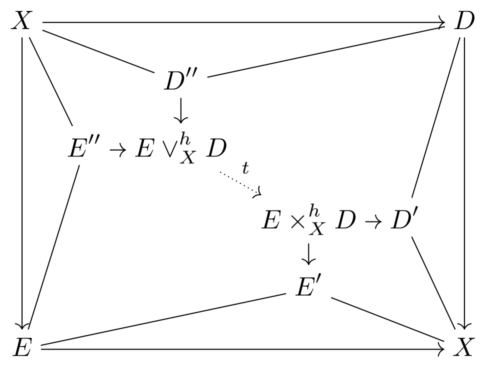
Remark 1.3.12. Please note, that for the following Definition 1.3.13 we refer to the convetion of Definition A.1.18 of considering hocolim and hocolim as functors to the category of simplicial presheaves.

Definition 1.3.13. Let X be a simplicial presheaf. For morphisms $X \to E$ and $X \to D$ in sPre, one sets $E \vee^h_X D = \mathrm{hocolim}(E \leftarrow X \to D)$ as the *homotopy wedge* of E and D over X. For morphisms $E \to X$ and $D \to X$ in sPre one defines $E \times^h_X D = \mathrm{holim}(D \to X \leftarrow E)$ as the *homotopy product* of E and D over X.

If $X \to E \to X$ and $X \to D \to X$ are the identities respectively, one defines

$$E \wedge^h_X D = \mathrm{hocofib}(E \vee^h_X D \xrightarrow{t} E \times^h_X D)$$

where t is induced by the diagram



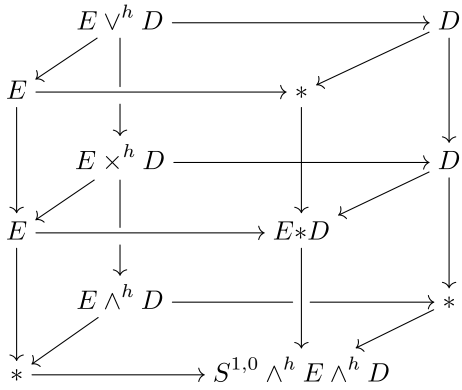
as the *homotopy smash product* of E and D over the space X. Finally, one sets $E * D = E *^h_X D = \mathrm{hocolim}(E' \leftarrow E \times^h_X D \rightarrow D')$ as the *(homotopy) join* of E and D over X. If X is the base scheme S, one drops the subscripted X from the notation.

Remark 1.3.14. Due to the Cogueing lemma and the Glueing lemma A.1.22, all the operations of Definition 1.3.13 preserve weak equivalences in both arguments. For the pointed motivic homotopical category and $X = S$, the categorical wedge is weakly equivalent to the the homotopy wedge since all object are cofibrant in the $\mathbb{A}^1$-local injective model. The categorical product is is weakly equivalent to the the homotopy product as $R^{\mathbb{A}^1}$ preserves products. To identify the smash products, one has to observe that the canonical map $E \vee D \rightarrow E \times D$ is a cofibration in one of the models, which holds by monoidality (confer Remark A.4.6) or by direct observation.

The reason why we have to introduce these notions is, that the author is not able to show that the over model categories $\mathrm{sPre}(S)/X$ (confer Remark 1.3.11) are monoidal with respect to the product over X. The difficulty is, that for a smooth morphism $X \rightarrow S$ in $\mathcal{S}\mathrm{m}_S$ and two morphisms $E \rightarrow X$ and $D \rightarrow X$ in $\mathcal{S}\mathrm{m}_S$, the scheme $E \times_X D$ does not have to be smooth over S. Although one may define the categorical wedge over X, the categorical product over X and hence the categorical smash product over X, the author does not know in general whether they model their homotopical analogues in the over model $\mathrm{sPre}(S)/X$. However, in the cases we are interested in, this holds true and will be considered in Section 3.3.

Lemma 1.3.15. For pointed simplicial presheaves E and D, there is an isomorphism $E * D \cong S^{1,0} \wedge^h E \wedge^h D$ in the pointed motivic homotopy category. The same holds for the over model structures on $\mathrm{sPre}(S)/X_*$.

Proof. For a diagram



consisting of cofiber sequences in all but the front right vertical column and homotopy pushouts in the upper and the middle horizontal square, one gets that the front right vertical column is a cofiber sequence if and only if the bottom square is a homotopy pushout [Doe98, Lemma 1.4]. This implies the result as $* \to E{*}D \to S^{1,0} \wedge^h E \wedge^h D$ is a cofiber sequence if and only if its latter morphism is an isomorphism in the homotopy category. □

Remark 1.3.16. Considering the Zariski (and thus Nisnevich) distinguished square

$$\begin{array}{ccc} (\mathbb{A}^n - 0) \times (\mathbb{A}^1 - 0) & \longrightarrow & \mathbb{A}^n \times (\mathbb{A}^1 - 0) \\ \downarrow & & \downarrow \\ (\mathbb{A}^n - 0) \times \mathbb{A}^1 & \longrightarrow & \mathbb{A}^{n+1} - 0 \end{array}$$

for $n \geq 1$, one gets an isomorphism $(\mathbb{A}^n - 0) * S^{1,1} \cong \mathbb{A}^{n+1} - 0$, hence $\mathbb{A}^n - 0 \cong S^{2n-1,n}$ and therefore $S^{1,0} \wedge (\mathbb{A}^n - 0) \cong (\mathbb{P}^1)^{\wedge n}$ in the pointed motivic homotopy category.

Lemma 1.3.17. The following statements are true with respect to the pointed motivic homotopical category, pointed simplicial presheaves X and Y and a motivic fiber sequence $F \to E \to X$.

(1) $X * * \cong *$ and $X * S^0 \cong S^{1,0} \wedge X$,
(2) $S^{1,0} \wedge (X \times Y) \cong (X * Y) \vee (S^{1,0} \wedge X) \vee (S^{1,0} \wedge \Sigma Y)$,
(3) There is a motivic fiber sequence $\Sigma\Omega X \to X \vee X \to X$,
(4) There is a motivic fiber sequence $\Omega X * \Omega Y \to X \vee Y \to X \times Y$,
(5) There is a motivic fiber sequence $F * \Omega X \to \mathrm{hocofib}(F \to E) \to X$,
(6) There is a motivic fiber sequence $\Omega X * \Omega X \to \Sigma\Omega X \to X$

Proof. All these statements are abstract facts about suitable model categories in which Mather's cube theorem 1.3.9 holds. Statement (1) is clear. Statement (2) is [Doe98, Corollary 2.13] and (3) follows directly from Puppe's theorem 1.3.8 applied to the diagram $* \leftarrow X \to *$ over X. The assertion (4) is [Doe98, Proposition 4.6] and statement (5) is [Doe98, Corollary 4.3]. Proposition (6) follows from (5) applied to the fiber sequence $\Omega X \to * \to X$. □

2. **Stable motivic homotopy theory**

2.1. The stable motivic homotopy category

Remark 2.1.1. We are going to apply the construction of Section A.5 in order to get a stable version of the pointed motivic homotopy category with respect to the smash product with $S^{2,1}$ (which is motivically weak equivalent to $\mathbb{P}^1$). From this it follows the invertibility of the smash with both, the simplicial cycle $S^{1,0}$ and the algebraic cycle $S^{1,1} = \mathbb{G}_m$.

Remark 1.1.24 introduces the motivic $(\mathcal{I}_*, \mathcal{J}_*)$-cofibrantly generated, proper, combinatorial, simplicial, monoidal and pointed model structure on $\mathrm{sPre}(S)_*$ with a cofibrant unit S^0. It is well stabilizable by [DRØ03b, Corollary 2.16]. In order to get a cofibrant object T for the method of Section A.5, one uses the simplicial mapping cylinder construction of Lemma A.3.6 with respect to the monomorphism $(\mathbb{A}^1 - 0)_* \rightarrowtail \mathbb{A}^1_*$ to get a factorization $(\mathbb{A}^1 - 0)_* \rightarrowtail Cyl \xrightarrow{\sim} \mathbb{A}^1_*$. The pushout

$$\begin{array}{ccc} (\mathbb{A}^1 - 0)_* & \rightarrowtail & Cyl \\ \downarrow & & \downarrow \\ * & \rightarrowtail & T \end{array}$$

yields a cofibrant object T which is as a pushout of finitely presentable objects again finitely presentable. Since this pushout is a homotopy pushout as well, the homotopy cofibers of the two horizontal morphisms coincide. Hence, there is an isomorphism $T \cong \mathbb{P}^1$ in the pointed motivic homotopy category (confer Remark 1.3.16).

The object T is weakly equivalent to a symmetric object in the sense of Definition A.5.17 by [Jar00a, Lemma 3.13] (confer also [DRØ03b, Lemma 2.24]) and therefore the conclusions of Theorem A.5.19 hold.

Definition 2.1.2. As observed in the previous Remark 2.1.1, there is by the results of Section A.5 a proper, combinatorial, simplicial and pointed model structure on the category $\mathcal{S}p_T$ (or $\mathcal{S}p_T(S)$) of *motivic T-spectra* (or *motivic spectra*), the *unstable motivic model structure*. Moreover, there is a proper, combinatorial, simplicial and pointed model structure on the category $\mathcal{S}p_T$, the *stable motivic model structure* and its homotopy category is denoted by $\mathcal{SH}$.

Remark 2.1.3. Even though the stable motivic homotopy category $\mathcal{SH}$ can be equipped with a symmetric monoidal structure in a reasonable way as done below, the category of motivic spectra $\mathcal{S}p_T$ does not serve as a good model for this symmetric monoidal structure in the sense of Theorem A.2.7 [Lew91].

The category $\mathcal{S}p_T^\Sigma$ (or $\mathcal{S}p_T^\Sigma(S)$) of *motivic symmetric T-spectra* as constructed in [Hov01b] may be equipped with a closed symmetric monoidal structure [Hov01b, Theorem 8.3] such that there is a monoidal adjunction

$$\Sigma_T^\infty : \mathrm{sPre}_* \rightleftarrows \mathcal{S}p_T^\Sigma : (-)_0.$$

Moreover, there is an adjunction

$$\mathcal{V} : \mathcal{S}p_T \rightleftarrows \mathcal{S}p_T^\Sigma : \mathcal{U}$$

relating the ordinary and the symmetric motivic T-spectra and factorizing $(\Sigma_T, (-)_0)$ through its non-symmetric analogue. The category $\mathcal{S}p_T^\Sigma$ may be furnished with a left proper combinatorial, simplicial and monoidal pointed *stable motivic model structure* as done in [Hov01b, Theorem 8.11] such that $(\Sigma_T, (-)_0)$ becomes a monoidal Quillen adjunction. The stable model structure on motivic symmetric T-spectra is also right proper which does not follow from the general arguments of [Hov01b] and may be seen as follows: Replacing injective objects by unstably cofibrant objects, the proof of [Jar00a, Proposition 4.8] can be adopted to show that $\mathcal{U} : \mathcal{S}p_T^\Sigma \to \mathcal{S}p_T$ reflects stable weak equivalences. Hence, the projective analogue of the properness argument [Jar00b, Theorem 12] holds. Moreover, the projective version of [Jar00b, Corollary 11] and [Jar00b, Theorem 13] imply that $(\mathcal{V},\mathcal{U})$ is a Quillen equivalence.

Lemma 2.1.4 (Stability lemma)**.** Let X and Y be motivically cofibrant and finitely presentable pointed simplicial presheaves. Then, there exists an isomorphism

$$u : \Sigma_T^\infty X \xrightarrow{\cong} \Sigma_T^\infty Y$$

in the stable motivic homotopy category $\mathcal{SH}$ if and only if there exists a natural number n and an isomorphism

$$v : \Sigma_T^n X \xrightarrow{\cong} \Sigma_T^n Y$$

in the pointed motivic homotopy category $\mathcal{H}_*$.

Proof. For a motivically cofibrant and finitely presentable pointed simplicial presheaf X and a motivic T-spectrum E, there is an isomorphism

$$\begin{aligned}
\langle \Sigma_T^\infty X, E\rangle_{\mathcal{SH}} &\cong \langle \Sigma_T^\infty X, \Theta_T^\infty RE\rangle_{\mathcal{SH}} \\
&\cong \langle X, (\Theta_T^\infty RE)_0\rangle_{\mathcal{H}_*} \\
&\cong \langle X, \operatorname{colim}_n \Omega_T^n RE_n\rangle_{\mathcal{H}_*} \\
&\cong \pi_0 \mathrm{s}\mathcal{S}\mathrm{et}_*(X, \operatorname{colim}_n \Omega_T^n RE_n) \\
&\cong \operatorname{colim}_n \pi_0 \mathrm{s}\mathcal{S}\mathrm{et}_*(X, \Omega_T^n RE_n) \\
&\cong \operatorname{colim}_n \langle X, \Omega_T^n RE_n\rangle_{\mathcal{H}_*} \\
&\cong \operatorname{colim}_n \langle \Sigma_T^n X, RE_n\rangle_{\mathcal{H}_*} \\
&\cong \operatorname{colim}_n \langle \Sigma_T^n X, E_n\rangle_{\mathcal{H}_*}
\end{aligned}$$

where R is a fibrant replacement functor for the unstable model structure (respectively, denoted by abuse of notation by the same symbol, for the pointed motivic model) and where the fifth isomorphism exists as the object $X \wedge \Delta_*^n$ is finitely presentable [DRØ03b, Lemma 2.5]. The isomorphism above is natural with respect to morphisms of pointed simplicial presheaves between motivically cofibrant and finitely presentable pointed simplicial presheaves X and with respect to morphisms in $\mathcal{SH}$ between motivic T-spectra E.

Let Y be another motivically cofibrant and finitely presentable pointed simplicial presheaf and consider the special case of $E = \Sigma_T^\infty Y$. The morphisms of the filtered diagram for the colimit on the right-hand side of the isomorphism are given by

$$\langle \Sigma_T^n X, R\Sigma_T^n Y\rangle_{\mathcal{H}_*} \xrightarrow{\mathcal{L}\Sigma} \langle \Sigma_T \Sigma_T^n X, \Sigma R\Sigma_T^n Y\rangle_{\mathcal{H}_*} \xrightarrow{\sigma_*} \langle \Sigma_T^{n+1} X, R\Sigma_T^{n+1} Y\rangle_{\mathcal{H}_*}$$

(confer Definition A.5.15) and hence the isomorphism from the right to the left is on components of the colimit given as

$$\begin{aligned}
\langle \Sigma_T^n X, \Sigma_T^n Y\rangle_{\mathcal{H}_*} &\xrightarrow{\mathcal{L}\Sigma_T^\infty} \langle \Sigma_T^\infty \Sigma_T^n X, \Sigma_T^\infty \Sigma_T^n Y\rangle_{\mathcal{SH}} \\
&\cong \langle \Sigma_T^n \Sigma_T^\infty X, \Sigma_T^n \Sigma_T^\infty Y\rangle_{\mathcal{SH}} \\
&\xrightarrow{\mathcal{L}[-n]} \langle \Sigma_T^\infty X, \Sigma_T^\infty Y\rangle_{\mathcal{SH}},
\end{aligned}$$

(confer Remark A.5.8 and Theorem A.5.19) which is again natural with respect to morphisms of pointed simplicial presheaves between motivically cofibrant and finitely presentable pointed simplicial presheaves X and Y. The statement follows now from naturality and the fact that the identity is represented by the identity for some n. □

Corollary 2.1.5. Let $f : X \to Y$ be a morphism of motivically cofibrant and finitely presentable pointed simplicial presheaves. Then $\Sigma_T^\infty f$ is a stable motivic weak equivalence if and only if there exists a natural number n such that $\Sigma_T^n f$ is a motivic weak equivalence.

Remark 2.1.6. If S is the spectrum of the complex numbers, there is a *complex realization functor*

$$Re_{\mathbb{C}} : \mathrm{sPre}(\mathrm{Spec}\,\mathbb{C}) \to \mathcal{T}\mathrm{op}$$

into the category of compactly generated topological spaces given as the left Kan extension of the functor $\mathcal{S}\mathrm{m}_S \times \Delta \to \mathcal{T}\mathrm{op}$ sending $(U, [n])$ to $U(\mathbb{C}) \times |\Delta^n|$ where $U(\mathbb{C})$ is the set of complex points of U with the analytic topology. This functor is symmetric monoidal with respect to the product. It has a right adjoint given by $\mathcal{S}\mathrm{ing}_{\mathbb{C}}(X) = \mathrm{hom}_{\mathcal{T}\mathrm{op}}((-)(\mathbb{C}) \times \Delta^{(\text{-})}, X)$ and induces a functor on the pointed categories which is denoted likewise and is symmetric monoidal with respect to the smash product. By a similar argument as in the proof of [PPR07, Theorem A.4.1], both functors are left Quillen functors with respect to the motivic model structure.

There is a (symmetric monoidal) model structure on the category of topological (symmetric) $\mathbb{C}P^1$-spectra whose homotopy category is equivalent as a symmetric monoidal and triangulated category to the ordinary stable homotopy category of spaces [PPR07, Theorem A.7.1] and the complex realization functor $Re_{\mathbb{C}}$ induces a (strict symmetric monoidal) left Quillen functor from the model category of (symmetric) motivic $\mathbb{P}^1$-spectra (confer [Hor06, Theorem 1.12]) into the former (symmetric) one [PPR07, Theorem A.7.2].

Definition 2.1.7. Let X be an element of $\mathcal{S}p_T$. For $a \geq b \geq 0$ one defines the *a-th (T-)stable homotopy sheaf with weight b* of X as

$$\pi^{\mathrm{st}}_{a,b}(X) = a\langle \Sigma_T^\infty((-)_* \wedge S^{a,b}), X\rangle_{\mathcal{SH}}$$

and with the usual convention of smashing with the respective negative indices on the right-hand side, this sheaf can be defined for all integers a and b. A spectrum X is called *(T-)stably contractible* if the morphism $X \to S$ is a stable weak equivalence.

One writes just $\pi^{\mathrm{st}}_{a,b}$ for the group $\pi^{\mathrm{st}}_{a,b}(\mathbb{S})(S) = \langle S^{a,b}, \mathbb{S}\rangle_{\mathcal{SH}}$.

Remark 2.1.8. For a motivic T-spectrum X and integers $a \geq b \geq 0$, one calculates

$$\begin{aligned}
\pi^{\mathrm{st}}_{a,b}(X) &\cong a\langle \Sigma_T^\infty((-)_* \wedge S^{a,b}), X\rangle_{\mathcal{SH}} \\
&\cong a\langle (-)_* \wedge S^{a,b}, (\Theta_T^\infty RX)_0\rangle_{\mathbb{A}^1} \\
&\cong a\langle (-)_* \wedge S^{a,b}, \operatorname*{colim}_m \Omega_T^m RX_m\rangle_{\mathbb{A}^1} \\
&\cong \operatorname*{colim}_m a\langle (-)_* \wedge S^{a,b}, \Omega_T^m RX_m\rangle_{\mathbb{A}^1} \\
&\cong \operatorname*{colim}_m a\langle (-)_* \wedge S^{a+2m,b+m}, X_m\rangle_{\mathbb{A}^1} \\
&\cong \operatorname*{colim}_m \pi^{\mathbb{A}^1}_{a+2m,b+m}(X_m)
\end{aligned}$$

where R is a fibrant replacement functor for the unstable model structure. The fourth equality holds as the object $U_* \wedge S^{a,b} \wedge \Delta^n_*$ is finitely presentable for any U in $\mathcal{S}\mathrm{m}_S$ by [DRØ03b, Lemma 2.5]. In accordance with the usual convention, we may define the sheaf $\pi^{\mathrm{st}}_{a,b}(X)$ with respect to all integers a and b and it holds

$$\pi^{\mathrm{st}}_{a-2l,b-l}(X) \cong \pi^{\mathrm{st}}_{a,b}(X[l]).$$

A morphism $f : X \to Y$ of spectra is a stable weak equivalence if and only if it induces an isomorphism $\pi^{\mathrm{st}}_{a,b}(f)$ for all integers a and b [Hov99, Theorem 7.3.1].

Remark 2.1.9. Let S be the spectrum of a perfect field k. As far as the author is informed, there is not much known about the groups $\pi^{\mathrm{st}}_{a,b}$. The Unstable $\mathbb{A}^1$-connectivity theorem together with Corollary 1.2.30 imply that $\pi^{\mathrm{st}}_{a,b} \cong 0$ whenever $a < b$. For $a = b$ Morel proved in [Mor04, Theorem 6.2.1] and [Mor12, Corollary 5.42] the existence of an isomorphism $\pi^{\mathrm{st}}_{a,a} \cong K^{MW}_{-a}$ (which can be promoted into an isomorphism of a-graded rings) where it should be recalled that K^{MW}_0 is isomorphic to the Grothendieck-Witt ring $GW(k)$.

2.2. A consequence of the motivic Hurewicz theorem

Remark 2.2.1. Let the base scheme S be the spectrum of a perfect field. Applying the canonical monoidal adjunction $\mathbb{Z} : \mathcal{S}\mathrm{et} \rightleftarrows \mathcal{A}\mathrm{b} : U$ degree-wise induces a monoidal adjunction $\mathbb{Z} : \mathrm{s}\mathcal{S}\mathrm{et} \rightleftarrows \mathrm{s}\mathcal{A}\mathrm{b} : U$ where the the right adjoint factorizes through the subcategory of Kan complexes. This can be promoted into a monoidal Quillen adjunction furnishing $\mathrm{s}\mathcal{A}\mathrm{b}$ with the model structure where weak equivalences and fibrations are detected by the right adjoint [GJ99, Theorem III.2.8]. As a simplicial abelian group is canonically pointed, one has a monoidal Quillen adjunction $\mathrm{s}\mathcal{S}\mathrm{et}_* \rightleftarrows \mathrm{s}\mathcal{A}\mathrm{b}$ with left adjoint $\tilde{\mathbb{Z}}(* \to X) = \mathbb{Z}(X)/\mathbb{Z}(*)$. This construction is compatible with the canonical monoidal Quillen equivalence from simplicial sets to pointed simplicial sets. The *Dold-Kan correspondence* is an equivalence of categories

$$N : \mathrm{s}\mathcal{A}\mathrm{b} \rightleftarrows \mathcal{C}\mathrm{h}_{\geq 0} : K$$

with the property that $\pi_n(X, 0) \cong H_n(NX)$. The homotopy groups of a simplicial abelian group do not depend on the basepoint [GJ99, Corollary III.2.3]. Furthermore, one has an adjunction

$$i : \mathcal{C}\mathrm{h}_{\geq 0} \rightleftarrows \mathcal{C}\mathrm{h} : t$$

where i is the inclusion and t is the *good truncation* defined by $t(X)_n = X_n$ if $n \geq 1$ and $t(X)_0 = \ker(X_0 \to X_{-1})$, both preserving isomorphisms on homology.

Applying this to the category $\mathcal{C}\mathrm{h}(S) \cong \mathcal{C}\mathrm{h}(\mathcal{F}\mathrm{un}(\mathcal{S}\mathrm{m}^{\mathrm{op}}_S, \mathcal{A}\mathrm{b})) \cong \mathcal{F}\mathrm{un}(\mathcal{S}\mathrm{m}^{\mathrm{op}}_S, \mathcal{C}\mathrm{h})$, the composition of the above-mentioned adjunctions yields an adjunction

$$\tilde{\mathbb{Z}} : \mathrm{sPre}(S)_* \rightleftarrows \mathcal{C}\mathrm{h}(S) : U$$

denoted in this way by abuse of notation. There are again several viewpoints for model structures on $\mathcal{C}\mathrm{h}(S)$ as for instance [Jar03], developing a model structure on $\mathcal{C}\mathrm{h}(S)$ as a stabilization in the sense of Section A.5.

According to [Fau06, Theorem 6.1], there is a proper, cofibrantly generated, monoidal and pointed model structure on $\mathcal{C}\mathrm{h}(S)$ where the weak equivalences are the morphisms inducing homology isomorphisms on all (canonical) points. In particular, a morphism of chain complexes is a weak equivalence if and only if it induces an isomorphism on all homology sheaves $aH_n(-)$. The adjunction $(\tilde{\mathbb{Z}}, U)$ is a Quillen adjunction with respect to the local projective model structure on $\mathrm{sPre}(S)_*$ and the just defined *stalkwise model structure* on $\mathcal{C}\mathrm{h}(S)$ since the left adjoint sends generating cofibrations to cofibrations and weak equivalences to weak equivalences. The stalkwise model structure is a Bousfield localization of a projective model structure on $\mathcal{C}\mathrm{h}(S)$ (confer [CH02] and [Hov99, Section 2.3]).

On the category $\mathcal{C}\mathrm{h}'(S) = \mathcal{C}\mathrm{h}(\mathrm{Shv}^{\mathcal{A}\mathrm{b}}(S))$ of chain complexes of sheaves $\mathcal{S}\mathrm{m}_S^{\mathrm{op}} \to \mathcal{A}\mathrm{b}$, one has the *injective model structure* as $\mathrm{Shv}^{\mathcal{A}\mathrm{b}}(S)$ is a Grothendieck category (id est a cocomplete abelian category in which filtered colimits commute with finite limits with an object G, called a generator, such that $\hom(G, -)$ is faithful) [Hov01a]. Sheafification defines a Quillen equivalence

$$a : \mathcal{C}\mathrm{h}(S) \rightleftarrows \mathcal{C}\mathrm{h}'(S) : i$$

where $\mathcal{C}\mathrm{h}(S)$ is furnished with the stalkwise model structure [Fau06, Proposition 6.2]. In the injective model $\mathcal{C}\mathrm{h}'(S)$, every object is cofibrant and the fibrant objects are sometimes denoted as those complexes of injective sheaves which are K*-injective* (confer [Spa88]). In particular, any bounded above complex of injective sheaves is fibrant [Hov01a, Proposition 2.12] and there is a computable fibrant replacement given by the right Cartan-Eilenberg resolution (confer [Wei94, 5.7.9]).

One advantage of the stalkwise model structure on $\mathcal{C}\mathrm{h}(S)$ however is the monoidality with respect to the *canonical tensor product of chain complexes*

$$(\tilde{\mathbb{Z}}(X) \otimes \tilde{\mathbb{Z}}(Y))_n = \bigoplus_i \tilde{\mathbb{Z}}(X)_i \otimes \tilde{\mathbb{Z}}(Y)_{n-i}$$

(confer [Fau06, Lemma 4.4]). The adjunction $(\tilde{\mathbb{Z}}, U)$ is not a strong monoidal adjunction with respect to this canonical tensor product. Fortunately, the *Eilenberg-Zilber theorem* [GJ99, Theorem IV.2.4] implies the existence of a weak equivalence $\tilde{\mathbb{Z}}(X) \otimes \tilde{\mathbb{Z}}(Y) \to \tilde{\mathbb{Z}}(X \wedge Y)$ yielding an isomorphism $aH_n(X) \cong aH_{n+1}(\Sigma X)$ for a pointed simplicial presheaf X together with the isomorphism $\tilde{Z}(\Sigma X) \cong \tilde{Z}(X)[1]$ where $[1]$ denotes a shift of the chain complex to the left, id est $C[1]_n = C_{n-1}$.

For a pointed simplicial presheaf X, one defines the *n-th reduced local (Nisnevich) homology sheaf* as

$$\begin{aligned} \tilde{H}_n^s(X) &= aH_n(\tilde{\mathbb{Z}}(X)) \\ &\cong a[\tilde{\mathbb{Z}}[S^n \wedge (-)_*], \tilde{\mathbb{Z}}[X]] \\ &\cong a\langle S^n \wedge (-)_*, U\tilde{\mathbb{Z}}(X)\rangle_s \\ &\cong \pi_n^s(U\tilde{\mathbb{Z}}(X)) \end{aligned}$$

where we do not have to make a fibrant replacement before applying the right adjoint as it preserves weak equivalences. Setting $H_n^s(X) = \tilde{H}_n^s(X_*)$ as the *n-th local (Nisnevich) homology sheaf*, one has $\tilde{H}_n^s(X) \cong \ker(H_n^s(X) \to H_n^s(\mathrm{Spec}(k)))$ and since $H_n^s(\mathrm{Spec}(k))$ is trivial for $n \geq 1$, the reduced and the non-reduced local homology sheaves coincide for those indices. The (reduced) homology sheaves are invariant under local weak equivalences by [Jar03, Lemma 1.1]. For a pointed simplicial presheaf X, the *Hurewicz morphism* $\pi_n^s(X) \to \tilde{H}_n^s(X)$ is induced by the unit of the adjunction $(\tilde{\mathbb{Z}}, U)$.

The Bousfield localization of the stalkwise model structure on $\mathcal{C}\mathrm{h}(S)$ at the set $\{\mathbb{Z}(U \times \mathbb{A}^1) \to \mathbb{Z}(U)\}$, where we implicitly choose a skeleton of $\mathcal{S}\mathrm{m}_S$ without changing the notation, provides a model structure which we call by abuse of notation the *motivic model structure* as well. The adjunction $(\tilde{\mathbb{Z}}, U)$ becomes a Quillen adjunction with respect to the motivic model structures by [Hir03, Theorem 3.3.20]. Due to the same theorem, one obtains an equivalent homotopy category from localizing the injective model structure on chain complexes of shaves $\mathcal{C}\mathrm{h}'(S)$ at the set $\{a\mathbb{Z}(U \times \mathbb{A}^1) \to a\mathbb{Z}(U)\}$ (confer also [Mor12, Lemma 5.17]). An argument for the existence of those Bousfield localizations is the following: The categories $\mathcal{C}\mathrm{h}(S)$ and $\mathcal{C}\mathrm{h}'(S)$ are locally presentable (confer [Bek00, Proposition 3.10] and the fact that the category of chain complexes on a Grothendieck category is again a Grothendieck category) and the stalkwise model structure on $\mathcal{C}\mathrm{h}(S)$ as well as the injective model structure on $\mathcal{C}\mathrm{h}(S)'$ are left proper and cofibrantly generated (confer [Fau06, Theorem 6.1] and [Hov01a, Theorem 2.2]). Hence, [Bar10, Theorem 4.7] based on an argument of Jeffrey Smith applies.

Let $R^{\mathbb{A}^1}$ be a fibrant replacement functor for the motivic model structure on $\mathcal{C}\mathrm{h}(S)$. For a pointed simplicial presheaf X, one defines the *n-th reduced* $\mathbb{A}^1$*-(Nisnevich) homology sheaf* as

$$\begin{aligned}\tilde{H}_n^{\mathbb{A}^1}(X) &= aH_n(R^{\mathbb{A}^1}\tilde{\mathbb{Z}}(X))\\ &\cong a[\tilde{\mathbb{Z}}(S^n \wedge (-)_*), R^{\mathbb{A}^1}\tilde{\mathbb{Z}}(X)]\\ &\cong a\langle S^n \wedge (-)_*, UR^{\mathbb{A}^1}\tilde{\mathbb{Z}}(X)\rangle_s\\ &\cong \pi_n^s(UR^{\mathbb{A}^1}\tilde{\mathbb{Z}}(X)).\end{aligned}$$

Setting $H_n^{\mathbb{A}^1}(X) = \tilde{H}_n^{\mathbb{A}^1}(X_*)$ as the *n-th* $\mathbb{A}^1$*-(Nisnevich) homology sheaf*, one obtains $\tilde{H}_n^{\mathbb{A}^1}(X) \cong \ker(H_n^{\mathbb{A}^1}(X) \to H_n^{\mathbb{A}^1}(\mathrm{Spec}(k)))$ and since $H_n^{\mathbb{A}^1}(\mathrm{Spec}(k))$ is trivial for indices $n \geq 1$ (confer [Mor12, Definition 5.28]), the reduced and the non-reduced $\mathbb{A}^1$-homology sheaves coincide for those indices. The (reduced) homology sheaves are invariant under motivic weak equivalences [Mor12, p.188]. The *motivic Hurewicz morphism* $\pi_n^{\mathbb{A}^1}(X) \to \tilde{H}_n^{\mathbb{A}^1}(X)$ for pointed simplicial presheaves X is induced by the unit of the derived adjunction of $(\tilde{\mathbb{Z}}, U)$.

Theorem 2.2.2. (Morel, Motivic Hurewicz theorem) Let S be the spectrum of a perfect field. Let n be a positive integer and X a pointed $(n-1)$-$\mathbb{A}^1$-connected simplicial presheaf. If $n \geq 2$ or if $n = 1$ and $\pi_1^{\mathbb{A}^1}(X)$ is an abelian sheaf, then

(1) $\tilde{H}_i^{\mathbb{A}^1}(X) \cong 0$ for all $i \in \{0, \ldots, n-1\}$,
(2) the motivic Hurewicz morphism $\pi_n^{\mathbb{A}^1}(X) \to \tilde{H}_n^{\mathbb{A}^1}(X)$ is an isomorphism,
(3) the motivic Hurewicz morphism $\pi_{n+1}^{\mathbb{A}^1}(X) \to \tilde{H}_{n+1}^{\mathbb{A}^1}(X)$ is an epimorphism.

Proof. This is [Mor12, Theorem 5.56] together with Remark [Mor12, 5.57] for simplicial sheaves. The result follows since homotopy sheaves are invariant under sheafification as they are invariant under local weak equivalences. □

Remark 2.2.3. Even though the order of this text suggests the opposite, the Motivic Hurewicz theorem 2.2.2 implies the Unstable $\mathbb{A}^1$-connectivity theorem 1.2.20 [Mor12, Theorem 5.37]. This is the case since for a pointed simplicial presheaf X, one has an isomorphism $\tilde{H}_n^{\mathbb{A}^1}(X) \cong \tilde{H}_{n+1}^{\mathbb{A}^1}(\Sigma X)$ as the localization of the model structure on $\mathcal{C}\mathrm{h}(S)$ behaves well with respect to the suspension [Mor12, Remark 5.29].

Remark 2.2.4. Let F be a pointed simplicial presheaf and $i_* : S^0 \to F_*$ the canonical map induced by the basepoint. If $[S, F]_{\mathbb{A}^1} \cong *$, then then induced map $\langle S^0, i_*\rangle_{\mathbb{A}^1}$ is an isomorphism. This may be seen as follows: Choose the motivic model structure and suppose that F is motivically fibrant. If we could prove that $F_* = F \sqcup *$ is motivically fibrant, there would be an isomorphism

$$\begin{aligned}[*, F_*]_{\mathbb{A}^1} &\cong \pi_0 \mathrm{s}\mathcal{S}\mathrm{et}(*, F \sqcup *)\\ &\cong \pi_0(\mathrm{s}\mathcal{S}\mathrm{et}(*, F) \sqcup \mathrm{s}\mathcal{S}\mathrm{et}(*, *))\\ &\cong \pi_0 \mathrm{s}\mathcal{S}\mathrm{et}(*, F) \sqcup \pi_0 \mathrm{s}\mathcal{S}\mathrm{et}(*, *))\\ &\cong [*, F]_{\mathbb{A}^1} \sqcup [*, *]_{\mathbb{A}^1}\end{aligned}$$

and the morphism $\langle S^0, S^0\rangle_{\mathbb{A}^1} \to \langle S^0, F_*\rangle_{\mathbb{A}^1}$ would by Lemma A.4.3 be isomorphic to the morphism $[*, *_*]_{\mathbb{A}^1} \to [*, F_*]_{\mathbb{A}^1}$. This would imply the result.

In order to show that F_* is motivically fibrant, if F is motivically fibrant, one observes, that the simplicial presheaf F_* is flasque (confer Remark 1.1.12) and hence local projective fibrant as a coproduct

$$\begin{array}{ccc} F(X) \sqcup *(X) & \longrightarrow & F(U) \sqcup *(U) \\ \downarrow & & \downarrow \\ F(Y) \sqcup *(Y) & \longrightarrow & F(W) \sqcup *(W) \end{array}$$

of homotopy pullback squares of simplicial sets is again a homotopy pullback square [Rez98, Lemma 7.2]. It remains to show that

$$\mathrm{s}\mathcal{S}\mathrm{et}(U, F_*) \xrightarrow{pr^*} \mathrm{s}\mathcal{S}\mathrm{et}(U \times \mathbb{A}^1, F_*)$$

is a weak equivalence for all objects U of $\mathcal{S}\mathrm{m}_S$. This morphism is isomorphic to

$$F(U) \sqcup * \xrightarrow{pr^* \sqcup \mathrm{id}} F(U \times \mathbb{A}^1) \sqcup *$$

which implies the result by the assumption that F is motivically fibrant.

Proposition 2.2.5. Let S be the spectrum of a perfect field. Let F be a pointed simplicial presheaf and $i_* : S^0 \to F_*$ the canonical map induced by the basepoint. If $\pi_0^{\mathbb{A}^1}(i_*)(S) = \langle S^0, i_* \rangle_{\mathbb{A}^1}$ is an isomorphism, then $\pi_0^{\mathrm{st}}(\Sigma_T^\infty i_*)(S) \cong \langle \mathbb{S}_T, \Sigma_T^\infty i_* \rangle_{\mathcal{SH}}$ is an isomorphism as well.

Proof. We will use frequently and without explicitly mentioning it that the evaluation at S is a point. As i_* is a section, it induces a monomorphism on stable homotopy sheaves evaluated at S. Hence, it is to show that there exists an $M \geq 0$ such that for all $m \geq M$ the morphism

$$\pi_{2m,m}^{\mathbb{A}^1}(S^{2m,m} \wedge S^0)(S) \to \pi_{2m,m}^{\mathbb{A}^1}(S^{2m,m} \wedge F_*)(S)$$

induced by i_* is an epimorphism (confer Remark 2.1.8). Set $M = 2$ and let $m \geq M$. We have to show that $\pi_{2m-1,m}^{\mathbb{A}^1}(\mathrm{hofib}^{\mathbb{A}^1})(S)$ is trivial. This triviality in turn is implied by the triviality of $\pi_{m-1}^{\mathbb{A}^1}(\mathrm{hofib}^{\mathbb{A}^1})(S)$ by Corollary 1.2.30. The latter would follow from

$$\pi_m^{\mathbb{A}^1}(\Sigma^m(S^{m,m} \wedge S^0))(S) \to \pi_m^{\mathbb{A}^1}(\Sigma^m(S^{m,m} \wedge F_*))(S)$$

since $\pi_{m-1}^{\mathbb{A}^1}(S^{2m,m} \wedge S^0)(S)$ is trivial by Corollary 1.2.21.

The Motivic Hurewicz theorem 2.2.2 provides the diagram

$$\begin{array}{ccc} \pi_m^{\mathbb{A}^1}(\Sigma^m(S^{m,m} \wedge R^{\mathbb{A}^1}S^0))(S) & \longrightarrow & \pi_m^{\mathbb{A}^1}(\Sigma^m(S^{m,m} \wedge R^{\mathbb{A}^1}F_*))(S) \\ \cong\downarrow & & \downarrow\cong \\ \tilde{H}_m^{\mathbb{A}^1}(\Sigma^m(S^{m,m} \wedge R^{\mathbb{A}^1}S^0))(S) & \longrightarrow & \tilde{H}_m^{\mathbb{A}^1}(\Sigma^m(S^{m,m} \wedge R^{\mathbb{A}^1}F_*))(S) \\ \cong\uparrow & & \uparrow\cong \\ \tilde{H}_0^{\mathbb{A}^1}(S^{m,m} \wedge R^{\mathbb{A}^1}S^0)(S) & \longrightarrow & \tilde{H}_0^{\mathbb{A}^1}(S^{m,m} \wedge R^{\mathbb{A}^1}F_*)(S) \end{array}$$

as an m-th suspension is locally $(m-1)$-connected and thus $(m-1)$-$\mathbb{A}^1$-connected by the Unstable $\mathbb{A}^1$-connectivity theorem 1.2.20. The wanted result would follow from the upper horizontal map being an epimorphism due to the fact that the functor $S^{2m,m} \wedge -$ preserves motivic weak equivalences. Hence, it suffices to show that the lower horizontal map is an epimorphism.

By assumption, $\pi_0^s(R^{\mathbb{A}^1}S^0)(S) \to \pi_0^s(R^{\mathbb{A}^1}F_*)(S)$ is an epimorphism where $R^{\mathbb{A}^1}$ denotes a motivic fibrant replacement functor. From the topological fact that smashing a π_0-epimorphism with an arbitrary space is again a π_0-epimorphism, it follows that $\pi_0^s(S^{m,m} \wedge R^{\mathbb{A}^1}S^0)(S) \to \pi_0^s(S^{m,m} \wedge R^{\mathbb{A}^1}F_*)(S)$ is an epimorphism.

By the topological fact, that a π_0-surjection induces an $\tilde{H}_0$-surjection, we know that there is an epimorphism $\tilde{H}_0^s(S^{m,m} \wedge R^{\mathbb{A}^1}S^0)(S) \to \tilde{H}_0^s(S^{m,m} \wedge R^{\mathbb{A}^1}F_*)(S)$ and it suffices to show that for some fibrant replacement functor $R^{\mathbb{A}^1}$ the natural transformation $\tilde{H}_0^s(-)(S) \to \tilde{H}_0^s(R^{\mathbb{A}^1}(-))(S)$ preserves epimorphisms.

We use the localized injective model on chain complexes of sheaves $\mathcal{C}\mathrm{h}'(S)$ discussed in Remark 2.2.1. Let $f : C \to D$ be a morphism between fibrant chain complexes of sheaves which are (-1)-connected (id est they have non-zero homology sheaves only in non-negative degrees) and suppose that f induces an epimorphism on aH_0 evaluated at S.

In the proof of [Mor12, Lemma 5.18], Fabien Morel gives a construction of a specific fibrant replacement functor $R^{\mathbb{A}^1}(C) = \operatorname{colim}_n (L_{\mathbb{A}^1}^{(1)} \circ R^s)^n(C)$ for a chain complex C of sheaves. As R^s induces an isomorphism of homology sheaves and homology sheaves commute with filtered colimits, it suffices to show that $aH_0(-)(S) \to aH_0(L_{\mathbb{A}^1}^{(1)}(-))(S)$ preserves epimorphisms (confer also the proof of Theorem 1.2.20).

By definition, $L_{\mathbb{A}^1}^{(1)}C$ is the cone (id est a homotopy cofiber)

$$\begin{array}{ccc} \underline{\mathrm{hom}}(\tilde{\mathbb{Z}}(\mathbb{A}^1), C) & \xrightarrow{ev_1} & C \\ \downarrow & & \downarrow \\ * & \longrightarrow & L_{\mathbb{A}^1}^{(1)}C \end{array}$$

of the canonical evaluation morphism ev_1 where $\mathbb{A}^1$ is pointed at zero. One gets a diagram of long exact sequences

$$\begin{array}{ccccccccccc} \ldots \to & aH_0(\underline{\mathrm{hom}}..C) & \to & aH_0(C) & \to & aH_0(L_{\mathbb{A}^1}^{(1)}C) & \to & aH_{-1}(\underline{\mathrm{hom}}..C) & \to & aH_{-1}(C) & \to \ldots \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \ldots \to & aH_0(\underline{\mathrm{hom}}..D) & \to & aH_0(D) & \to & aH_0(L_{\mathbb{A}^1}^{(1)}D) & \to & aH_{-1}(\underline{\mathrm{hom}}..D) & \to & aH_{-1}(D) & \to \ldots \end{array}$$

and by the Five-lemma $aH_0(L_{\mathbb{A}^1}^{(1)}f)$ is an epimorphism if $aH_{-1}(\underline{\mathrm{hom}}(\tilde{\mathbb{Z}}(\mathbb{A}^1), f))$ is an epimorphism.

We want to show that $aH_{-1}(\underline{\mathrm{hom}}(\tilde{\mathbb{Z}}(\mathbb{A}^1), f))(S)$ is an epimorphism.

Since there is a cofiber sequence $\mathbb{Z} \to \mathbb{Z}(X) \to \tilde{\mathbb{Z}}(X)$, [Hov99, Proposition 6.5.3.(a)] provides that $[\mathbb{Z}(X), E[n]] \cong 0$ implies $[\tilde{\mathbb{Z}}(X), E[n]] \cong 0$. Choosing the monoidal stalkwise model as described in Remark 2.2.1, one has

$$\begin{aligned} [\tilde{\mathbb{Z}}(X), E[n]] &\cong [\tilde{\mathbb{Z}}(X)[-n], E] \\ &\cong [\tilde{\mathbb{Z}}(S^{-n,0}) \otimes \tilde{\mathbb{Z}}(X), E] \\ &\cong [\tilde{\mathbb{Z}}(S^{-n,0}), \underline{\mathrm{hom}}(\tilde{\mathbb{Z}}(X), E)] \\ &\cong aH_{-n}(\underline{\mathrm{hom}}(\tilde{\mathbb{Z}}(X), E))(S). \end{aligned}$$

For a chain complex E, let $\bar{E}$ be the associated cochain complex defined by $\bar{E}_n = E_{-n}$. With respect to our non-canonical convention $E[1]_n = E_{n-1}$ for the shift of chain complexes, there is an isomorphism

$$\mathbb{H}^n(X, \bar{E}) \cong [\mathbb{Z}(X), E[n]]$$

by [MV99, Proposition 2.1.16] for an (injective) fibrant chain complex E.

Consider the cone (id est the homotopy cofiber) $0 \to C \to D \to E \to 0$ of f. A long exact sequence argument shows that E is 0-connected (id est a chain complex having non-zero homology sheaves only in strictly positive degrees). This means for the associated cochain complexes, that $0 \to \bar{C} \to \bar{D} \to \bar{E} \to 0$ is a short exact sequence of cochain complexes and $aH^q(\bar{E}) = aH_{-q}(E) \cong 0$ for $q \geq 0$.

We want to show that $\mathbb{H}^1(\mathbb{A}^1, \bar{C}) \to \mathbb{H}^1(\mathbb{A}^1, \bar{D})$ is an epimorphism. By [TT90, Lemma D.3], one gets a long exact sequence

$$\ldots \to \mathbb{H}^1(\mathbb{A}^1, \bar{C}) \to \mathbb{H}^1(\mathbb{A}^1, \bar{D}) \to \mathbb{H}^1(\mathbb{A}^1, \bar{E}) \to \ldots$$

of hypercohomology groups and it suffices to show that $\mathbb{H}^1(\mathbb{A}^1, \bar{E})$ is zero. There is a spectral sequence $H^p(X, H^q(\bar{E})) \Rightarrow \mathbb{H}^{p+q}(X, \bar{E})$ [Wei94, Application 5.7.10] and thus it is to show that $H^p(\mathbb{A}^1, H^{1-p}(\bar{E}))$ is zero. The group $H^p(\mathbb{A}^1, H^{1-p}(\bar{E}))$ can only be nonzero, if $1 - p < 0$ or equivalently $p \geq 2$ but since the Nisnevich cohomological dimension of a $\mathbb{A}^1$ is less or equal to its Krull dimension [TT90, Lemma E.6.c], this groups vanishes as well and the result follows. □

2.3. Motivic connectivity of morphisms

Definition 2.3.1. For $n \geq 0$, a morphism $f : X \to Y$ of pointed simplicial presheaves is called $\mathbb{A}^1$-*n-connected* if $p^* R^{\mathbb{A}^1}(f)$ is n-connected in the topological sense for every canonical point p^*, this is for every basepoint $x : \Delta^0 \to p^* R^{\mathbb{A}^1}(X)$ the induced maps $\pi_i(p^* R^{\mathbb{A}^1}(f), x)$ are isomorphisms for all indices $i \in \{0, \ldots, n-1\}$ and $\pi_n(p^* R^{\mathbb{A}^1}(f), x)$ is surjective.

Remark 2.3.2. A morphism $f : X \to Y$ of pointed simplicial presheaves into an $\mathbb{A}^1$-connected space Y is $\mathbb{A}^1$-n-connected if and only if its homotopy fiber $\mathrm{hofib}^{\mathbb{A}^1}(f)$ is $(n-1)$-connected, id est

$$\mathrm{conn}^{\mathbb{A}^1}(f) = \mathrm{conn}^{\mathbb{A}^1}(\mathrm{hofib}^{\mathbb{A}^1}(f)) + 1.$$

This definition conflicts with [Mor12, Definition 5.54] where the author defines a morphism into an $\mathbb{A}^1$-connected space to be $\mathbb{A}^1$-n-connected if and only if its homotopy fiber $\mathrm{hofib}^{\mathbb{A}^1}(f)$ is n-connected. This is the reason why our Corollary 2.3.9 looks like an improvement of [Mor12, Theorem 5.60] but it is not.

Remark 2.3.3. In Remark 1.2.7, it was mentioned that a diagram

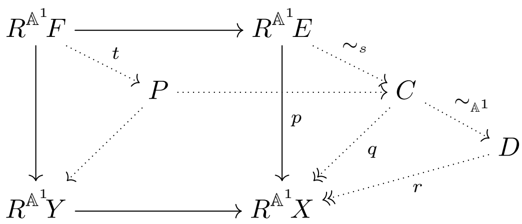
$$\begin{array}{ccc} F & \longrightarrow & E \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

is an $\mathbb{A}^1$-homotopy pullback diagram if and only if

$$\begin{array}{ccc} R^{\mathbb{A}^1}F & \longrightarrow & R^{\mathbb{A}^1}E \\ \downarrow & & \downarrow \\ R^{\mathbb{A}^1}Y & \longrightarrow & R^{\mathbb{A}^1}X \end{array}$$

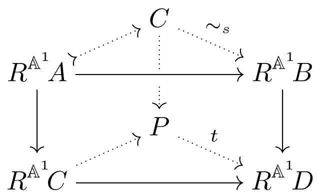
is a local homotopy pullback diagram:

The *if* direction is clear and for the other direction, consider the solid $\mathbb{A}^1$-homotopy pullback diagram



and factorize p as a local weak equivalence followed by a local injective fibration q. It is to show that t is a local weak equivalence where P is the categorical pullback of the indicated diagram. Factorize q as a motivic weak equivalence followed by an $\mathbb{A}^1$-local injective fibration r. The map $C \to D$ is a local weak equivalence as D is $\mathbb{A}^1$-local injective fibrant. Lemma A.3.10 implies, that q is an $\mathbb{A}^1$-local injective fibration. Thus by assumption, t is an $\mathbb{A}^1$-local weak equivalence but since the object P is $\mathbb{A}^1$-local injective fibrant, it is also a local weak equivalence.

The above observations may not work for motivic homotopy pushouts: A local homotopy pushout square is an $\mathbb{A}^1$-homotopy pushout square but conversely, if the solid diagram



is an $\mathbb{A}^1$-homotopy pushout square and one factorizes the upper horizontal map as a cofibration followed by a local weak equivalence, there is no need for the motivic weak equivalence t out of the indicated categorical pushout P to be a local weak equivalence as P does not have to be $\mathbb{A}^1$-local injective fibrant.

Lemma 2.3.4. Let S be the spectrum of a perfect field and $n \geq 0$ be an integer. Let X and Y be two pointed simplicial presheaves such that Y is $\mathbb{A}^1$-connected and let $f : X \to Y$ be a morphism. If f is $\mathbb{A}^1$-n-connected, then its homotopy cofiber is $\mathbb{A}^1$-n-connected as well, id est

$$\mathrm{conn}^{\mathbb{A}^1}(\mathrm{hocofib}(f)) \geq \mathrm{conn}^{\mathbb{A}^1}(f),$$

Proof. We may assume that f is an $\mathbb{A}^1$-local injective fibration, between $\mathbb{A}^1$-local injective fibrant objects. The statement is true for $n = 0$ since Y is locally connected. Hence, we suppose that $n \geq 1$. Since Y is an $\mathbb{A}^1$-1-connected simplicial presheaf, one has $\mathrm{conn}^{\mathbb{A}^1}(\mathrm{hofib}^{\mathbb{A}^1}(f)) + 1 = \mathrm{conn}^{\mathbb{A}^1}(f)$. The local homotopy fiber of f is an $\mathbb{A}^1$-homotopy fiber and therefore $\mathrm{conn}^s(\mathrm{hofib}^s(f)) = \mathrm{conn}^{\mathbb{A}^1}(\mathrm{hofib}^{\mathbb{A}^1}(f))$. Consider the local homotopy pushout square

$$\begin{array}{ccc} X & \longrightarrow & * \\ {\scriptstyle f}\big\downarrow & & \big\downarrow \\ Y & \longrightarrow & \mathrm{hocofib}(f). \end{array}$$

From the topological observation that a pushout of an n-connected map is n-connected [Str11, Problem 12.41.(a)], one gets

$$\mathrm{conn}^s(\mathrm{hocofib}(f)) = \mathrm{conn}^s(\Omega^s \mathrm{hocofib}(f)) + 1 \geq \mathrm{conn}^s(\mathrm{hofib}^s(f)) + 1.$$

Finally, the Unstable $\mathbb{A}^1$-connectivity theorem 1.2.20 yields

$$\mathrm{conn}^{\mathbb{A}^1}(\mathrm{hocofib}(f)) \geq \mathrm{conn}^s(\mathrm{hocofib}(f)).$$

□

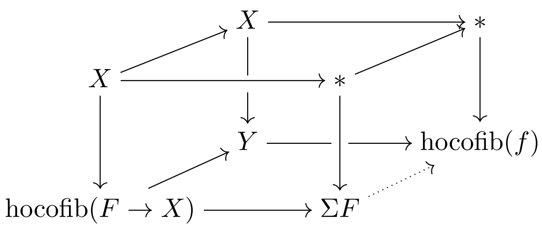
Lemma 2.3.5. Let S be the spectrum of a perfect field. Let X, Y be pointed simplicial presheaves such that X and Y are $\mathbb{A}^1$-1-connected. Let $f : X \to Y$ be a morphism. Then it holds

$$\mathrm{conn}^{\mathbb{A}^1}(\mathrm{hocofib}(f)) = \mathrm{conn}^{\mathbb{A}^1}(f).$$

Proof. The previous Lemma 2.3.4 shows that $\mathrm{conn}^{\mathbb{A}^1}(\mathrm{hocofib}(f)) \geq \mathrm{conn}^{\mathbb{A}^1}(f)$. We may assume that f is an $\mathbb{A}^1$-local injective fibration, between $\mathbb{A}^1$-local injective fibrant objects. Let F denote the $\mathbb{A}^1$-homotopy fiber of f which is also the local homotopy fiber of f and set $\mathrm{conn}^{\mathbb{A}^1}(F) = n-1$. From the long exact sequence of $\mathbb{A}^1$-homotopy sheaves and from the connectivity assumptions on the spaces X and Y we know that $n \geq 1$ and that there is an epimorphism $\pi_2^{\mathbb{A}^1}(Y) \twoheadrightarrow \pi_1^{\mathbb{A}^1}(F)$ which implies that $\pi_1^{\mathbb{A}^1}(F)$ is abelian. Hence, the Motivic Hurewicz theorem 2.2.2 and Remark 2.2.3 yield that $\mathrm{conn}^{\mathbb{A}^1}(\Sigma F) = \mathrm{conn}^{\mathbb{A}^1}(F) + 1$. We assume that

$$\mathrm{conn}^{\mathbb{A}^1}(\mathrm{hocofib}(f)) > \mathrm{conn}^{\mathbb{A}^1}(F) + 1 = n$$

and show that this leads to a contradiction. Consider the cube



where the front and the back face are local homotopy pushouts. The indicated induced map is locally at least as connected as the minimum of the other diagonal maps [Str11, Proposition 12.37], id est

$$\mathrm{conn}^s(\mathrm{hocofib}(F \to X) \to Y) \leq \mathrm{conn}^s(\Sigma F \to \mathrm{hocofib}(f)) = \mathrm{conn}^s(G) + 1$$

where G denotes the local homotopy fiber of the morphism $\Sigma F \to \mathrm{hocofib}(f)$. The existence of the Ganea fiber sequence of Lemma 1.3.17.(5) for the local model implies

$$\begin{aligned} \mathrm{conn}^s(\mathrm{hocofib}(F \to X) \to Y) &= \mathrm{conn}^s(F * \Omega X) + 1 \\ &= \mathrm{conn}^s(S^1 \wedge F \wedge \Omega X) + 1 \\ &\geq \mathrm{conn}^s(F) + 3. \end{aligned}$$

Hence, it holds $\mathrm{conn}^{\mathbb{A}^1}(G) \geq \mathrm{conn}^s(G) \geq \mathrm{conn}^s(F) + 2 = \mathrm{conn}^{\mathbb{A}^1}(F) + 2$ by the Unstable $\mathbb{A}^1$-connectivity theorem 1.2.20. We assumed $\pi_{n+1}^{\mathbb{A}^1}(\mathrm{hocofib}(f)) \cong 0$ and the above estimation yields $\pi_{n+1}^{\mathbb{A}^1}(G) \cong 0$. This implies $\pi_{n+1}^{\mathbb{A}^1}(\Sigma F) \cong \pi_n^{\mathbb{A}^1}(F) \cong 0$ by the long exact sequence of $\mathbb{A}^1$-homotopy sheaves for the local fiber sequence $G \to \Sigma F \to \mathrm{hocofib}(f)$ which is in fact an $\mathbb{A}^1$-fiber sequence by Theorem 1.2.16 since $\mathrm{hocofib}(f)$ is locally 1-connected as a local homotopy pushout of 1-connected spaces. □

Corollary 2.3.6. Let S be the spectrum of a perfect field and $n \geq 0$ an integer. Consider a motivic homotopy pushout diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle p} \\ C & \longrightarrow & D \end{array}$$

of pointed simplicial presheaves where A, B and C are $\mathbb{A}^1$-1-connected. Then g is $\mathbb{A}^1$-n-connected if and only if p is $\mathbb{A}^1$-n-connected.

Proof. As observed in Remark 2.3.3, we may assume that A, B and C are $\mathbb{A}^1$-local injective fibrant, f is a monomorphism and D is the pushout of the diagram. Hence, D is locally 1-connected and $\mathbb{A}^1$-1-connected by the Unstable $\mathbb{A}^1$-connectivity theorem 1.2.20. The result then follows from the previous Lemma 2.3.5. $\square$

Corollary 2.3.7. Let S be the spectrum of a perfect field and $n \geq 1$ be an integer. Let X, Y be $\mathbb{A}^1$-1-connected pointed simplicial presheaves. Let $f : X \to Y$ be a morphism. If $\tilde{H}_i^{\mathbb{A}^1}(f)$ is an isomorphism for all $i \in \{1, \dots, n\}$, then $\pi_i^{\mathbb{A}^1}(f)$ is an isomorphism for all $i \in \{1, \dots, n\}$.

Theorem 2.3.8. Let S be the spectrum of a perfect field. Let $n, m \geq 2$ be integers and consider the solid diagram

$$\begin{array}{ccccc} A & & \xrightarrow{\quad f \quad} & & B \\ & \overset{t}{\searrow} & & \nearrow & \\ {\scriptstyle g}\downarrow & & Q & & \downarrow{\scriptstyle p} \\ & \swarrow & & & \\ C & & \xrightarrow{\quad q \quad} & & D \end{array}$$

of pointed simplicial presheaves which is assumed to be a motivic homotopy pushout. Let A be $\mathbb{A}^1$-1-connected, B and C be $\mathbb{A}^1$-2-connected, the morphism f be $\mathbb{A}^1$-m-connected and g be $\mathbb{A}^1$-n-connected. Then, an induced map t into an $\mathbb{A}^1$-homotopy pullback Q of the diagram $C \to P \leftarrow B$ is $\mathbb{A}^1$-$(n+m-1)$-connected. In particular, the induced morphism

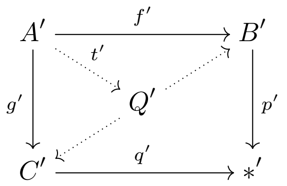
$$\pi_i^{\mathbb{A}^1}(\mathrm{hofib}^{\mathbb{A}^1}(A \xrightarrow{g} C)) \to \pi_i^{\mathbb{A}^1}(\mathrm{hofib}^{\mathbb{A}^1}(B \xrightarrow{p} D))$$

of $\mathbb{A}^1$-homotopy sheaves is an isomorphism for $0 \leq i \leq (n + m - 2)$ and an epimorphism for $i = (n + m - 1)$.

Proof. Corollary 2.3.6 implies that q is $\mathbb{A}^1$-m-connected and p be $\mathbb{A}^1$-n-connected.

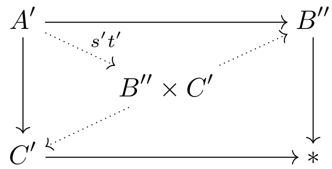
Choose for instance the $\mathbb{A}^1$-local injective model. Factorize the basepoint of D as a motivic weak equivalence followed by an $\mathbb{A}^1$-local injective fibration $* \xrightarrow{\sim} *' \twoheadrightarrow D$.

Pulling back the whole main square along the fibration $*' \twoheadrightarrow D$ yields a diagram



where all spaces are $\mathbb{A}^1$-1-connected as they are $\mathbb{A}^1$-homotopy fibers of $\mathbb{A}^1$-2-connected morphisms. The square consisting of Q', B', C' and $*'$ is again an $\mathbb{A}^1$-homotopy pullback since $*' \twoheadrightarrow D$ was a fibration. The primed analogues of the maps in the initial square have weakly equivalent $\mathbb{A}^1$-homotopy fibers and hence the same $\mathbb{A}^1$-connectivity respectively. Therefore, it suffices to show that t' is $\mathbb{A}^1$-$(n+m-1)$-connected. Factorizing the composition of p' with $*' \xrightarrow{\sim} *$ as a motivic weak equivalence followed by an $\mathbb{A}^1$-local injective fibration $B' \xrightarrow{\sim} B'' \twoheadrightarrow *$ and taking the categorical pullback of $C' \to * \leftarrow B''$ induces a motivic weak equivalence $s' : Q' \xrightarrow{\sim} B'' \times C'$ and it suffices to show that $s't' : A' \to B'' \times C'$ is $\mathbb{A}^1$-$(n+m-1)$-connected and we already know that it is $\mathbb{A}^1$-1-connected as its domain and its codomain are $\mathbb{A}^1$-1-connected.

Mather's cube theorem 1.3.9 implies that the outer square consisting of A', B', C' and $*'$ is a motivic homotopy pushout and hence the outer square of

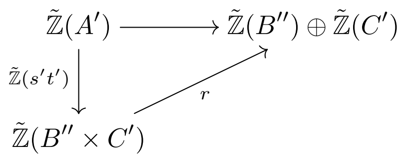


is a motivic homotopy pushout as well. We are going to identify $s't'$ with the canonical morphism from the wedge into the product on $\mathbb{A}^1$-homology sheaves.

It can be assumed that all the objects are $\mathbb{A}^1$-injective fibrant simplicial sheaves, by applying the sheafification functor which commutes with products and preserves homotopy pushouts as a left Quillen functor. The left Quillen functor $\tilde{\mathbb{Z}}$ to the $\mathbb{A}^1$-localized model structure on chain complexes of sheaves $\mathcal{C}\mathrm{h}'$ preserves homotopy pushouts (confer Remark 2.2.1 and Remark A.3.11). Hence, the canonical morphism

$$\mathbb{Z}(A') \to \tilde{\mathbb{Z}}(B'') \oplus \tilde{\mathbb{Z}}(C')$$

obtained by the product property of the direct sum in $\mathcal{C}\mathrm{h}'$ induces an isomorphism on all $aH_i(R^{\mathbb{A}^1}(-))$. There is a factorization



again obtained by the product property of the direct sum with respect to the morphisms $\tilde{\mathbb{Z}}(pr_{B''})$ and $\tilde{\mathbb{Z}}(pr_{C'})$. The canonical morphism

$$\tilde{\mathbb{Z}}(B'') \oplus \tilde{\mathbb{Z}}(C') \xrightarrow{\sim} \tilde{\mathbb{Z}}(B'' \vee C')$$

induces an isomorphism on all $aH_i(R^{\mathbb{A}^1}(-))$ and the composite

$$\tilde{\mathbb{Z}}(B'' \times C') \xrightarrow{r} \tilde{\mathbb{Z}}(B'') \oplus \tilde{\mathbb{Z}}(C') \xrightarrow{\sim} \tilde{\mathbb{Z}}(B'' \vee C')$$

admits a section by

$$\tilde{\mathbb{Z}}(u) : \tilde{\mathbb{Z}}(B'' \vee C') \to \tilde{\mathbb{Z}}(B'' \times C')$$

where $u : B'' \vee C' \to B'' \times C'$ is the canonical morphism. This identifies $s't'$ with the canonical morphism u from the wedge into the product on $\mathbb{A}^1$-homology sheaves.

Hence, it suffices by Corollary 2.3.7 to show, that $\tilde{H}_i^{\mathbb{A}^1}(u)$ is an isomorphism for all $i \in \{1, \ldots, n+m-2\}$ and that $\tilde{H}_{n+m-1}^{\mathbb{A}^1}(u)$ is an epimorphism. By the long exact sequence, this is equivalent to $\tilde{H}_i^{\mathbb{A}^1}(\mathrm{hocofib}(u)) \cong 0$ for $i \in \{1, \ldots, n+m-1\}$.

As $B'' \times C'$ is $\mathbb{A}^1$-1-connected and $\mathbb{A}^1$-local injective fibrant, the (homotopy) cofiber of u is $\mathbb{A}^1$-1-connected. Hence we may use the Motivic Hurewicz theorem 2.2.2 and it suffices to show the $\mathbb{A}^1$-$(n+m-1)$-connectivity of $\mathrm{hocofib}(u) \sim B'' \wedge C'$. From a topological cellular argument, one gets that $B'' \wedge C'$ is locally $(n+m-1)$-connected and by the Unstable $\mathbb{A}^1$-connectivity theorem 1.2.20, its $\mathbb{A}^1$-connectivity is at least $(n+m-1)$. □

Corollary 2.3.9. Let S be the spectrum of a perfect field, $n \geq 2$ an integer and A an $(n-1)$-$\mathbb{A}^1$-connected pointed simplicial presheaf. Then, the morphism

$$\pi_i^{\mathbb{A}^1}(A) \to \pi_{i+1}^{\mathbb{A}^1}(\Sigma A)$$

of $\mathbb{A}^1$-homotopy sheaves is an isomorphism for $0 \leq i \leq 2n-2$ and an epimorphism for $i = 2n-1$.

Proof. This follows from the previous Theorem 2.3.8 setting $B = C \cong *$. □

3. **The motivic J-homomorphism**

3.1. Principal bundles and fiber bundles

We assume the base scheme S to be the spectrum of a field k for the whole section. The symbol τ always denotes the Zariski, the Nisnevich, the étale or the fppf (fidèlement plat et de présentation finie) Grothendieck topology. All these topologies are subcanonical and refine each other in the opposite given order [Jan87, 5.3.7].

Definition 3.1.1 ([Jan87, Chapter 2])**.** A *group object* in a complete category $\mathcal{C}$ is an object G of $\mathcal{C}$ together with a morphism

$$m : G \times G \to G,$$

called the *multiplication*, a morphism

$$e : * \to G$$

called the *unit* and a morphism

$$(-)^{-1} : G \to G$$

called the *inverse* such that the three obvious diagrams encoding associativity, the meaning of the identity and the meaning of the inverse commute. A group object in $\mathcal{S}\mathrm{m}_S$ is called a *group scheme* and a group object in Pre is called a *group presheaf*. As the Yoneda embedding behaves well with respect to products, the embedding of a group scheme (respectively a group presheaf) into the category of simplicial presheaves is a group object in sPre and will be called a group scheme (respectively a group presheaf) as well by abuse of notation.

A *(left) action* of a group presheaf G on a simplicial presheaf X is a morphism $\alpha : G \times X \to X$ such that both the diagram

$$\begin{array}{ccc} G \times G \times X & \xrightarrow{\mathrm{id} \times \alpha} & G \times X \\ {\scriptstyle m \times \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \alpha} \\ G \times X & \xrightarrow{\quad\alpha\quad} & X \end{array}$$

and the diagram

$$\begin{array}{ccc} X & & \\ \downarrow & \searrow\!\!\!\!= & \\ * \times X & & \\ {\scriptstyle e \times \mathrm{id}}\downarrow & & \\ G \times X & \xrightarrow{\quad\alpha\quad} & X \end{array}$$

commute. Alternatively, a left action (respectively a right action) may be defined as a morphism $\alpha : G \times X \to X$ such that α induces a left action (respectively a right action) of the group $G(U)$ on the (possibly empty) set $X(U)_n$ for all n and U of $\mathcal{S}\mathrm{m}_S$. By abuse of notation, we write gx for $\alpha(U)(g,x)_n$. As above, if G and X are representable, we speak of an action of a group scheme on the scheme X. These definitions may be adopted for simplicial sheaves on $\mathcal{S}\mathrm{m}_S$ with respect to a topology τ (or sheaves on k-Alg by Remark 1.0.2) by considering the underlying simplicial presheaves.

An action α of G on X is called *free*, if $\alpha(U)_n$ is a free action for every n and every U, this is from $gx = hx$, it follows $g = h$.

Remark 3.1.2. Let G be a group presheaf and let E be a simplicial presheaf with a right G-action. The action is free if and only if the morphism

$$E \times G \to E \times E$$

given on sections by $(x, g) \mapsto (x, xg)$ is a monomorphism. For example, if G is a group presheaf and H is a *subgroup presheaf*, id est $H(U)$ is a subgroup of $G(U)$ for every U, the action $G \times H \to G$ given by the multiplication is a free action.

Definition 3.1.3. Let α be a right action of a group presheaf G on a simplicial presheaf E. The object defined by the pushout

$$\begin{array}{ccc} aE \times aG & \xrightarrow{a\alpha} & aE \\ {\scriptstyle pr}\downarrow & & \downarrow \\ aE & \longrightarrow & aE/aG \end{array}$$

in the category of simplicial τ-sheaves (which is the same as the sheafification $a_\tau(E/X)$ of the simplicial presheaf quotient) viewed as an object of sPre via the inclusion functor is called the *τ-orbit space* and is denoted by $E/\!\!/_\tau G$.

Remark 3.1.4. With the notations of the previous Definition 3.1.3, the morphism $aE \to E/\!\!/_\tau G$ is always an epimorphism in the category of simplicial τ-sheaves. The canonical simplicial presheaf morphism $E/G \to E/\!\!/_\tau G$ induced by the unit of the adjunction (a_τ, i) is always a monomorphism, id est E/G is separated [Skj09, Proposition 1.13]. If α is a free right action of aG on aE, the pushout of the previous

Definition 3.1.3 is also a pullback as the monomorphism $aE \times aG \rightarrowtail aE \times aE$ defines an equivalence relation [MM92, Theorem Appendix.1.1] and there is an isomorphism

$$aE \times aG \xrightarrow{\cong} aE \times_{E/\!\!/_{\tau}G} aE$$

of simplicial presheaves (we omit the inclusion functor i by abuse of notation).

Remark 3.1.5. The existence of the isomorphism in the previous Remark 3.1.4 does not imply that the orbit space $E/\!\!/_{\tau}G$ with respect to a topology τ coincides with the orbit space in another topology [Skj09, Lemma 3.6]. It is an interesting and non-trivial question to decide whether a quotient $E/\!\!/_{\tau}G$ with respect to a certain Grothendieck topology is actually representable by an object of $\mathcal{S}\mathrm{m}_S$. There are more tools available to decide this question, the finer the topology is and the subcanonical fppf topology which is even finer than the étale topology turns out to be helpful (confer [DG70, Chapitre III] or [Jan87]). If an orbit space $E/\!\!/_{\tau}G$ is a simplicial sheaf in a finer topology τ', one has $E/\!\!/_{\tau}G \cong E/\!\!/_{\tau'}G$ as simplicial presheaves by definition. In particular, if the orbit space $E/\!\!/_{\tau}G$ is representable by an object of $\mathcal{S}\mathrm{m}_S$, it is the orbit space for any finer topology τ'.

Remark 3.1.6. We briefly recall some definitions and statements of category and topos theory. If $\mathcal{C}$ is complete category, a *regular monomorphism*, is a morphism $f : A \to B$ which fits into an equalizer diagram

$$A \xrightarrow{f} B \rightrightarrows C$$

whereas a monomorphism $f : A \to B$ just fits into a a pullback diagram

$$\begin{array}{ccc} A & = & A \\ \| & & \downarrow f \\ A & \overset{f}{\rightarrowtail} & B. \end{array}$$

For a cocomplete category, a *regular epimorphism* is defined analogously. A regular monomorphism is in particular a monomorphism. The analogous statement holds for epimorphisms.

From Giraud's theorem [MM92, Theorem Appendix.1.1], one deduces that every topos is complete and cocomplete [Bor94c, Proposition 3.4.3], every monomorphism is a regular monomorphism [Bor94c, Proposition 3.4.11] and every epimorphism is a regular epimorphism [Bor94c, Proposition 3.4.13].

Let $\mathcal{C}$ be a complete and cocomplete category.

(1) A pullback of a monomorphism is a monomorphism and a pushout of an epimorphism is an epimorphism.
(2) If $\mathcal{C}$ is a topos, a pushout of a monomorphism is a monomorphism and a pullback of an epimorphism is an epimorphism [LS06, Lemma 18].
(3) If the the solid diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & Q \nearrow \!\!\swarrow & \downarrow \\ C & \longrightarrow & P \end{array}$$

is a pushout, then the pullback diagram Q-B-C-P is also a pushout diagram. The dual statement holds as well.

(4) If $\mathcal{C}$ is a topos,

$$\begin{array}{ccc} A & \rightarrowtail & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

is a pushout diagram and $A \rightarrowtail B$ monomorphisms, then the diagram is also a pullback diagram [Joh02, Corollary A.2.4.3].

(5) If the diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow{\scriptstyle p} \\ B & \overset{p}{\twoheadrightarrow} & D \end{array}$$

is a pullback diagram and $p : B \twoheadrightarrow D$ is a regular epimorphism, then the diagram is also a pushout diagram [LS06, Lemma 7].

Definition 3.1.7. Let G be a group presheaf, $p : E \to X$ a morphism of simplicial presheaves and let α be a right action of G on E. The morphism p (together with the action α) is called a *principal G-morphism* if the diagram

$$\begin{array}{ccc} E \times G & \xrightarrow{\alpha} & E \\ {\scriptstyle pr}\downarrow & & \downarrow{\scriptstyle p} \\ E & \xrightarrow{p} & X \end{array}$$

is commutative and a pullback. This can also be phrased as the property that p is G-equivariant (where X has the trivial action) and $E \times G \to E \times_X E$ is an isomorphism.

Remark 3.1.8. Since $E \times_X E \rightarrowtail E \times E$ is always a monomorphism, it follows that the action α in the Definition 3.1.7 of a principal G-morphism $p : E \to X$ is free. Conversely, for a free right action of aG on aE, the morphism $aE \to E /\!\!/_{\tau} G$ is an epimorphism of τ-sheaves by Remark 3.1.6 and a principal G-morphism by Remark 3.1.4. This relates the definition of a principal G-morphism to the notion of [MV99, Section 4.1].

A pullback of a principal G-morphism along a morphism $U \to X$ is again a principal G-morphism.

Definition 3.1.9. A morphism $p : E \to X$ of simplicial presheaves is called *τ-locally trivial* if for all $U \in \mathcal{S}\mathrm{m}_S$ and all morphisms $U \to X$ there exists a *trivializing τ-covering* $\{U_i \to U \mid i \in I\}$ such that for all $i \in I$ there exists a simplicial presheaf F and the diagram

$$\begin{array}{ccccc} p^{-1}(U_i) & & \xrightarrow[\cong]{\phi} & & U_i \times F \\ & {\scriptstyle p}\searrow & & \swarrow{\scriptstyle pr} & \\ & & U_i & & \end{array}$$

where $p^{-1}(U_i) = U_i \times_X E$ commutes.

A principal G-morphism is called a *principal G-bundle* with respect to a topology τ (or a *principal τ-G-bundle*), if p is locally trivial with respect to that topology. A *morphism $f : E \to D$ of principal τ-G-bundles* over the same simplicial presheaf X is a G-equivariant morphism over X. For a topology τ, let $H^1_\tau(X, G)$ denote the isomorphism classes of principal G bundles over X.

Let X be a pointed simplicial presheaf. A diagram $F \to E \xrightarrow{p} X$ of simplicial presheaves is called a *τ-fiber bundle* with respect to a topology τ, if it is defined by the pullback diagram

$$\begin{array}{ccc} F & \longrightarrow & E \\ \downarrow & & \downarrow{\scriptstyle p} \\ * & \longrightarrow & X \end{array}$$

and if p is locally trivial with respect to the topology τ. A τ-fiber bundle $F \to E \xrightarrow{p} X$ is called a *smooth τ-fiber bundle*, if the objects F, E and X are from $\mathcal{S}\mathrm{m}_S$ and p is a smooth morphism.

Remark 3.1.10. If a morphism of simplicial presheaves is τ-locally trivial, it is locally trivial with respect to a finer topology τ'.

Any pullback of a locally trivial morphism is again locally trivial. Therefore by Remark 3.1.8, any pullback of a principal G-bundle is again a principal G-bundle.

Any morphism of principal G-bundles over X is an isomorphism as this property can be tested locally [Jar11, Lemma 2.26]. For an X from $\mathcal{S}\mathrm{m}_S$, the object $H^1_\tau(X,G)$ may be seen as a candidate for the first non-abelian sheaf cohomology, since it can be identified with the identically denoted sheaf cohomology group for a separated X and a sheaf G of abelian groups [Mil80, Proposition 4.6].

A principal τ-G-bundle $p : E \to X$ in the category of pointed simplicial presheaves, defines a τ-fiber bundle $G \to E \to X$ by the two pullback diagrams

$$\begin{array}{ccccc} G & \longrightarrow & E \times G & \xrightarrow{\alpha} & E \\ \downarrow & & {\scriptstyle pr}\downarrow & & \downarrow{\scriptstyle p} \\ * & \longrightarrow & E & \xrightarrow{p} & X. \end{array}$$

The diagram $G \to E \to X$ of simplicial presheaves will by abuse of notation also be denoted as a *principal G-bundle* in the following.

Consider a morphism $p : E \to X$ in $\mathcal{S}\mathrm{m}_S$. The embedding into sPre is Zariski-locally trivial if and only if there exists a Zariski covering $\{U_i \to X \mid i \in I\}$ such that for all $i \in I$ the diagram

$$\begin{array}{ccc} p^{-1}(U_i) & \xrightarrow[\cong]{\phi} & U_i \times F \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle pr} & \\ & U_i & \end{array}$$

commutes.

Remark 3.1.11. A morphism $p : E \to X$ between objects from $\mathcal{S}\mathrm{m}_S$ is a principal Zariski-G-bundle (with respect to a right action α of G on E) if and only if it is a *Zariski-G-torsor* (with respect to α) in the sense of [GW10, Definition 11.9], this is

(1) for every element $U \to X$ of a Zariski covering, the action of $G(U)$ on the set $E(U)$ is free and transitive and
(2) there exists a Zariski covering $\{U_i \to X\}$ such that each $E(U_i)$ is non-empty,

as the existence of a (global) section of p is equivalent to its triviality. For any other Grothendieck topology, the definition and the statement are analogous.

Remark 3.1.12. Even though Remark 3.1.5 draws attention to the possible difference of quotients for different topologies considered in this text, there is no ambiguity for a Zariski-locally trivial situation: Consider a topology τ' finer than a topology τ, and let α be a free right action of a group presheaf G from $\mathcal{S}\mathrm{m}_S$ on a simplicial presheaf E from $\mathcal{S}\mathrm{m}_S$. The pushout

$$\begin{array}{ccc} E \times G & \xrightarrow{a\alpha} & E \\ {\scriptstyle pr}\downarrow & & \downarrow{\scriptstyle p} \\ E & \longrightarrow & E/\!\!/_{\tau'} G \end{array}$$

defines the quotient in the category of simplicial τ'-sheaves. If the epimorphism p is τ-locally trivial, it is an epimorphism in the category of simplicial τ-sheaves as well as a projection is always an epimorphism and those are detected locally [Jar11, Lemma 2.26]. By Remark 3.1.6, the diagram is then a pushout in the category of simplicial τ-sheaves which implies that the quotients (viewed as simplicial presheaves) coincide by Remark 3.1.5.

Remark 3.1.13. A Grothendieck pre-topology Cov on a complete category $\mathcal{C}$ defines a Grothendieck pre-topology $\mathrm{C\overline{ov}}$ where

$$\mathrm{C\overline{ov}}(X) = \{\{U_i \to X\} \mid \{U_i \to X\} \text{ is refined by some element in } \mathrm{Cov}(X)\}$$

called the *saturation* of Cov. If two Grothendieck pre-topologies refine each other, they have the same saturation. Even though for every covering $\{U_i \to X\}$ of a pre-topology Cov on $\mathcal{C}$, the morphism $\coprod ayU_i \twoheadrightarrow ayX$ is an epimorphism of $\mathrm{Shv}(\mathcal{C})$, the converse may not be true. However, a morphism

$$\coprod_i ayU_i \to ayX$$

is an epimorphism of $\mathrm{Shv}(\mathcal{C})$ if and only if $\{U_i \to X\}$ is an element of $\mathrm{C\overline{ov}}(X)$ [MM92, Corollary III.7.7].

Lemma 3.1.14. In the category sShv of simplicial sheaves with respect to a topology τ, a principal G-morphism $p : E \to X$ between objects from $\mathcal{S}\mathrm{m}_S$ is τ-locally trivial if and only if it is an epimorphism.

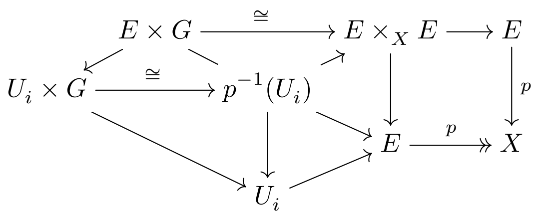
Proof. A locally trivial morphism is an epimorphism in the category of simplicial sheaves as the projection out of a product is an epimorphism and being an epimorphism is a local property by [Jar11, Lemma 2.26].

For the other implication, let $p : E \to X$ be a principal G-morphism of objects from $\mathcal{S}\mathrm{m}_S$ which is an epimorphism. Consider the diagram

$$\begin{array}{ccccc} E \times G & \xrightarrow{\cong} & E \times_X E & \longrightarrow & E \\ & \searrow^{pr} & \downarrow & & \downarrow{\scriptstyle p} \\ & & E & \overset{p}{\twoheadrightarrow} & X \end{array}$$

and pick a τ-covering $\{U_i \to E\}$ of E. Since by Remark 3.1.13, the map $\coprod U_i \to E$ is an epimorphism of τ-sheaves, so is the composition $\coprod U_i \to E \to X$. Then, Remark 3.1.13 implies, that $\{U_i \to X\}$ is a covering of X in the saturation of the pre-topology τ. Hence, it is refined by some τ-covering $\{U_i \to X\}$ of X, which we denote likewise, by abuse of notation. All in all, there is a τ-covering $\{U_i \to X\}$ of X, such that each $U_i \to X$ factorizes through $p : E \to X$.

Pulling back along $U_i \to X$ defines the diagram

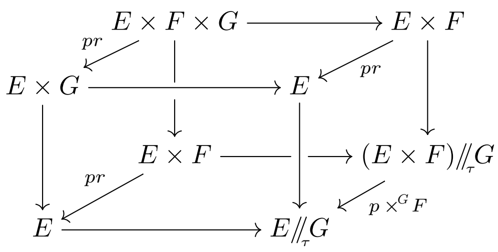


which implies the result. $\square$

Definition 3.1.15. Let α be a free right action of a group presheaf G on a simplicial presheaf E. Let β be a (not necessarily free) left action of G on a simplicial presheaf F. The morphism

$$(E \times F) \times G \to E \times F$$

given on sections by $(e, f, g) \mapsto (eg, g^{-1}f)$ defines a free right action on $E \times F$ whose orbit space $(E \times F)/\!\!/_{\tau} G$ with respect to a topology τ is denoted by $E \times^G F$ (ignoring the topology τ in the notation). The diagram



provides a canonical morphism $p \times^G F : E \times^G F \to E/\!\!/_{\tau} G$.

Lemma 3.1.16. Let G be a τ-group sheaf and let F be a simplicial τ-sheaf. Let $G \to E \xrightarrow{p} X$ be a principal τ-G-bundle, E and X be objects from $\mathcal{S}\mathrm{m}_S$ and let F be equipped with a left G-action. Then

$$F \to E \times^G F \xrightarrow{p \times^G F} X$$

is a τ-fiber bundle.

Moreover, if F is a group sheaf and $\varphi : G \to F$ a group homomorphism (id est a homomorphism for each section) through which G acts on F from the left, then the above fiber bundle is a principal τ-F-bundle where F acts on $E \times^G F$ from the right by (the sheafification of) $[e, f]f' = [e, ff']$.

Proof. Since p is a principal τ-G-bundle, one has $X \cong E/\!\!/_{\tau} G$. The bottom diagram

$$\begin{array}{ccc} E \times F & \longrightarrow & (E \times F)/G \\ \downarrow & & \downarrow \\ E & \longrightarrow & E/G \end{array}$$

of the cube in Definition 3.1.15 with quotients taken in the category of simplicial presheaves is a pullback, since it is a pullback in each simplicial degree and in every section (confer [Jan87, 5.14.3]). The sheafification functor with respect to the topology τ commutes with pullbacks and thus the diagram of the associated sheaves is a pullback as well.

With the same reasoning as in the proof of Lemma 3.1.14, there is a τ-covering $\{U_i \to X\}$ factorizing through p. Hence, the pullback diagram

$$\begin{array}{ccccc} U_i \times F & \longrightarrow & E \times F & \longrightarrow & (E \times F)/\!\!/_{\tau} G \\ \downarrow & & \downarrow & & \downarrow {\scriptstyle p \times^G F} \\ U_i & \longrightarrow & E & \xrightarrow{p} & E/\!\!/_{\tau} G \end{array}$$

implies that $p \times^G F$ is τ-locally trivial and $F \to E \times^G F \to X$ is a τ-fiber bundle.

If F group sheaf, the above-mentioned action of F on $E \times^G F$ is well defined and free which may be tested section-wise: From $[e, f]f' = [e, f]$, which means that there exists a g such that $(eg, \varphi(g^{-1})ff') = (e, f)$, it follows that f' is the unit element because F acts freely on itself. Since the action of G on $E \times F$ commutes with the action of F on $E \times F$, applying the sheafification functor with respect to the topology τ yields $(E \times^G F)/\!\!/_{\tau} F \cong E/\!\!/_{\tau} G \cong X$ (confer [Jan87, I.5.18]). ☐

Definition 3.1.17. In the situation of Lemma 3.1.16, on says that

$$F \to E \times^G F \xrightarrow{p \times^G F} X$$

is an *extension of the structure group* of

$$G \to E \xrightarrow{p} X$$

by $\varphi : G \to F$. Conversely, p is a *reduction of the structure group* of $p \times^G F$ by φ.

Lemma 3.1.18 ([Jan87, I.5.16.1])**.** Let G be a Zariski-group sheaf, $G \to E \to X$ be a principal Zariski-G-bundle, F, E and X be from $\mathcal{S}\mathrm{m}_S$ and let F be equipped with a left G-action. Then $E \times^G F$ comes from $\mathcal{S}\mathrm{m}_S$.

Lemma 3.1.19. Let G be a τ-group sheaf, $\varphi : H \hookrightarrow G$ a τ-subgroup sheaf. Let moreover E and X be objects from $\mathcal{S}\mathrm{m}_S$ and $G \to E \xrightarrow{p} X$ be a principal τ-G-bundle. Then, there exists a pullback diagram

$$\begin{array}{ccccc} & & E \times^H G & \longrightarrow & E \\ & & {\scriptstyle q \times^H G} \downarrow & & \downarrow {\scriptstyle p} \\ G/\!\!/_{\tau} H & \longrightarrow & E/\!\!/_{\tau} H & \xrightarrow{f} & X \end{array}$$

of simplicial presheaves, where the lower horizontal morphisms constitute a τ-fiber bundle and where $q \times^H G$ is the extension of the structure group of the canonical principal τ-H-bundle $q : E \to E/\!\!/_{\tau} H$.

Proof. Equip $G/\!\!/_{\tau} H$ with the canonical left G-action given on simplicial presheaves by $G \times G/H \to G/H$ by $g'(gH) = g'gH$. Lemma 3.1.16 provides a τ-fiber bundle

$$G/\!\!/_{\tau} H \to E \times^G (G/\!\!/_{\tau} H) \xrightarrow{f} X.$$

A free right action on $E \times G$ by $G \times H$ is given by $(e, g)(g', h) = (eg', g'^{-1}gh)$. The two induced actions by G and H on $E \times G$ respectively commute and hence the quotient $(E \times G)/(G \times H)$ may be taken successively. This shows $(E \times G/H)/G \cong E/H$ and f is identified with the canonical map $E/\!\!/_{\tau} H \to E/\!\!/_{\tau} G \cong X$ after sheafification. Hence, there is a τ-fiber bundle

$$G/\!\!/_{\tau} H \to E/\!\!/_{\tau} H \xrightarrow{f} X.$$

As the action $E \times H \xrightarrow{\mathrm{id} \times \varphi} E \times G \to E$ of H on E is free, it defines a principal τ-H-bundle $q : E \to E/\!\!/_{\tau} H$. We have to show, that $q \times^H G$ is the pullback of f and p because this implies already that it is τ-locally trivial. The diagram

$$\begin{array}{ccc} (E \times G)/H & \longrightarrow & (E \times G)/G \\ \downarrow & & \downarrow \\ E/H & \longrightarrow & E/G \end{array}$$

of simplicial presheaves is a pullback and hence its sheafification is a pullback as well. This implies the result. ☐

Lemma 3.1.20. If in the situation of the above Lemma 3.1.19, the topology τ is the Zariski topology and if $G/\!\!/_{\tau} H$ is an object from $\mathcal{S}\mathrm{m}_S$, then also $E/\!\!/_{\tau} H$ is an object from $\mathcal{S}\mathrm{m}_S$.

Proof. This follows from the isomorphism $E \times^G (G/\!\!/_{\tau} H) \cong E/\!\!/_{\tau} H$ of the previous Lemma 3.1.19 and Lemma 3.1.18. ☐

3.2. Fiber sequences from bundles

We assume the base scheme S to be the spectrum of a field k for the whole section.

Lemma 3.2.1. For pointed simplicial presheaves X and F where X is from $\mathcal{S}\mathrm{m}_S$, a Zariski-fiber bundle $F \to E \xrightarrow{p} X$ defines a local fiber sequence.

Proof. This is either implied by Theorem 1.2.11 considering stalks or differently by [Rez98, Theorem 5.1]: By (P4), p is locally sharp if the base change along the epimorphism $\coprod U_i \to X$ for a trivializing Zariski-covering $\{U_i \to X\}$ is locally sharp. By (P5), a coproduct of locally sharp maps is again sharp and hence it suffices to show that the projections $U_i \times F \to U_i$ are locally sharp. By (P2), the morphism $F \to *$ is locally sharp and hence the projections are locally sharp as sharp morphisms are stable under basechange by Remark 1.2.10. ☐

Remark 3.2.2. Of course, the previous Lemma 3.2.1 requires the bundle $F \to E \to X$ only to be Nisnevich-locally trivial. It is observed in [Wen07, Proposition 3.3.7], that even an étale-fiber bundle of objects from $\mathcal{S}\mathrm{m}_S$ induces a local fiber sequence.

Remark 3.2.3. We are interested in the fact whether certain Zariski-fiber bundles $F \to E \to X$ induce motivic fiber sequences. By Corollary 1.2.15, this is for example the case if X is an $\mathbb{A}^1$-local scheme. As $X \to \mathcal{S}\mathrm{ing}(X)$ is a motivic weak equivalence and the singular functor of Remark 1.1.21 commutes with pullbacks, one may as well ask if $\mathcal{S}\mathrm{ing}(F) \to \mathcal{S}\mathrm{ing}(E) \to \mathcal{S}\mathrm{ing}(X)$ is a motivic fiber sequence because this pre-fiber sequence belongs to the same equivalence class as $F \to E \to X$.

Definition 3.2.4. For an integer $n \geq 1$, one sets

$$GL_n = \operatorname{Spec} k[x_{11}, \ldots, x_{ij}, \ldots, x_{nn}, t]/(det(x_{ij})t - 1)$$

as the *general linear group scheme* over S. This is a group object of $\mathcal{S}\mathrm{m}_S$ by means of the canonical Hopf algebra structure given by the comultiplication Δ defined by

$x_{ij} \mapsto \sum_{k=1}^{n} y_{ik} \otimes z_{kj}$, the counit $x_{ij} \mapsto \delta_{ij}$, and the antipode $(x_{ij}) \mapsto \frac{Adj(x_{ij})}{det(x_{ij})}$. It has the *group of monomial matrices*

$$NT_n = \coprod_{\sigma \in \Sigma_n} \operatorname{Spec} k[x_{11}, \ldots, x_{ij}, \ldots, x_{nn}, t] \Bigg/ \begin{pmatrix} x_{ij}, \text{ if } j \neq \sigma(i) \\ det(x_{ij})t - 1 \end{pmatrix}$$

as a closed subgroup.

Theorem 3.2.5 (Morel, Moser, Wendt)**.** Let $n \geq 1$, and let F, E and X be pointed simplicial presheaves from $\mathcal{S}\mathrm{m}_S$. Let moreover $GL_n \to E \to X$ be a principal Zariski-GL_n-bundle and let F be equipped with a left GL_n-action. Then, the Zariski-fiber bundle

$$F \to E \times^{GL_n} F \xrightarrow{p \times^{GL_n} F} X$$

of Lemma 3.1.16 defines a motivic fiber sequence. In particular, the principal Zariski-GL_n-bundle $GL_n \to E \to X$ defines a motivic fiber sequence.

Proof. The proof of a more general statement can be found in [Wen11] but we will briefly recall the structure of the argument.

By Lemma 3.1.18, the simplicial presheaf $E \times^{GL_n} F$ comes from $\mathcal{S}\mathrm{m}_S$. First, one shows that there is a local fiber sequence (confer Remark 3.2.3)

$$\mathcal{S}\mathrm{ing}(F) \to \mathcal{S}\mathrm{ing}(E \times^{GL_n} F) \xrightarrow{q} \mathcal{S}\mathrm{ing}(X)$$

and q is locally sharp with fiber $\mathcal{S}\mathrm{ing}(F)$: A filtered colimit of fibrations of simplicial sets is again a fibration [TV07, Proposition 2.2.1] and therefore, it suffices by [Rez98, Theorem 5.1.(P1)] to prove, that for a smooth local ring A the morphism

$$\mathcal{S}\mathrm{ing}(E \times^{GL_n} F)(A) \to \mathcal{S}\mathrm{ing}(X)(A)$$

is a fibration of simplicial sets with fiber $\mathcal{S}\mathrm{ing}(F)(A)$. This is done as in the proof of [Wen11, Proposition 4.7] or [Wen07, Proposition 5.4.4] following [GJ99, Corollary V.2.7]. In the same statements, it is observed that this local fiber sequence is a motivic fiber sequence if the local fiber sequence $\mathcal{S}\mathrm{ing}(GL_n) \to \mathcal{S}\mathrm{ing}(E) \to \mathcal{S}\mathrm{ing}(X)$ is a motivic fiber sequence. The stability of $\mathbb{A}^1$-sharp morphisms under pullbacks as described in Remark 1.2.10 implies that it suffices to prove that the local fiber sequence $\mathcal{S}\mathrm{ing}(GL_n) \to E\mathcal{S}\mathrm{ing}(GL_n) \to B\mathcal{S}\mathrm{ing}(GL_n)$ is a motivic fiber sequence. Now we can apply the Theorem [Wen11, Theorem 5.3] of Fabien Morel and Lukas-Fabian Moser, asserting that $B\mathcal{S}\mathrm{ing}(GL_n) \sim_s \mathcal{S}\mathrm{ing}(Gr_{(n,\infty)})$ is $\mathbb{A}^1$-local. □

3.3. Definition of a motivic J-homomorphism

A object X of $\mathcal{S}\mathrm{m}_S$ is assumed to be connected (as a scheme) in the following unless no other indication is made.

Remark 3.3.1. We follow [GW10, Chapter 11] for the definition of a vector bundle: Let X be an object of $\mathcal{S}\mathrm{m}_S$ and $n \geq 0$ an integer. A *vector bundle of rank* n on X is a morphism $p : E \to X$ in $\mathcal{S}\mathrm{m}_S$ which is Zariski-locally trivial, id est there exists a Zariski covering $\{U_i \to X \mid i \in I\}$ by affine schemes $U_i \cong \operatorname{Spec} A_i$ such that for all $i \in I$ the diagram

$$\begin{array}{ccc} p^{-1}(U_i) & \xrightarrow[\cong]{\phi_i} & U_i \times \mathbb{A}^n \\ & {\scriptstyle p} \searrow \qquad \swarrow {\scriptstyle pr} & \\ & U_i & \end{array}$$

commutes such that for all pairs (i, j) of $I \times I$, the isomorphism

$$\operatorname{Spec} A_i \otimes A_j[x_1, \ldots, x_n] = U_i \cap U_j \times \mathbb{A}^n \xrightarrow{\phi_i \phi_j^{-1}} U_i \cap U_j \times \mathbb{A}^n = \operatorname{Spec} A_i \otimes A_j[x_1, \ldots, x_n]$$

is induced by a linear $A_i \otimes A_j$-algebra isomorphism, id est an $A_i \otimes A_j$-algebra automorphism φ with $\varphi(x_k) = \sum_l a_{kl} x_l$ for a_{kl} in $A_i \otimes A_j$.

A vector bundle $p : E \to X$ is an object of $\mathcal{S}\mathrm{m}_X$ [Gro61, Corollaire 1.3.7] and often only written as E by abuse of notation.

The category $\mathcal{V}b(X)$ of *vector bundles on* X has as objects the vector bundles on X and the morphisms $E \to E'$ of $\mathcal{S}\mathrm{m}_S$ which are linear restricted to some (simultaneously trivializing) Zariski covering of X as morphisms. The symbol $\mathcal{V}b(X)_n$ denotes the category of vector bundles of rank n on X.

The *Whitney sum* $E \oplus E'$ of a vector bundle E of rank n and a vector bundle E' of rank m on X, is defined by the categorical product of $\mathcal{S}\mathrm{m}_X$, id est there is a pullback diagram

$$\begin{array}{ccc} E \oplus E' & \longrightarrow & E \\ \downarrow & \searrow^{p \oplus p'} & \downarrow p \\ E' & \xrightarrow{p'} & X \end{array}$$

in Sm_S and $E \oplus E'$ is a vector bundle of rank $n + m$ on X.

There is an equivalence of categories [GW10, Proposition 11.7]

$$\begin{array}{rcl} \mathcal{V}b(X)^{\mathrm{op}} & \rightleftarrows & \{\text{coherent and locally free } \mathcal{O}_X\text{-modules}\} \\ E & \mapsto & \hom_{\mathcal{O}_X\text{-Mod}}(\Gamma(-, E), \mathcal{O}_X) = \Gamma(-, E)^\vee \\ \mathbb{V}(M) = \operatorname{Spec}_X(\operatorname{Sym}_{\mathcal{O}_X})(M) & \leftarrow\!\shortmid & M \end{array}$$

respecting the rank on connected components and we define a sequence

$$A \hookrightarrow E \to B$$

of two morphisms of vector bundles over X to be *exact* if the corresponding sequence $\Gamma(-, B)^\vee \to \Gamma(-, E)^\vee \to \Gamma(-, A)^\vee$ is an exact sequence of $\mathcal{O}_X$-modules. In particular, it is implied that the morphism $A \hookrightarrow E$ is a closed immersion. As the category of $\mathcal{O}_X$-modules is abelian and has the category of coherent and locally free $\mathcal{O}_X$-modules as an additive subcategory, the category $\mathcal{V}b(X)$ can be equipped with the structure of an exact category through the above-mentioned (anti-)equivalence. For a non-affine X, the category $\mathcal{V}b(X)$ may not be split exact [Wei12, Example I.5.4].

Remark 3.3.2. There is an one-to-one correspondence [GW10, Section 11.6]

$$\begin{array}{rcl} \mathcal{V}b(X)_n & \rightleftarrows & \{\text{principal Zariski-}GL_n\text{-bundles}\} \\ E \xrightarrow{p} X & \mapsto & \hat{E} = \underline{\mathrm{Iso}}_{\mathcal{V}b(X)}(\mathbb{A}^n_X, E) \to X \\ \hat{E} \times^{GL_n} \mathbb{A}^n \xrightarrow{\hat{p} \times^{GL_n} \mathbb{A}^n} X & \leftarrow\!\shortmid & GL_n \to \hat{E} \xrightarrow{\hat{p}} X \end{array}$$

where the map from the right to the left-hand side is the associated bundle construction of Lemma 3.1.16 with the canonical left action $GL_n \times \mathbb{A}^n \to \mathbb{A}^n$. The sheaf $\underline{\mathrm{Iso}}_{\mathcal{V}b(X)}(\mathbb{A}^n_X, E) \to X$ comes with a canonical morphism to X and defines a principal Zariski-GL_n-bundle (confer Remark 3.1.11) where on a trivializing U_i, the right action of $GL_n(U_i)$ on $\hat{E}(U_i) \cong \mathrm{Iso}_{\mathcal{V}b(U_i)}(\mathbb{A}^n_{U_i}, \mathbb{A}^n_{U_i})$ is given canonically.

Hence, the set of isomorphism classes of vector bundles of rank n on X from $\mathcal{S}\mathrm{m}_S$ is isomorphic to $H^1_{\mathcal{Z}\mathrm{ar}}(X, GL_n)$. By a variant of Hilbert's theorem 90 [MV99, Lemma 4.3.6], one has $H^1_{\mathcal{Z}\mathrm{ar}}(X, GL_n) \cong H^1_{\mathcal{N}\mathrm{is}}(X, GL_n)$ and moreover [MV99, Proposition

4.1.15] yields $[X, BGL_n]_s \cong H^1_{\mathcal{N}\mathrm{is}}(X, GL_n)$ given by pulling back an universal principal GL_n-bundle $EGL_n \to BGL_n$.

As $BGL_1 \cong \mathbb{P}^\infty$ is $\mathbb{A}^1$-local, the isomorphism $H^1_{\mathcal{Z}\mathrm{ar}}(X, GL_n) \cong [X, BGL_n]_s$ promotes to an isomorphism $H^1_{\mathcal{Z}\mathrm{ar}}(X, GL_1) \cong [X, BGL_1]_{\mathbb{A}^1}$. Unfortunately, a counterexample [AD08, Section 2] shows that this is not true for $n \geq 2$.

Definition 3.3.3 ([Wei12, Definition 7.1])**.** Let X be an object of $\mathcal{S}\mathrm{m}_S$ and define the *(zeroth)(algebraic) K-Theory* of X as the abelian group

$$K^0(X) = \mathbb{Z}[\mathcal{V}b(X)]/\langle [E] = [A] + [B] \text{ for an exact sequence } A \hookrightarrow E \twoheadrightarrow B \rangle.$$

One has $0 = [0]$, $[A \oplus B] = [A] + [B]$ and $[A] = [B]$ if $A \cong B$. There is an universal morphism

$$[-] : \mathcal{V}b(X) \to K^0(X)$$

which means, that for every morphism $\varphi : \mathcal{V}b(X) \to G$ into an abelian group G, with the property that $\varphi(E) = \varphi(A) + \varphi(B)$ for an exact sequence $A \hookrightarrow E \twoheadrightarrow B$, there is an unique factorization

$$\begin{array}{ccc} \mathcal{V}b(X) & \xrightarrow{\varphi} & G \\ {\scriptstyle [-]}\downarrow & \nearrow_{\varphi'} & \\ K^0(X) & & \end{array}$$

where $\varphi'(\sum a_i[E_i]) \mapsto \sum a_i \varphi(E_i)$. The morphism $[-]$ is not injective in general.

If X is connected as a scheme, there is a surjection $rank : K^0(X) \to \mathbb{Z}$ induced by the rank function and a splitting $\mathbb{Z} \to K^0(X)$ defined by $m - n \mapsto [\mathbb{A}^m_X] - [\mathbb{A}^n_X]$. The abelian group given as the kernel of the rank function is called the *reduced (zeroth)(algebraic) K-Theory* of X

$$\tilde{K}^0(X) = K^0(X)/\langle [\mathbb{A}^n_X] \rangle.$$

For every morphism $\varphi : \mathcal{V}b(X) \to G$ into an abelian group G with the property that $\varphi(E) = \varphi(A) + \varphi(B)$ for an exact sequence $A \hookrightarrow E \twoheadrightarrow B$ and $\varphi(\mathbb{A}^n_X) = 0$ for all $n \geq 0$, there is an unique factorization through $\mathcal{V}b(X) \to K^0(X) \to \tilde{K}^0(X)$.

Remark 3.3.4. In view of the end of Remark 3.3.2, it is possibly surprising that one gets isomorphisms

$$K^0(X) \cong \langle X_+, BGL_\infty \times \mathbb{Z} \rangle_{\mathbb{A}^1}$$

and

$$\tilde{K}^0(X) \cong \langle X_+, BGL_\infty \rangle_{\mathbb{A}^1}.$$

The author does not know if the object BGL_∞ is $\mathbb{A}^1$-local (confer [MV99, Proposition 4.3.10]).

Definition 3.3.5. Let X be an object of $\mathrm{sPre}(S)_*$ and define the commutative monoid

$$\mathcal{S}\mathrm{ph}^+(X) = \left(\left\{ \begin{array}{c} \text{(isomorphism classes of)} \\ \text{motivic fiber sequences} \\ S^{a,b} \to E \to X \\ \text{where } a \geq b \geq 0 \end{array} \right\},\ *^h_X,\ S^{0,0} \to S^{0,0}_X \to X \right)$$

with the homotopy join over X as defined in Definition 1.3.13. This operation is well defined and closed by [Doe98, Proposition 4.2] using Theorem 1.3.7. By [Wen10, Proposition 4.5], the class involved is in fact a set.

Definition 3.3.6. Let $p : E \to X$ be a vector bundle over a scheme X of $\mathcal{S}\mathrm{m}_S$ and $z : X \hookrightarrow E$ its zero section. The object $S(E) = E \setminus z(X) \cong \hat{E} \times^{GL_n} (\mathbb{A}^n \setminus \{0\})$ of $\mathcal{S}\mathrm{m}_S$ together with its structure morphism to X is called the *spherical bundle associated to the vector bundle E.*

The pushout diagram

$$\begin{array}{ccc} S(E) & \longrightarrow & E \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathrm{Th}(E) \end{array}$$

of simplicial presheaves defines a pointed simplicial presheaf $\mathrm{Th}_S(E)$ (or $\mathrm{Th}(E)$), called the *Thom space* of E. The suspension spectrum $\Sigma_T^\infty \mathrm{Th}(E)$ of $\mathcal{S}p_T(S)$ is called the *Thom spectrum* and is abbreviated by $\mathbb{T}h_S(E)$ (or $\mathbb{T}h(E)$).

Remark 3.3.7. The diagram in the last Definition 3.3.6 is a motivic homotopy pushout diagram as $S(E) \to E$ is a monomorphism in $\mathcal{S}\mathrm{m}_S$.

As the pullback of a vector bundle $E \to X$ along a morphism $Y \to X$ in $\mathcal{S}\mathrm{m}_S$ is a vector bundle $D \to Y$, the pullback of the spherical bundle $S(E) \to X$ is the spherical bundle $S(D) \to Y$. From the universal property of a pullback, one gets moreover, that if $S(E) \to X$ has a section, then $S(D) \to Y$ has a section as well.

Choose the base scheme S to be X and suppose that $S(E) \to X$ admits a section $X \to S(E)$ through which $S(E)$ can be considered as an object of $\mathrm{sPre}(X)_*$. Then, one has $\mathrm{Th}_X(E) \sim S_X^{1,0} \wedge_X S(E)$ in $\mathrm{sPre}(X)_*$ by Lemma A.4.10. In particular, it holds $\mathrm{Th}_X(\mathbb{A}_X^n) \sim S_X^{2n,n}$ in $\mathrm{sPre}(X)_*$, even for $n = 0$. If $S(E)' = (X \to S(E))$ is another section of $S(E) \to X$ and therefore a different object of $\mathrm{sPre}(X)_*$, one has $S_X^{1,0} \wedge_X S(E) \sim S_X^{1,0} \wedge_X S(E)'$ in $\mathrm{sPre}(X)_*$.

Definition 3.3.8. The abelian group

$$\widetilde{\mathcal{S}\mathrm{ph}}(X) = \mathbb{Z}[\mathcal{S}\mathrm{ph}^+(X)] \Big/ \left\langle \begin{array}{c} [\mathbb{A}^n \setminus \{0\} \to S(\mathbb{A}_X^n) \to X],\ n \geq 1, \\ [A *_X^h B] - [A] - [B] \end{array} \right\rangle$$

will be called the *group of motivic spherical fiber sequences* on X.

Remark 3.3.9. Let $t : X \to S$ be an object of $\mathcal{S}\mathrm{m}_S$. As t is smooth, there is an adjunction

$$t_\sharp : \mathrm{sPre}(X) \to \mathrm{sPre}(S) : t^*$$

where the right adjoint t^* is given by $t^*(F) = yX \times_{yS} F$ and where the left adjoint $t_\sharp$ is a left Kan extension of the functor $c : \mathcal{S}\mathrm{m}_X \times \Delta \to \mathrm{sPre}(S)$ sending $(E \to X, [n])$ to $y(E \to X \xrightarrow{t} S) \times \Delta^n$ along the Yoneda embedding $y : \mathcal{S}\mathrm{m}_X \times \Delta \to \mathrm{sPre}(X)$. This adjunction is not a Quillen adjunction with respect to the $\mathbb{A}^1$-local injective model structure [MV99, Example 3.1.22] but the situation is better using the motivic model as it was observed in [Rön05, Proposition 2.19]: In the objectwise projective setting, the cofibrations are identified as $U \times \partial\Delta^n \to U \times \Delta^n$ by Remark 1.1.4 and thus are preserved by $t_\sharp$. The same holds for the acyclic cofibrations. Hence, the adjunction is a Quillen adjunction for the objectwise projective structure [Hir03, Proposition 8.5.3]. The functor t^* preserves colimits. To show the Quillen adjointness for the motivic structure, it suffices by [Dug01, Corollary A.2] and Lemma A.3.10 to check that t^* preserves motivically fibrant objects and those can be identified using Remark A.3.9 (confer also [Ayo07b, Théorème 4.5.10]).

The Lemma A.4.3 provides a pointed analogue of the Quillen adjunction $(t_\sharp, t^*)$ which will be denoted by $(t_\sharp^\circ, t_\circ^*)$. Following [Hu01, Proposition 2.8] or [Rön05, Proposition 2.7], there is an isomorphism $t_\sharp^\circ(E \wedge_X t_\circ^* D) \cong t_\sharp^\circ E \wedge D$ in sPre_* natural in E of $\mathrm{sPre}(X)_*$ and D of sPre_*, called the *projection formula.* This implies $t_\sharp^\circ t_\circ^*(D) \cong X_* \wedge D$. Moreover, monoidality yields a natural isomorphism $t_\circ^*(S^{a,0} \wedge D) \cong S_X^{a,0} \wedge_X t_\circ^* D$ in $\mathrm{sPre}(X)_*$.

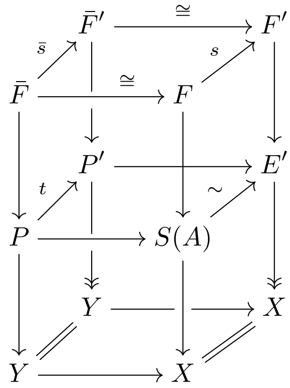
If X is pointed by $b : S \to X$, there is a natural transformation $b^*(E) \to t_\sharp(E)$ of functors $\mathrm{sPre}(X) \to \mathrm{sPre}(S)$ defined on representants by sending $S \times_X E \to S$ to $E \to S$ by the projection and being the identity on Δ^n. Alternatively, this could be defined as the composition $b^* \xrightarrow{\mathrm{id} \circ \eta} b^* \circ (t^* \circ t_\sharp) \cong (b^* \circ t^*) \circ t_\sharp \cong t_\sharp$, since $t \circ b = \mathrm{id}_S$ where η is the unit of the adjunction $(t_\sharp, t^*)$. Using the naturality for the basepoint $X \to E$ and then taking the categorical cofibers yields a natural transformation $b_\circ^*(E) \to t_\sharp^\circ(E)$ as well.

Lemma 3.3.10. Let X and Y be objects in sPre_* from $\mathcal{S}\mathrm{m}_S$ and let $A \to X$ be a vector bundle over X such that $S(A)$ admits a (fixed) X-section and may hence be considered as an object of $\mathrm{sPre}(X)_*$. Assume that Y is $\mathbb{A}^1$-connected. Then, the (categorical) pullback diagram

$$\begin{array}{ccc} P & \longrightarrow & S(A) \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \longrightarrow & X \end{array}$$

in $\mathrm{sPre}(S)_*$ is an $\mathbb{A}^1$-homotopy pullback diagram.

Proof. By Remark 3.3.7, $P \cong S(D) \to Y$ is the spherical bundle associated to a vector bundle $D \to Y$. Hence, the fiber sequences $\mathrm{hofib}^s(p) \to S(A) \to X$ and $\mathrm{hofib}^s(q) \to S(D) \to X$ are both motivic fiber sequences by Theorem 3.2.5. The diagram in question is a local homotopy pullback as p is locally sharp by Lemma 3.2.1. Consider the diagram



obtained by first factorizing p as a motivic weak equivalence $S(A) \to E'$ followed by a motivic fibration $E' \to X$. The space P' is the (categorical) pullback of $E' \to X$ along $Y \to X$. The differently decorated F are the (categorical) fibers of the morphisms below them respectively. It is to show that t is a motivic weak equivalence. As $F \to S(A) \to X$ is a motivic fiber sequence, s is a motivic weak equivalence and hence $\bar{s}$ is a motivic weak equivalence. The result now follows by a long exact sequence argument in the same way as in the proof of Proposition 1.2.24. □

Lemma 3.3.11. Let X be an object in sPre_* from $\mathcal{S}\mathrm{m}_S$ and let $A \to X$ and $B \to X$ be two vector bundles over X such that $S(A)$ and $S(B)$ admit a (fixed) X-section and may hence be considered as objects of $\mathrm{sPre}(X)_*$. Then, one has

$$S(A \oplus B) \sim S(A) \wedge_X S(B) \wedge_X S_X^{1,0}$$

in $\mathrm{sPre}(X)_*$. If $S(A)$ and $S(B)$ are considered as objects of sPre_* via the base-point $S \to X$, one has moreover

$$\begin{array}{rcl} S(A \oplus B) & \sim & S(A) \wedge_X^h S(B) \wedge_X^h S_X^{1,0} \\ & \sim & S(A) *_X^h S(B) \end{array}$$

in sPre_* over X, if $S(B)$ is $\mathbb{A}^1$-connected or if $S(B) \to X$ is a trivial spherical bundle $S(\mathbb{A}_X^n) \to X$ with $n \geq 1$.

Proof. The diagram

$$\begin{array}{ccc} S(A) \vee_X S(B) & \longrightarrow & S(A) \vee_X B \\ \downarrow & & \downarrow \\ A \vee_X S(B) & \longrightarrow & A \vee_X B \end{array}$$

is a homotopy pushout diagram in $\mathrm{sPre}(X)_*$ as the functor $- \vee_X -$ preserves pushout diagrams in both variables and in a topos, the pushout of a monomorphism is a monomorphism (confer Remark 3.1.6). The diagram

$$\begin{array}{ccc} S(A) \times_X S(B) & \longrightarrow & S(A) \times_X B \\ \downarrow & & \downarrow \\ A \times_X S(B) & \longrightarrow & S(A \oplus B) \end{array}$$

is a Zariski distinguished square in $\mathcal{S}\mathrm{m}_X$ as the property of a diagram to be a pullback in $\mathcal{S}\mathrm{m}_X$ is checked locally and there is a pullback diagram

$$\begin{array}{ccc} \mathbb{A}^a \setminus \{0\} \times \mathbb{A}^b \setminus \{0\} & \longrightarrow & \mathbb{A}^a \setminus \{0\} \times \mathbb{A}^b \\ \downarrow & & \downarrow \\ \mathbb{A}^a \times \mathbb{A}^b \setminus \{0\} & \longrightarrow & (\mathbb{A}^a \times \mathbb{A}^b) \setminus \{0\}. \end{array}$$

Since a Zariski distinguished square defines a homotopy pushout for the local models by Remark 1.1.13 and hence for the $\mathbb{A}^1$-local model structures, the categorical pushout diagram

$$\begin{array}{ccc} S(A) \wedge_X S(B) & \longrightarrow & S(A) \wedge_X B \\ \downarrow & & \downarrow \\ A \wedge_X S(B) & \longrightarrow & S(A \oplus B) \big/ A \vee_X B \end{array}$$

obtained by taking the cofibers of the morphism from the wedges to the products respectively, is a motivic homotopy pushout diagram. As $A \vee_X B \sim X$ in $\mathrm{sPre}(X)_*$ [Doe98, Lemma 1.4] and since the lower left corner and the upper right corner of this homotopy pushout diagram are both weakly equivalent to X, one gets the relation $S(A \oplus B) \sim S(A) \wedge_X S(B) \wedge_X S_X^{1,0}$ in $\mathrm{sPre}(X)_*$.

For the second assertion, one first notes that $S(A) *_X^h S(B)$ is weakly equivalent to $S(A) \wedge_X^h S(B) \wedge_X^h S_X^{1,0}$ by Lemma 1.3.15 and the simplicial structure of sPre/X_*.

Hence, one has to show that $S(A \oplus B) \sim S(A) \wedge^h_X S(B) \wedge^h_X S^{1,0}_X$ in sPre/X_*. The left Quillen functor $t_\sharp : \mathrm{sPre}(X) \to \mathrm{sPre}(S)$ of Remark 3.3.9 preserves homotopy colimits and hence $t_\sharp(S(A) \vee_X S(B) \vee_X S^{1,0}_X) \sim S(A) \vee^h_X S(B) \vee^h_X S^{1,0}_X$. One would be done by the same reason, if $t_\sharp(S(A) \times_X S(B) \times_X S^{1,0}_X) \sim S(A) \times^h_X S(B) \times^h_X S^{1,0}_X$ holds as the canonical morphism $Y \vee_X Z \to Y \times_X Z$ is a monomorphism in $\mathrm{sPre}(S)$. A pullback diagram

$$\begin{array}{ccc} Y \times_X Z & \longrightarrow & Y \\ \big\downarrow & & \big\downarrow \\ Z & \longrightarrow & X \end{array}$$

in $\mathcal{S}\mathrm{m}_X$ is also a pullback diagram in $\mathcal{S}\mathrm{m}_S$ and hence there is an isomorphism $t_\sharp(S(A) \times_X S(B) \times_X S^{1,0}_X) \cong S(A) \times_X S(B) \times_X S^{1,0}_X$ in sPre_*. Therefore, it suffices to show that the categorical pullback

$$\begin{array}{ccc} S(A) \times_X S(B) & \longrightarrow & S(A) \\ \big\downarrow & & \big\downarrow \\ S(B) & \longrightarrow & X \end{array}$$

is an $\mathbb{A}^1$-homotopy pullback. If $S(B)$ is $\mathbb{A}^1$-connected, this is implied by Lemma 3.3.10. If $n \geq 1$ is an integer and $S(B) \to X$ is a trivial spherical bundle $S(\mathbb{A}^n_X) \to X$, the result is obtained by Corollary 1.2.14. □

Lemma 3.3.12. Let S be the spectrum of a perfect field. Let moreover X be an $\mathbb{A}^1$-connected object in sPre_* from $\mathcal{S}\mathrm{m}_S$ and $E \to X$ a vector bundle. Then, the object $S(E \oplus \mathbb{A}^2_X)$ is $\mathbb{A}^1$-connected.

Proof. By Lemma 3.3.11, one has $S(E \oplus \mathbb{A}^2_X) \sim S(E \oplus \mathbb{A}^1_X) \wedge_X S^{1,1}_X \wedge_X S^{1,0}_X$ in $\mathrm{sPre}(X)_*$. As smashing with $S^{1,0}_X$ is a model for the simplicial suspension in $\mathrm{sPre}(X)_*$ by Lemma A.4.10, the solid diagram

$$\begin{array}{ccccc} S(E \oplus \mathbb{A}^1_X) \wedge_X S^{1,1}_X & \longrightarrow & X & \overset{\sim}{\dashrightarrow} & R^{\mathbb{A}^1}X \\ \big\downarrow & & \big\downarrow & & \\ X & \longrightarrow & S(E \oplus \mathbb{A}^2_X) & & \\ \sim \big\downarrow \vdots & & & & \\ R^{\mathbb{A}^1}X & & & & \end{array}$$

is a motivic homotopy pushout in $\mathrm{sPre}(S)_*$ since the left Quillen functor $t_\sharp$ of Remark 3.3.9 preserves homotopy pushouts. Let $S(E \oplus \mathbb{A}^1_X) \wedge_X S^{1,1}_X \to X' \to R^{\mathbb{A}^1}X$ be a factorization of the upper horizontal composition as a motivic cofibration followed by a motivic acyclic fibration. The motivically fibrant objects $R^{\mathbb{A}^1}X$ and X' are both locally connected and for this reason, so is the pushout P of the diagram $R^{\mathbb{A}^1}X \leftarrow S(E \oplus \mathbb{A}^1_X) \wedge_X S^{1,1} \to X'$. Therefore, it is $\mathbb{A}^1$-connected by the Unstable $\mathbb{A}^1$-connectivity theorem 1.2.20. As P and $S(E \oplus \mathbb{A}^2_X)$ are motivically weakly equivalent, the statement follows. □

Corollary 3.3.13. Let S be the spectrum of a perfect field, X an $\mathbb{A}^1$-connected object in sPre from $\mathcal{S}\mathrm{m}_S$ and let $E \to X$ be a vector bundle. Two motivic fiber sequences $S^{a,b} \to S(E) \to X$ which differ only by the choice of a basepoint $S \to S(E)$ are identified in $\widetilde{\mathcal{S}\mathrm{ph}}(X)$.

Lemma 3.3.14. Let X be an object in sPre_* from $\mathcal{S}\mathrm{m}_S$ and let $A \hookrightarrow E \to B$ be an exact sequence of vector bundles over X such that $S(A)$, $S(E)$ and $S(E)$ admit a (fixed) X-section and may hence be considered as objects of sPre_* via the basepoint $S \to X$. Suppose that $S(B)$ is $\mathbb{A}^1$-connected. Then, there exists an $n \geq 1$ such that

$$S(A) *^h_X S(B) *^h_X S(\mathbb{A}^n_X) \sim S(E) *^h_X S(\mathbb{A}^n_X)$$

in sPre_* over X.

Proof. One should first note that we may omit the bracketing on the left-hand side as the homotopy join is associative by [Doe98, Theorem 4.8], which uses Mather's cube theorem 1.3.9.

In the proof of [Rön10, Proposition 2.1], Oliver Röndigs constructs with the help of [Ayo07a, Théorème 1.5.18] an isomorphism

$$\Sigma^\infty_T(\mathrm{Th}_X(A) \wedge_X \mathrm{Th}_X(B)) \cong \mathbb{T}h_X(E)$$

in the stable homotopy category $\mathcal{SH}(X)$ for an exact sequence $A \hookrightarrow E \to B$ of vector bundles over X. The Stability lemma 2.1.4 asserts that there is an integer $n \geq 0$ and an isomorphism

$$\mathrm{Th}_X(A) \wedge_X \mathrm{Th}_X(B) \wedge_X S^{2n,n}_X \cong \mathrm{Th}_X(E) \wedge_X S^{2n,n}_X$$

in $\mathcal{H}(X)_*$ as the object $\mathrm{Th}_X(A) \wedge_X \mathrm{Th}_X(B)$ and the object $\mathrm{Th}_X(E)$ of $\mathrm{sPre}(X)_*$ are both finitely presentable [Jar00a, Lemma 2.2]. Smashing with the sphere $S^{1,1}_X$ and using Remark 3.3.7 yields an isomorphism

$$(S^{1,0}_X \wedge_X S(A)) \wedge_X (S^{1,0}_X \wedge_X S(B)) \wedge_X S^{2(n+1)-1,n+1} \cong (S^{1,0}_X \wedge_X S(E)) \wedge_X S^{2(n+1)-1,n+1}$$

in $\mathcal{H}(X)_*$ and therefore

$$S(A \oplus B \oplus \mathbb{A}^{n+1}_X) \cong S(E \oplus \mathbb{A}^{n+1}_X)$$

by Lemma 3.3.11 which is also an isomorphism in $\mathcal{H}(S)_*$ using the left Quillen functor $t_\sharp$ of Remark 3.3.9. The claimed result now follows from Lemma 3.3.11 as $S(B)$ is $\mathbb{A}^1$-connected. □

Definition 3.3.15 (A motivic J-homomorphism)**.** Let S be the spectrum of a perfect field and X an $\mathbb{A}^1$-connected object in sPre_* from $\mathcal{S}\mathrm{m}_S$. The morphism

$$\begin{array}{rcl} J : \tilde{K}^0(X) & \to & \widetilde{\mathcal{S}\mathrm{ph}}(X) \\ {[E]} & \mapsto & [\mathbb{A}^{n+2} \setminus \{0\} \to S(E \oplus \mathbb{A}^2_X) \to X] \end{array}$$

of abelian groups is well defined by the Lemma 3.3.16 below and it is called the *motivic J-homomorphism.*

Lemma 3.3.16. The motivic J-homomorphism of Definition 3.3.15 is well defined. If $E \to X$ is a vector bundle with a nowhere vanishing section $S \to X$, then

$$J([E]) = [\mathbb{A}^n \setminus \{0\} \to S(E) \to X]$$

where $S(E)$ is pointed by the composition of the basepoint $S \to X$ with this section.

Proof. The map

$$\varphi : \mathcal{V}b(X) \to \widetilde{\mathcal{S}\mathrm{ph}}(X)$$

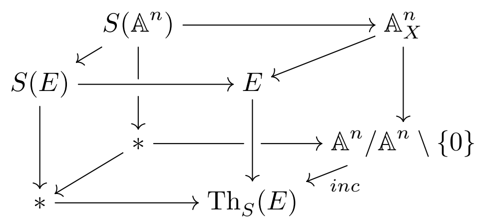
defined by $\varphi(E) = S(E \oplus \mathbb{A}^2_X)$ factorizes through $\tilde{K}(X)$ if $\varphi(E) = \varphi(A) *^h_X \varphi(B)$ for every exact sequence $A \hookrightarrow E \to B$ and $\varphi(\mathbb{A}^n_X) = 0$ for $n \geq 0$. The map φ sends trivial vector bundles $\mathbb{A}^n_X$ to zero. Let $A \hookrightarrow E \to B$ be an exact sequence of vector bundles over X. Then, also $A \oplus \mathbb{A}^2 \hookrightarrow E \oplus \mathbb{A}^4 \to B \oplus \mathbb{A}^2$ is an exact sequence of vector bundles over X. The object $S(B \oplus \mathbb{A}^2)$ of sPre_* is $\mathbb{A}^1$-connected by Lemma 3.3.12. Therefore, Lemma 3.3.14 implies that $\varphi(E) = \varphi(A) *^h_X \varphi(B)$ and the motivic J-homomorphism is well defined. The second claim of the lemma follows from Corollary 3.3.13. □

Remark 3.3.17. There are other ways to define a motivic J-homomorphism which we do not want to consider in this text. For example, an element of GL_n may be considered as an isomorphism $\mathbb{A}^n \to \mathbb{A}^n$ mapping $\mathbb{A}^n \setminus \{0\}$ to itself. Therefore, one gets a morphism $GL_n \to \underline{\mathrm{hom}}(\mathbb{A}^n/\mathbb{A}^n \setminus \{0\}, \mathbb{A}^n/\mathbb{A}^n \setminus \{0\})$ and by composing with a fibrant replacement functor in the last entry a morphism $\psi_n : GL_n \to \Omega^{2n,n} S^{2n,n}$. After stabilizing, the domain of this morphism may be related to K-Theory. As in topology, one could hope to get information about motivic homotopy groups of the spheres by investigating the motivic J-homomorphism.

4. **Vanishing results for the J-homomorphism**

4.1. A relation with Thom classes

Definition 4.1.1. Let X be an object of sPre_* from $\mathcal{S}\mathrm{m}_S$ and $E \to X$ a vector bundle over X. A *fiber inclusion* into the Thom space of E is given as a morphism $inc : \mathbb{A}^n/\mathbb{A}^n \setminus \{0\} \to E/S(E) = \mathrm{Th}_S(E)$ in sPre_* defined by the diagram



of simplicial presheaves.

A *Thom class* for $E \to X$ is a morphism $u : \mathbb{T}h_S(E) \to \Sigma^\infty_T(\mathbb{A}^n/\mathbb{A}^n \setminus \{0\})$ in the stable homotopy category $\mathcal{SH}(S)$ such that the composition

$$\Sigma^\infty_T(\mathbb{A}^n/\mathbb{A}^n \setminus \{0\}) \xrightarrow{\Sigma^\infty_T(inc)} \mathbb{T}h_S(E) \xrightarrow{u} \Sigma^\infty_T(\mathbb{A}^n/\mathbb{A}^n \setminus \{0\})$$

is an isomorphism in $\mathcal{SH}(S)$ for some fiber inclusion *inc*.

Remark 4.1.2. Let $F \xrightarrow{i} E \xrightarrow{p} X$ be a smooth fiber bundle in sPre, where X, E and F are objects from $\mathcal{S}\mathrm{m}_S$. Consider E_* as an object of $\mathrm{sPre}(X)_*$. The morphism $b^*_\circ(E_*) \to t^\circ_\sharp(E_*)$ in sPre_* discussed in Remark 3.3.9, is $i_* : F_* \to E_*$, where in the latter expression F_* and E_* are considered as objects of sPre_*. The morphism $b^*(E) \to t_\sharp(E)$ in sPre is the fiber inclusion $i : F \to E$. It is a little confusing, that for a pointed X, one may view $t_\sharp$ as a functor $\mathrm{sPre}(X)_* \to \mathrm{sPre}_*$ as well but $t_\sharp$ is different from $t^\circ_\sharp$. There is a natural factorization $b^*_\circ(E) \to t_\sharp(E) \to t^\circ_\sharp(E)$ in this case as shown on representants.

Lemma 4.1.3. Let X be an object of sPre_* from $\mathcal{S}\mathrm{m}_S$, A an object from sPre_* and let $E \to X$ be a morphism of sPre_* such that there exists a section $X \to E$. Then, there is an isomorphism

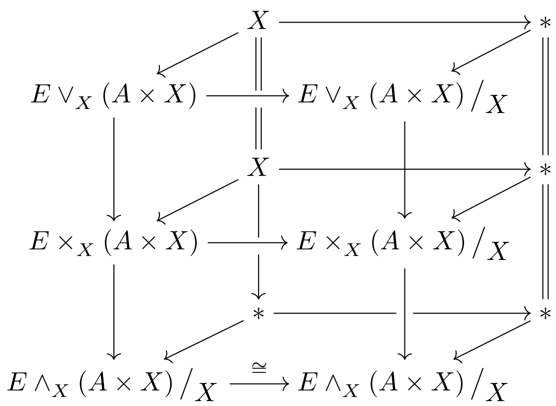
$$E \wedge_X (A \times X) \big/ _X \cong E/X \wedge A$$

in sPre_* which is natural in A and all the appearing operations are models for their homotopical analogues (confer Definition 1.3.13).

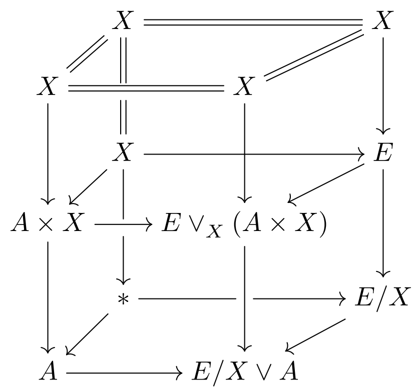
Proof. The pushout diagram

$$\begin{array}{ccc} E \vee_X (A \times X) & \longrightarrow & E \times_X (A \times X) \\ \downarrow & & \downarrow \\ X & \longrightarrow & E \wedge_X (A \times X) \end{array}$$

shows that the cofibers of the two horizontal maps are isomorphic in sPre_* and the diagram



where all the horizontal squares are pushouts and where all the vertical edges are cofiber sequences shows, that $E \wedge_X (A \times X) \big/ _X$ is the quotient in sPre_* of $E/X \times A$ by $E \vee_X (A \times X) \big/ _X$ but the latter may be identified with $E/X \vee A$ by the diagram



in sPre_*. The statement about the homotopical operations is obtained as in the proof of Lemma 3.3.11 since the projection $A \times X \to X$ is $\mathbb{A}^1$-sharp by Corollary 1.2.14. As the morphism $X \to E$ is assumed to be a section, it is a monomorphism and hence an $\mathbb{A}^1$-injective cofibration, guaranteeing that the wedge sum models its homotopical analogue. □

Lemma 4.1.4. Let X be an object in sPre_* from $\mathcal{S}\mathrm{m}_S$ which is connected, let $a \geq 1$ be an integer and let $E \to X$ be a vector bundle over X of rank n with a nowhere vanishing section $S \to X$. Set $a = n + b$. Then, there are commutative diagrams

$$\begin{array}{ccc} \mathbb{A}^b \setminus \{0\} & \xrightarrow{\quad inc \quad} & S(E \oplus \mathbb{A}^a_X) \\ \sim \downarrow & & \downarrow q_X \\ (\mathbb{A}^n/\mathbb{A}^n \setminus \{0\}) \wedge (\mathbb{A}^a \setminus \{0\}) & \xrightarrow{inc \wedge \mathrm{id}} & \mathrm{Th}_S(E) \wedge (\mathbb{A}^a \setminus \{0\}) \end{array}$$

and

$$\begin{array}{ccc} \mathbb{A}^b/\mathbb{A}^b \setminus \{0\} & \xrightarrow{\quad inc \quad} & \mathrm{Th}_S(E \oplus \mathbb{A}^a_X) \\ \sim \downarrow & & \downarrow \sim \\ (\mathbb{A}^n/\mathbb{A}^n \setminus \{0\}) \wedge (\mathbb{A}^a/\mathbb{A}^a \setminus \{0\}) & \xrightarrow{inc \wedge \mathrm{id}} & \mathrm{Th}_S(E) \wedge (\mathbb{A}^a/\mathbb{A}^a \setminus \{0\}) \end{array}$$

in sPre_*.

Proof. The diagram of Lemma 3.3.11 may be enlarged to the right to obtain the two categorical pushout diagrams

$$\begin{array}{ccccc} S(E) \wedge_X (\mathbb{A}^a_X \setminus \{0\}) & \longrightarrow & S(E) \wedge_X \mathbb{A}^a_X & \xrightarrow{\quad \sim \quad} & X \\ \downarrow & & \downarrow & & \downarrow a \\ E \wedge_X (\mathbb{A}^a_X \setminus \{0\}) & \longrightarrow & S(E \oplus \mathbb{A}^a_X) \big/ E \vee_X \mathbb{A}^a_X & \xrightarrow{\sim} & \mathrm{Th}_X(E) \wedge_X (\mathbb{A}^a_X \setminus \{0\}) \end{array}$$

in sPre_*. The monomorphisms in sPre/X are the monomorphisms in sPre over X and since sPre/X is cartesian closed, the functor $- \wedge_X (\mathbb{A}^a_X \setminus \{0\})$ preserves monomorphisms. As $S(E) \to E$ is a monomorphism and a pushout of a monomorphism is a monomorphism (confer Remark 3.1.6), the two diagrams above are homotopy pushout diagrams. The categorical cofiber of the morphism a may be identified as $\mathrm{Th}_S(E) \wedge (\mathbb{A}^a \setminus \{0\})$ by the previous Lemma 4.1.3. Let q_X be the composition $S(E \oplus \mathbb{A}^a_X) \to \mathrm{Th}_S(E) \wedge (\mathbb{A}^a \setminus \{0\})$ of the constructed morphisms in sPre_*. By the stability of colimits under basechange (confer Remark 1.3.1), the categorical pullback along the basepoint $S \to X$ before collapsing X yields the first commutative diagram of the statement. Equally, one could use Remark 4.1.2.

For the second assertion, a construction analogous to the above provides a commutative diagram

$$\begin{array}{ccc} \mathbb{A}^b & \xrightarrow{\quad inc \quad} & E \oplus \mathbb{A}^a_X \\ \sim \downarrow & & \downarrow q'_X \\ (\mathbb{A}^n/\mathbb{A}^n \setminus \{0\}) \wedge \mathbb{A}^a & \xrightarrow{inc \wedge \mathrm{id}} & \mathrm{Th}_S(E) \wedge \mathbb{A}^a \end{array}$$

in sPre_*.

The solid diagram

$$\begin{array}{ccccc}
X & = & X & \longrightarrow & * \\
\downarrow & & \downarrow & & \vdots \\
S(E \oplus \mathbb{A}^a_X) & \longrightarrow & E \oplus \mathbb{A}^a_X & \longrightarrow & \mathrm{Th}_S(E \oplus \mathbb{A}^a_X) \\
\downarrow & & \downarrow & & \vdots\,\sim \\
\mathrm{Th}_S(E) \wedge (\mathbb{A}^a \setminus \{0\}) & \longrightarrow & \mathrm{Th}_S(E) \wedge \mathbb{A}^a & \longrightarrow & \mathrm{Th}_S(E) \wedge (\mathbb{A}^a/\mathbb{A}^a \setminus \{0\})
\end{array}$$

consists of homotopy cofiber sequences. Let the pointed morphisms be the categorically induced ones. Then, [Doe98, Lemma 1.4] states that also the right vertical morphisms constitute a homotopy cofiber sequence, or in other words, there is a weak equivalence $\mathrm{Th}_S(E \oplus \mathbb{A}^a_X) \xrightarrow{\sim} \mathrm{Th}_S(E) \wedge (\mathbb{A}^a/\mathbb{A}^a \setminus \{0\})$ in sPre_*. By the same argument as above, the second commutative diagram of the statement is obtained. □

Proposition 4.1.5. Let X be a $\mathbb{A}^1$-connected object in sPre_* from $\mathcal{S}\mathrm{m}_S$ and let $E \to X$ be a vector bundle over X. Then $J(E) = 0$ if and only if there exists a Thom class for E.

Proof. Choose for example the motivic model for the motivic homotopical category. Suppose first that $J(E) = 0$. We may assume that E has a nowhere vanishing section, because otherwise we could choose $E \oplus \mathbb{A}^1$. Hence, there exist integers a, b (where we may assume $a, b \geq 2$) and an isomorphism in $\mathcal{H}_*$ making the diagram

$$\begin{array}{ccccc}
\mathbb{A}^n \setminus \{0\} *^h \mathbb{A}^a \setminus \{0\} & \xrightarrow{inc *^h inc} & S(E) *^h_X S(\mathbb{A}^a_X) & \searrow & \\
\cong\downarrow & & \cong\downarrow & & X \\
\mathbb{A}^b \setminus \{0\} & \xrightarrow{inc} & S(\mathbb{A}^b_X) & \nearrow &
\end{array}$$

of fiber sequences commutative. The upper fiber sequence is equivalent to the fiber sequence $\mathbb{A}^b \setminus \{0\} \xrightarrow{inc} S(E \oplus \mathbb{A}^a_X) \to X$ by Lemma 3.3.11 and Lemma A.6.7. As X is $\mathbb{A}^1$-connected, there is a morphism g in $\mathcal{H}_*$ such that the composition

$$\mathbb{A}^b \setminus \{0\} \xrightarrow{inc} S(E \oplus \mathbb{A}^a_X) \xrightarrow{g} \mathbb{A}^b \setminus \{0\}$$

is an isomorphism in $\mathcal{H}_*$ by Proposition 1.2.24. The commutative diagram

$$\begin{array}{ccccc}
\mathbb{A}^b \setminus \{0\} \wedge S^{1,0} & \xrightarrow{inc \wedge S^{1,0}} & S(E \oplus \mathbb{A}^a_X) \wedge S^{1,0} & \xrightarrow{g \wedge^{\mathcal{L}} S^{1,0}} & \mathbb{A}^b \setminus \{0\} \wedge S^{1,0} \\
\cong\uparrow & & \uparrow\cong & \nearrow g' & \\
\mathbb{A}^b/\mathbb{A}^b \setminus \{0\} & \xrightarrow{inc} & (E \oplus \mathbb{A}^a_X)/S(E \oplus \mathbb{A}^a_X) & &
\end{array}$$

in the homotopy category $\mathcal{H}_*$ where g' is defined as indicated, shows that the composition

$$\mathbb{A}^b/\mathbb{A}^b \setminus \{0\} \xrightarrow{inc} \mathrm{Th}_S(E \oplus \mathbb{A}^a_X) \xrightarrow{g'} \mathbb{A}^b/\mathbb{A}^b \setminus \{0\}$$

is an isomorphism in $\mathcal{H}_*$. Lemma 4.1.4 implies that the composition

$$(\mathbb{A}^n/\mathbb{A}^n \setminus \{0\}) \wedge S^{2a,a} \xrightarrow{inc \wedge \mathrm{id}} \mathrm{Th}_S(E) \wedge S^{2a,a} \xrightarrow{g'} (\mathbb{A}^n/\mathbb{A}^n \setminus \{0\}) \wedge S^{2a,a}$$

is an isomorphism in $\mathcal{H}_*$ and the result follows considering Σ^∞_T.

For the other direction, let

$$\Sigma_T^\infty(\mathbb{A}^n/\mathbb{A}^n\setminus\{0\}) \xrightarrow{\Sigma_T^\infty(inc)} \mathbb{T}h_S(E) \xrightarrow{u} \Sigma_T^\infty(\mathbb{A}^n/\mathbb{A}^n\setminus\{0\})$$

be an isomorphism in $\mathcal{SH}(S)$. Since all the objects involved are finitely presentable by [Jar00a, Lemma 2.2], the Stability lemma 2.1.4 asserts that there is an integer $c \geq 0$ and an isomorphism

$$(\mathbb{A}^n/\mathbb{A}^n\setminus\{0\})\wedge S^{2c,c} \xrightarrow{inc\wedge\mathrm{id}} \mathrm{Th}_S(E)\wedge S^{2c,c} \xrightarrow{g''} (\mathbb{A}^n/\mathbb{A}^n\setminus\{0\})\wedge S^{2c,c}$$

in $\mathcal{H}_*$. Smashing with $S^{1,1}$ yields an isomorphism

$$(\mathbb{A}^n/\mathbb{A}^n\setminus\{0\})\wedge(\mathbb{A}^a\setminus\{0\}) \xrightarrow{inc\wedge\mathrm{id}} \mathrm{Th}_S(E)\wedge(\mathbb{A}^a\setminus\{0\}) \xrightarrow{g'} (\mathbb{A}^n/\mathbb{A}^n\setminus\{0\})\wedge(\mathbb{A}^a\setminus\{0\})$$

where $a = c+1$ and by Lemma 4.1.4, there is an isomorphism

$$\mathbb{A}^b\setminus\{0\} \xrightarrow{inc} S(E\oplus\mathbb{A}^a_X) \xrightarrow{g} \mathbb{A}^b\setminus\{0\}$$

in $\mathcal{H}_*$. As X is $\mathbb{A}^1$-connected by the same Proposition 1.2.24, the fiber sequence

$$\mathbb{A}^b\setminus\{0\} \to S(E\oplus\mathbb{A}^a_X) \to X$$

is trivial, but as observed in the proof of the other direction, this fiber sequence is by Lemma 3.3.11 equivalent to the fiber sequence

$$\mathbb{A}^n\setminus\{0\} *^h \mathbb{A}^a\setminus\{0\} \to S(E) *^h_X S(\mathbb{A}^a_X) \to X$$

finishing the proof. □

4.2. Motivic Atiyah duality and the transfer

Remark 4.2.1. In Remark 2.1.3 we noted, that the category $\mathcal{S}p_T^\Sigma(S)$ of motivic symmetric T-spectra can be given the monoidal stable motivic model structure which will serve as a model for the stable motivic homotopy category $\mathcal{SH}(S)$ in the following. In order to keep the notation clean, we omit the symbol T from the index everywhere. Due to the closed symmetric monoidal structure, we have in particular an internal hom-functor $\underline{\mathrm{hom}}(-,-)$ taking values in $\mathcal{S}p^\Sigma$ (confer Definition A.2.1). Let $\mathbb{S}$ denote the symmetric T-sphere spectrum $\Sigma^\infty(S^0)$. Theorem A.2.7 furnishes the stable motivic homotopy category with the structure of a closed symmetric monoidal category with unit $\mathbb{S}$ and functors $\wedge^{\mathcal{L}}$ and $\underline{\mathrm{hom}}^{\mathcal{R}}(-,-)$.

Definition 4.2.2 ([May01a], [May01b])**.** Let X be a motivic symmetric T-spectrum. The object $DX = \underline{\mathrm{hom}}^{\mathcal{R}}(X,\mathbb{S})$ of $\mathcal{SH}(S)$ is called the *dual object of* X (or just the *dual of* X).

The adjoint

$$\epsilon_S^X : DX\wedge^{\mathcal{L}} X \to \mathbb{S}$$

of the identity morphism $\underline{\mathrm{hom}}^{\mathcal{R}}(X,\mathbb{S}) \to \underline{\mathrm{hom}}^{\mathcal{R}}(X,\mathbb{S})$ is the *evaluation map for* X.

The object X is called *strongly dualizable* (or just *dualizable*), if the morphism

$$\begin{aligned} \nu_S^X : DX\wedge^{\mathcal{L}} X &= \underline{\mathrm{hom}}^{\mathcal{R}}(X,\mathbb{S})\wedge^{\mathcal{L}} X \\ &\cong \underline{\mathrm{hom}}^{\mathcal{R}}(X,\mathbb{S})\wedge^{\mathcal{L}} \underline{\mathrm{hom}}^{\mathcal{R}}(\mathbb{S},X) \\ &\xrightarrow{\wedge^{\mathcal{L}}} \underline{\mathrm{hom}}^{\mathcal{R}}(X\wedge^{\mathcal{L}}\mathbb{S},\mathbb{S}\wedge^{\mathcal{L}} X) \\ &\cong \underline{\mathrm{hom}}^{\mathcal{R}}(X,X) \end{aligned}$$

is an isomorphism in $\mathcal{SH}(S)$.

If X is dualizable, the *coevaluation map for* X is the morphism

$$\eta_S^X : \mathbb{S} \xrightarrow{\mathrm{id}^\sharp} \underline{\mathrm{hom}}^{\mathcal{R}}(X,X) \xrightarrow{(\nu_S^X)^{-1}} DX \wedge^{\mathcal{L}} X \cong X \wedge^{\mathcal{L}} DX,$$

where $\mathrm{id}^\sharp$ is the adjoint of the identity.

For a suspension spectrum $X = \Sigma^\infty E_*$, the morphism

$$\mathrm{trf}_S^E : \mathbb{S} \xrightarrow{\eta_S^X} X \wedge^{\mathcal{L}} DX \cong DX \wedge^{\mathcal{L}} X \xrightarrow{\mathrm{id} \wedge^{\mathcal{L}} \Delta} DX \wedge^{\mathcal{L}} X \wedge^{\mathcal{L}} X \xrightarrow{\epsilon_S^X \wedge^{\mathcal{L}} \mathrm{id}} \mathbb{S} \wedge^{\mathcal{L}} X \cong X$$

in $\mathcal{SH}(S)$ is called the *transfer map for* E (or just the *transfer for* E). The composition

$$\chi(E) : \mathbb{S} \xrightarrow{\mathrm{trf}_S^E} X \xrightarrow{\mathrm{pinch}} \mathbb{S}$$

is called the *Euler characteristic of* E.

Remark 4.2.3. We recall some terms and statements from algebraic geometry additionally to the definitions in Remark 1.0.1. Every base scheme X is noetherian and in particular quasi-compact [Liu02, Exercise 2.3.14]. Hence, every Zariski-fiber bundle $F \to E \to X$ has a finite trivializing Zariski covering. Moreover, a Nisnevich-fiber bundle has a finite trivializing Nisnevich covering (confer [GLMS$^+$00, Lemma 12.2]).

If the base scheme S is a field, then every object of $\mathcal{S}\mathrm{m}_S$ is regular and a reduced scheme [Liu02, Theorem 4.2.16, 4.3.36]. Moreover, in this case the (finitely many) irreducible components of an object of $\mathcal{S}\mathrm{m}_S$ are exactly its connected components.

Remark 4.2.4. If $A \to B \to C$ is a cofiber sequence in $\mathcal{SH}(S)$, then if two of its objects are dualizable, then so is the third (confer [May01b, Theorem 1.9]). This was used in the proof of the Theorem [RØ06, Theorem 52] cited below as Theorem 4.2.6 in the following way: A Nisnevich distinguished square

$$\begin{array}{ccc} W & \overset{j}{\rightarrowtail} & Y \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ U & \underset{i}{\rightarrowtail} & X \end{array}$$

defines a homotopy pushout diagram in the motivic homotopy category. Hence, there are cofiber sequences $\Sigma^\infty W_* \to \Sigma^\infty U_* \to \Sigma^\infty C$ and $\Sigma^\infty Y_* \to \Sigma^\infty X_* \to \Sigma^\infty C$ with the same homotopy cofiber $\Sigma^\infty C$. Therefore, if three of the objects $\Sigma^\infty W_*$, $\Sigma^\infty U_*$, $\Sigma^\infty X_*$ and $\Sigma^\infty X_*$ are dualizable in $\mathcal{SH}(S)$, then so is the remaining one.

Suppose that X is an object of sPre_*. Then $\Sigma^\infty X$ is dualizable in $\mathcal{SH}(S)$ if and only if $\Sigma^\infty X_*$ is dualizable in $\mathcal{SH}(S)$ since there is a cofiber sequence $S^0 \to X_+ \to X$ and all the spheres $S^{a,b}$ are dualizable as they are invertible (confer [May01a, Definition 2.8]). Moreover, if two objects X and Y from $\mathcal{S}p^\Sigma$ are dualizable, then so is $X \wedge^{\mathcal{L}} Y$ [May01a, Proposition 2.7].

Theorem 4.2.5 ([Hu05, Theorem A.1])**.** Let S be the spectrum of a field and let $X \in \mathcal{S}\mathrm{m}_S$ be a smooth projective scheme. Then, X_* is dualizable.

Theorem 4.2.6 ([RØ06, Theorem 52])**.** Let S be the spectrum of a field of characteristic zero and let $X \in \mathcal{S}\mathrm{m}_S$ be a smooth quasi-projective scheme. Then, X_* is dualizable.

Remark 4.2.7. If S is the spectrum of a field of arbitrary characteristic, it is not known, whether for an affine scheme $X \hookrightarrow \mathbb{A}^n_S$ of $\mathcal{S}\mathrm{m}_S$, the spectrum $\Sigma^\infty X_*$ is dualizable in $\mathcal{SH}(S)$. With the following argument, one may show the dualizability of $\Sigma^\infty X_*$ in a concrete situation. Consider the two pullback diagrams

$$\begin{array}{ccccc} X & \overset{\circ}{\rightarrowtail} & \mathbb{P}X & \overset{+}{\leftarrowtail} & X_\infty \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{A}^n & \overset{\circ}{\rightarrowtail} & \mathbb{P}^n & \overset{+}{\leftarrowtail} & \mathbb{P}^{n-1} \end{array}$$

of schemes, where $\mathbb{P}X$ denotes the projective closure of X. Suppose $\mathbb{P}X$ is smooth and hence an object of $\mathcal{S}\mathrm{m}_S$. There is a homotopy cofiber sequence $X_* \rightarrowtail \mathbb{P}X_* \to \mathbb{P}X/X$ and Theorem 4.2.5 together with Remark 4.2.4 yield, that the suspension spectrum of $\mathbb{P}X_*$ is dualizable and therefore the suspension spectrum of X_* is dualizable, if the suspension spectrum of $\mathbb{P}X/X$ is dualizable. Suppose moreover, that X_∞ is smooth and hence an object of $\mathcal{S}\mathrm{m}_S$. Then, the Homotopy purity theorem [MV99, Theorem 3.2.23] shows, that the suspension spectrum of $\mathbb{P}X/(\mathbb{P}X \setminus X_\infty) = \mathbb{P}X/X$ is dualizable (confer [RØ06, Theorem 52]).

Remark 4.2.8. Let $p : E \to X$ be an object of $\mathcal{S}\mathrm{m}_X$. The Quillen adjunction

$$p^\circ_\sharp : \mathrm{sPre}(E)_* \to \mathrm{sPre}(X)_* : p^*_\circ$$

of Remark 3.3.9 can be promoted to a Quillen adjunction

$$p_\sharp : \mathcal{S}p^\Sigma(E) \to \mathcal{S}p^\Sigma(X) : p^*$$

of motivic symmetric spectra, where we omit the symbol $\circ$ from the notation as only the pointed setting is considered (confer [Ayo07b, Théorème 4.5.23] using [Ayo07b, Lemme 4.3.34] which works for the category $\mathcal{S}\mathrm{m}_S$ of this text). The projection formula of Remark 3.3.9 and the isomorphism $p^*_\circ T_X \cong T_E$ is used to construct the prolongation of $p^\circ_\sharp$ where for details we refer to [Ayo07b, Remarque 4.3.18]. In particular, one has $p_\sharp \Sigma^\infty \cong \Sigma^\infty p^\circ_\sharp$ and $p^*\Sigma^\infty \cong \Sigma^\infty p^*_\circ$. The functor p^* is strong symmetric monoidal by [Hov01b, Theorem 9.3]. By abuse of notation, we will use the same notation for the derived functors of $p_\sharp$ and p^* respectively.

Theorem 4.2.9 (confer [May95, Proposition III.1.9])**.** Let $p : E \to X$ be a morphism of objects from $\mathcal{S}\mathrm{m}_S$. If the object Y of $\mathcal{SH}(X)$ is dualizable, then so is p^*Y and there is an isomorphism

$$p^*DY \cong Dp^*Y$$

in $\mathcal{SH}(E)$.

4.3. A motivic version of Brown's trick

Remark 4.3.1. Let S be the spectrum of a field. In this section, we change our notation and let $\mathcal{S}\mathrm{m}_S$ be the category of smooth quasi-projective schemes of finite type over S to be able to use the results from [Ayo07a] and [Ayo07b]. The unpublished manuscript [Rön05] asserts though, that this restriction is not necessary if we use [Rön05, Corollary 3.14] instead of [Ayo07a, Corollaire 1.4.4].

Lemma 4.3.2. Let $F \to Y \xrightarrow{f} X$ be a smooth Nisnevich fiber bundle and suppose that $\Sigma^\infty F_*$ is dualizable in $\mathcal{SH}(S)$. Then, $\Sigma^\infty Y_*$ is dualizable in $\mathcal{SH}(X)$ and the transfer

$$\mathrm{trf}_X^Y : \Sigma^\infty X_* \to \Sigma^\infty Y_*$$

in $\mathcal{SH}(X)$ is defined.

Proof. Let $\{u_i : U_i \to X\}$ be a finite trivializing Nisnevich covering for the fiber bundle which exists by Remark 4.2.3. We have to show that the morphism

$$\begin{aligned} \nu_X^Y : \underline{\mathrm{hom}}_X^{\mathcal{R}}(\Sigma^\infty Y_*, \mathbb{S}_X) \wedge_X^{\mathcal{L}} \Sigma^\infty Y_* &\cong \underline{\mathrm{hom}}_X^{\mathcal{R}}(\Sigma^\infty Y_*, \mathbb{S}_X) \wedge_X^{\mathcal{L}} \underline{\mathrm{hom}}_X^{\mathcal{R}}(\mathbb{S}_X, \Sigma^\infty Y_*) \\ &\to \underline{\mathrm{hom}}_X^{\mathcal{R}}(\Sigma^\infty Y_* \wedge_X^{\mathcal{L}} \mathbb{S}_X, \mathbb{S}_X \wedge_X^{\mathcal{L}} \Sigma^\infty Y_*) \\ &\cong \underline{\mathrm{hom}}_X^{\mathcal{R}}(\Sigma^\infty Y_*, \Sigma^\infty Y_*) \end{aligned}$$

in $\mathcal{SH}(X)$ is an isomorphism where we write Y instead of $\Sigma^\infty Y_*$ in the upper index. By [Ayo07a, Corollaire 1.4.4], it suffices to show that $u_i^*(\nu_X^Y)$ is an isomorphism in $\mathcal{SH}(U_i)$ for every i. The functor u_i^* is strong symmetric monoidal by Remark 4.2.8 and since u_i is smooth, there is a natural isomorphism $u_i^*(\underline{\mathrm{hom}}_X^{\mathcal{R}}(E, D)) \cong \underline{\mathrm{hom}}_{U_i}^{\mathcal{R}}(u_i^* E, u_i^* D)$ by [Ayo07a, Proposition 2.3.54]. Therefore, it suffices to show, that $\nu_{U_i}^{u_i^*\Sigma^\infty Y_*}$ is an isomorphism, or in other words, that $u_i^*\Sigma^\infty Y_* \cong \Sigma^\infty u_{i\,\circ}^* Y_*$ is dualizable in $\mathcal{SH}(U_i)$. Since the bundle is trivial on the U_i, one has $u_{i\,\circ}^*(Y_*) \cong (U_i \times F)_*$ in $\mathrm{sPre}(U_i)_*$. Considering the composite morphism $U_i \to X \to S$ defined by $t \circ u_i$, the assertion follows from Theorem 4.2.9, as $\Sigma^\infty F_*$ is dualizable in $\mathcal{SH}(S)$ by assumption. □

Corollary 4.3.3. Let $F \to Y \xrightarrow{f} X$ be a smooth Nisnevich fiber bundle and suppose that $\Sigma^\infty F_*$ is dualizable in $\mathcal{SH}(S)$. Let $p : E \to X$ be a vector bundle and $q : D \to Y$ the pullback bundle along f. Then, $\Sigma^\infty S(D)_*$ is dualizable in $\mathcal{SH}(S(E))$ and the transfer

$$\mathrm{trf}_{S(E)}^{S(D)} : \Sigma^\infty S(E)_* \to \Sigma^\infty S(D)_*$$

in $\mathcal{SH}(S(E))$ is defined.

Proof. The Nisnevich fiber bundle $F \times (\mathbb{A}^n \setminus \{0\}) \to S(D) \to S(E)$ obtained by pullback along $S(E) \to X$ is smooth and $\Sigma^\infty F_* \wedge \Sigma^\infty(\mathbb{A}^n \setminus \{0\})_*$ is dualizable in $\mathcal{SH}(S)$ by Remark 4.2.4. Hence, the the Lemma 4.3.2 shows that $\Sigma^\infty S(D)_*$ is dualizable in $\mathcal{SH}(S(E))$. □

Remark 4.3.4. Consider the setting of Corollary 4.3.3. There is a commutative diagram

$$\begin{array}{ccc} p_\sharp p^* \Sigma^\infty Y_* & \xleftarrow{p_\sharp p^* \mathrm{trf}_X^Y} & p_\sharp p^* \Sigma^\infty X_* \\ \downarrow{\scriptstyle \varepsilon_{\Sigma^\infty Y_*}} & & \downarrow{\scriptstyle \varepsilon_{\Sigma^\infty X_*}} \\ \Sigma^\infty Y_* & \xleftarrow{\mathrm{trf}_X^Y} & \Sigma^\infty X_* \end{array}$$

in $\mathcal{SH}(X)$ where ε is the unit of the adjunction $(p_\sharp, p^*)$ of Remark 4.2.8.

Hence, by the compatibility of the transfer with p^* (confer Theorem 4.2.9), there is a commutative diagram

$$\begin{array}{ccc}
p_\sharp \Sigma^\infty S(D)_* & \xleftarrow{\;p_\sharp \operatorname{trf}_{S(E)}^{S(D)}\;} & p_\sharp \Sigma^\infty S(E)_* \\
{\scriptstyle \varepsilon'_{\Sigma^\infty Y_*}}\big\downarrow & & \big\downarrow{\scriptstyle \varepsilon'_{\Sigma^\infty X_*}} \\
\Sigma^\infty Y_* & \xleftarrow{\;\operatorname{trf}_X^Y\;} & \Sigma^\infty X_*
\end{array}$$

in $\mathcal{SH}(X)$. Applying the functor $t_\sharp : \mathcal{SH}(X) \to \mathcal{SH}(S)$ and omitting some decoration by abuse of notation, one has a commutative diagram

$$\begin{array}{ccc}
\Sigma^\infty S(D)_* & \xleftarrow{\;\operatorname{trf}\;} & \Sigma^\infty S(E)_* \\
{\scriptstyle \Sigma^\infty q_*}\big\downarrow & & \big\downarrow{\scriptstyle \Sigma^\infty p_*} \\
\Sigma^\infty Y_* & \xleftarrow{\;\operatorname{trf}\;} & \Sigma^\infty X_*
\end{array}$$

in $\mathcal{SH}(S)$. Hence, by taking the homotopy cofibers of the vertical morphisms and considering the natural transformation of the fiber inclusion of Remark 3.3.9, there is a commutative diagram

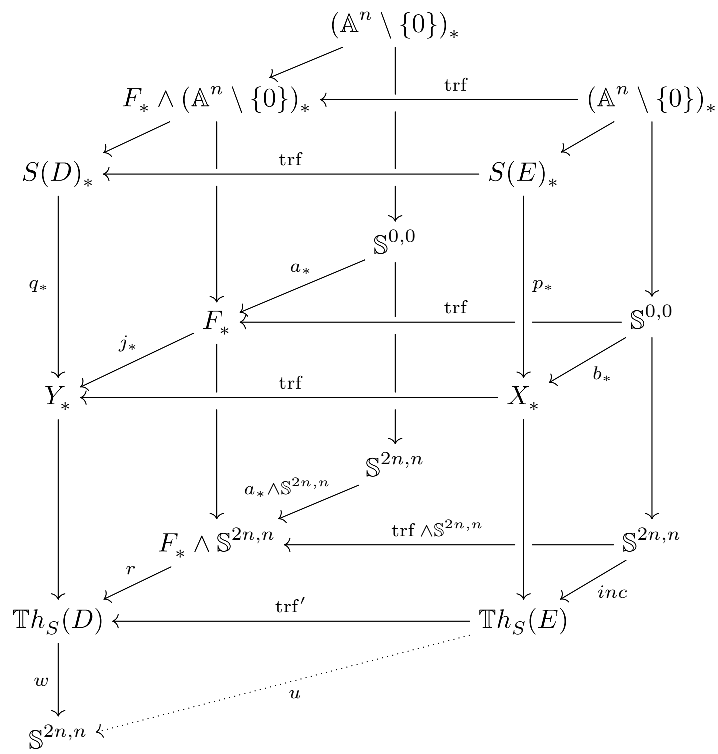
$$\begin{array}{ccccc}
 & & \Sigma^\infty F_* \wedge \Sigma^\infty S^{2n,n} & \xleftarrow{\;\operatorname{trf}'\;} & \Sigma^\infty S^{2n,n} \\
 & {\scriptstyle r}\swarrow & & \swarrow{\scriptstyle inc} & \\
\mathbb{T}h_S(D) & \xleftarrow{\;\operatorname{trf}'\;} & \mathbb{T}h_S(E) & &
\end{array}$$

in $\mathcal{SH}(S)$, where the upper morphism trf' is a suspension of trf_S^F in $\mathcal{SH}(S)$.

Theorem 4.3.5. Let S be the spectrum of a perfect field and F an $\mathbb{A}^1$-connected object from $\mathcal{S}\mathrm{m}_S$ such that $\Sigma^\infty F_*$ is dualizable in $\mathcal{SH}(S)$ with invertible Euler characteristic $\chi(F)$. Let moreover $F \to Y \to X$ be a smooth Nisnevich fiber bundle and $E \to X$ a vector bundle with pullback bundle $D \to Y$ along $Y \to X$. Suppose that X is $\mathbb{A}^1$-connected. Then, $J(D) = 0$ implies $J(E) = 0$.

Proof. Since the whole proof takes place in the stable motivic homotopy category $\mathcal{SH}(S)$, we omit writing the strong monoidal suspension spectrum functor Σ^∞ everywhere. Let $J(D) = 0$. By Proposition 4.1.5, there hence exists a Thom class $w : \mathbb{T}h_S(D) \to \mathbb{S}^{2n,n}$ for the vector bundle $D \to Y$. We have to show that there exists a Thom class $u : \mathbb{T}h_S(E) \to \mathbb{S}^{2n,n}$ for $E \to X$ since this implies $J(E) = 0$ by the same Proposition 4.1.5.

Consider the diagram



and define u as the indicated composition $w \circ \mathrm{trf}'$. We have to show that

$$\mathbb{S}^{2n,n} \xrightarrow{inc} \mathbb{T}h_S(E) \xrightarrow{u} \mathbb{S}^{2n,n}$$

is an isomorphism in $\mathcal{SH}(S)$ and by a Yoneda argument it suffices to show that this composition induces an isomorphism on $\langle \mathbb{S}^{2n,n}, -\rangle_{\mathcal{SH}(S)}$. In the remaining part of the proof, we will omit writing the $\mathcal{SH}(S)$ in the index of the homotopy classes. By the commutativity of the diagram above it is the same to show, that the composition $w \circ r \circ (\mathrm{trf} \wedge \mathbb{S}^{2n,n})$ induces an isomorphism on $\langle \mathbb{S}^{2n,n}, -\rangle$. The condition on the Euler characteristic of F is, that the composition $\mathrm{pinch} \circ \mathrm{trf}$ induces an isomorphism on $\langle \mathbb{S}^{0,0}, -\rangle$. The composition $\mathrm{pinch} \circ a_*$ is the identity on $\mathbb{S}^{0,0}$ and therefore an isomorphism on the homotopy classes $\langle \mathbb{S}^{0,0}, -\rangle$. Remark 2.2.4 and Proposition 2.2.5 yield, that a_* induces an isomorphism on $\langle \mathbb{S}^{0,0}, -\rangle$ and therefore pinch is an isomorphism on these homotopy classes as well. This implies that $\mathrm{trf} : \mathbb{S}^{0,0} \to F_*$ is an isomorphism on $\langle \mathbb{S}^{0,0}, -\rangle$. It suffices therefore by the invertibility of the suspension with $\mathbb{S}^{2n,n}$ to show, that $w \circ r$ is an isomorphism on $\langle \mathbb{S}^{2n,n}, -\rangle$. The composition $r \circ (a_* \wedge \mathbb{S}^{2n,n})$ is the fiber inclusion $\mathbb{S}^{2n,n} \to \mathrm{Th}_S(D)$ and by assumption, $w \circ r \circ (a_* \wedge \mathbb{S}^{2n,n})$ induces an isomorphism on $\langle \mathbb{S}^{2n,n}, -\rangle$. It was already observed that $(a_* \wedge \mathbb{S}^{2n,n})$ induces an isomorphism on these homotopy classes and hence it is implied that $w \circ r$ is an isomorphism on $\langle \mathbb{S}^{2n,n}, -\rangle$, showing the result. □

4.4. Dualizability of GL_n/NT

Theorem 4.4.1 (Ayoub, Röndigs [Rön10, Lemma 4.2])**.** Let S be the spectrum of a field and X a projective curve from $\mathcal{S}\mathrm{m}_S$. For any line bundle $E \to X$, one has $\mathbb{T}h_X(E \otimes E) \sim \mathbb{S}^{2,1}_X$ in $\mathcal{SH}(X)$.

Corollary 4.4.2. Let S be the spectrum of a perfect field and X a projective curve from $\mathcal{S}\mathrm{m}_S$ which is $\mathbb{A}^1$-connected. For any line bundle $E \to X$, one has

$$J(E \otimes E) = 0.$$

Proof. This can be done as in the proof of Lemma 3.3.14 by considering the vector bundle $(E \otimes E) \oplus \mathbb{A}^1_X \to X$ whose associated spherical bundle admits an X-section. □

Remark 4.4.3. There are exactly one example of a projective curve from $\mathcal{S}\mathrm{m}_S$ which is $\mathbb{A}^1$-connected and this is $\mathbb{P}^1$ [AM11, Proposition 2.1.12].

Remark 4.4.4. Let S be the spectrum of a field k. The algebraic group NT_n of monomial matrices introduced in Definition 3.2.4, acts freely from the right as a subgroup on GL_n. Set $A = \mathcal{O}_{GL_n}(GL_n)$, $B = \mathcal{O}_{NT}(NT)$ and consider the k-algebra

$$A^B = \{a \in A \mid \Delta(a) = a \otimes 1\}$$

of invariants. The group NT is reductive, id est the radical of the connected component of the unit is isomorphic to a product of $\mathbb{G}_m$. By the former Mumford conjecture which is now Haboush's theorem [Hab75, Theorem 5.2], it is geometrically reductive and therefore satisfies the assumption of Nagata's theorem [Dol94, Theorem 3.3] stating that A^B is finitely generated as a k-algebra (confer also [MFK94, Theorem A.1.0]).

Define GL_n/NT as the spectrum of the k-algebra A^B. It is shown in [MFK94, Theorem 1.1] that $p : GL_n \to GL_n/NT$ induced by the inclusion on k-algebras is a categorical quotient. By [Bor91, Theorem 6.8], there exists a geometric quotient for the canonical action of NT on GL_n. As a geometric quotient is in particular a categorical quotient (confer [Bor91, 6.16] or [MFK94, Proposition 0.1]), the two schemes coincide. In particular, by [Bor91, Theorem 6.8], the scheme GL_n/NT is smooth and quasi-projective and hence by the above observations an element of $\mathcal{S}\mathrm{m}_S$.

This geometric quotient GL_n/NT is also the fppf-quotient in the category of fppf-sheaves (confer Section 3.1) since [Jan87, 5.6.8] states, that the fppf-quotient is a scheme. Therefore, Lemma 3.1.14 together with Remark 3.1.6 imply, that the morphism $p : GL_n \to GL_n/NT$ is fppf-locally trivial.

As smoothness of a morphism may be checked fppf-locally by [Gro67, Proposition 17.7.4], the morphism p is smooth and hence also étale-locally trivial as it has sections étale-locally [Gro67, Corollaire 17.16.3]. The author does not know, if the morphism p is Nisnevich-locally trivial and therefore, if GL_n/NT is the Nisnevich-quotient $GL_n /\!\!/_{\mathcal{N}\mathrm{is}} NT$.

Lemma 4.4.5. Let S be the spectrum of an algebraically closed field k. Then, $[S, GL_n/NT]_{\mathbb{A}^1} \cong *$.

Proof. We know already, that the quotient morphism $p : GL_n \to GL_n/NT$ is étale-locally trivial. Let $a : S \to GL_n/NT$ be a k-rational point. As k is algebraically closed, the a may be lifted along some trivializing étale morphism $u_i : U_i \to GL_n/NT$ for p.

Together with the canonical section $U_i \to U_i \times NT$, one obtains a lift $b : S \to GL_n$ of a along p, which corresponds to an invertible matrix M with entries in k. This matrix may be written as a finite product $M = E_1 \dots E_k D$, where

$$E_i = \begin{pmatrix} 1 & & & & \\ & 1 & & \lambda_{m_i n_i} & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & d \end{pmatrix}$$

with $d \neq 0$ by [Lan02, Proposition 9.1]. Let $d : S \to GL_n$ be the k-rational point corresponding to the matrix D. One may construct an evident homotopy $\mathbb{A}^1 \to GL_n$ from D to the k-rational point $d_k : S \to GL_n/NT$ corresponding to $E_k D$ and therefore, the morphisms d and d_k are equal in the (unpointed) motivic homotopy category as $\mathbb{A}^1$ induces a cylinder object for the $\mathbb{A}^1$-local injective structure (confer Remark 1.1.20). After a finite iteration of this method, it is shown that the morphisms b and d are equal in the motivic homotopy category and, for the same reason, that the morphisms $a = pb$ and $pd : S \to GL_n/NT$ are equal in the motivic homotopy category. Since D is a monomial matrix, the morphism $pd : S \to GL_n/NT$ is the same as the morphism $pe : S \to GL_n/NT$ where $e : S \to GL_n$ corresponds to the identity matrix. This shows the result. □

Lemma 4.4.6. Let S be the spectrum of a field of characteristic zero. Then, the spectrum $\Sigma^\infty GL_n/NT_*$ is dualizable in $\mathcal{SH}(S)$.

Proof. Remark 4.4.4 shows that GL_n/NT is a quasi-projective scheme in $\mathcal{S}\mathrm{m}_S$ and hence the result follows from Theorem 4.2.6. □

Remark 4.4.7. Now, a strategy for proving vanishing results for the J-homomorphism could be the following: Let S be the spectrum of a perfect field k, let $X \in \mathcal{S}\mathrm{m}_S$ be an $\mathbb{A}^1$-connected scheme and $E \to X$ a vector bundle of rank n.
Find an algebraic subgroup H of GL_n such that:

(1a) The Zariski-quotient $G/\!\!/_{\mathcal{Z}\mathrm{ar}} H$ is an object of $\mathcal{S}\mathrm{m}_S$ or
(1b) the Nisnevich-quotients $G/\!\!/_{\mathcal{N}\mathrm{is}} H$ and $E/\!\!/_{\mathcal{N}\mathrm{is}} H$ are objects from $\mathcal{S}\mathrm{m}_S$.

Lemma 3.1.19 and Lemma 3.1.20 provide a vector bundle $D \to Y$, where $Y = E/\!\!/_{\mathcal{N}\mathrm{is}} H$ and a smooth Nisnevich-fiber bundle

$$\begin{array}{ccccc} & & D & \longrightarrow & E \\ & & {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ F & \longrightarrow & Y & \xrightarrow{f} & X \end{array}$$

where $F = G/\!\!/_{\mathcal{N}\mathrm{is}} H$.

(2) Show that $[S, F]_{\mathbb{A}^1}$ is trivial.

Moreover,

(3a) show that F is projective or
(3b) show that F is quasi-projective and the characteristic of k is zero or
(3c) show that F is affine and the method of Remark 4.2.7 applies.

In all of these cases (3 -), it is implied that $\Sigma^\infty F_*$ is dualizable in $\mathcal{SH}(S)$ by Theorem 4.2.5 or Theorem 4.2.6.

(4) Show that the Euler characteristic $\chi(F)$ is invertible.

If conditions (1)-(4) hold, Theorem 4.3.5 states, that $J(D) = 0$ implies $J(E) = 0$. The question of deciding whether $J(D) = 0$ can be considered as easier, since the vector bundle D admits a reduction of the structure group to H.

Therefore, it would be nice to find a large subgroup H of GL_n, fulfilling the properties. A naive solution would be $H = GL_1 \times GL_{n-1}$ since one may then split of a line bundle successively and has to show the vanishing of J only for line bundles as in Corollary 4.4.2. Unfortunately, the Euler characteristic of the quotient $G/\!\!/_{\mathcal{N}\text{is}}H$ does not seem to be invertible since the Euler characteristic of the topological quotients $O(2n)/(O(2) \times O(2n-2))$ and $U(n)/(U(1) \times U(n-1))$ is n.

The manuscript [Ebe] proposes $H = NT$, where NT denotes the normalizer of the standard maximal torus. The connected topological space $O(2n)/NT$ has in fact Euler characteristic one (confer [Ebe, Theorem 5.4]) and the observations in the reference can be applied to the complex case $U(n)/NT$ as well. Hence, NT could be a good candidate for a motivic reduction process as described above.

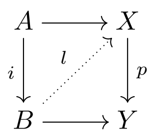
A. **Appendix**

This section contains preliminaries on model categories which are used throughout the whole text. Particularly monoidal structures on model categories are introduced and pointed model categories with their suspension and loop functors are described. Afterwards we focus on fiber and cofiber sequences in pointed model categories.

A.1. Preliminaries on model categories

Definition A.1.1 ([Hov99, Section 1.1])**.** Let $\mathcal{C}$ be a category. A *functorial factorization* (α, β) in $\mathcal{C}$ is a pair of endofunctors on the morphism category of $\mathcal{C}$ such that for every morphism f of $\mathcal{C}$ there is a factorization $f = \beta(f) \circ \alpha(f)$.

Let $i : A \to B$ and $p : X \to Y$ be morphisms in $\mathcal{C}$. One says that i fulfills the *left lifting property* (LLP) with respect to p and p fulfills the *right lifting property* (RLP) with respect to i if for every commutative solid diagram



a dotted arrow l exists such that everything commutes. In this case, l is called a *lift* of the diagram.

Definition A.1.2 ([Hov99, Definition 1.1.3])**.** Let $\mathcal{C}$ be a category. A *model structure* on $\mathcal{C}$ consists of three subcategories *we*, *cof*, fib of $\mathcal{C}$ and two functorial factorizations (α, β), (γ, δ) in $\mathcal{C}$ such that the following properties are fulfilled:

(**Two out of three**) Let f and g be two composable morphisms in $\mathcal{C}$. If two of the morphisms f, g or gf are in *we*, then so is the third.
(**Retracts**) Let f and g be morphisms in $\mathcal{C}$ and f a retract of g in the morphism category of $\mathcal{C}$. If g is in *we*, *cof* or fib, then so f lies in the same subcategory.
(**Liftings**) The morphisms in *we* $\cap$ *cof* have the LLP with respect to the morphisms in fib and the morphisms in *we* $\cap$ fib have the RLP with respect to the morphisms in *cof*.
(**Factorizations**) Let f be a morphisms in $\mathcal{C}$ then

$$\begin{array}{rcl} \alpha(f) \in \mathit{cof} & \text{and} & \beta(f) \in \mathit{we} \cap \text{fib}, \\ \gamma(f) \in \mathit{we} \cap \mathit{cof} & \text{and} & \delta(f) \in \text{fib}. \end{array}$$

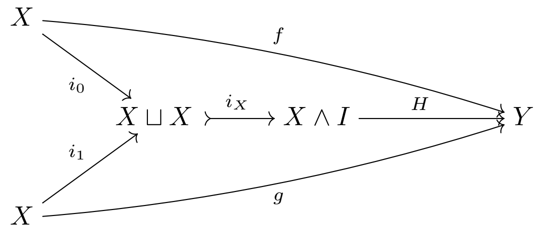
The morphisms in *we* are called *weak equivalences* and they are denoted by an arrow of the type $\xrightarrow{\sim}$ in the sequel. Morphisms in *cof* are denoted by $\rightarrowtail$ and called *cofibrations*. The morphisms in fib are denoted by $\twoheadrightarrow$ and called *fibrations*. A cofibration in *we* is called an *acyclic cofibration* and a fibration in *we* is called an *acyclic fibration*. A *bicomplete category* is a category having all limits and colimits of diagrams indexed by a small category. A pair consisting of a bicomplete category $\mathcal{C}$ and a model structure on $\mathcal{C}$ is called a *model category*.

Definition A.1.3 ([Hir03, Definition 8.1.15])**.** An object X of a model category $\mathcal{C}$ is called *cofibrant* if the map $\emptyset \to X$ from the initial object to X is a cofibration. Dually, X is called *fibrant* if the map $X \to *$ from X into the terminal object is a fibration.

A *cofibrant replacement functor* consists of a functor $Q : \mathcal{C} \to \mathcal{C}$ such that every QX is cofibrant and a natural weak equivalences $QX \to X$. Dually, a *fibrant replacement* *functor* consists of a functor $R : \mathcal{C} \to \mathcal{C}$ such that every RX is fibrant and a natural weak equivalences $X \to RX$.

Definition A.1.4 ([Hov99, Definition 1.2.4])**.** A *cylinder object* for an object X of a model category $\mathcal{C}$ is an object $X \wedge I$ of $\mathcal{C}$ together with a cofibration $i_X : X \sqcup X \to X \wedge I$ and a weak equivalence $p_X : X \wedge I \to X$ such that their composition is a factorization of the fold map $X \sqcup X \to X$.

Two morphisms $f, g : X \to Y$ in $\mathcal{C}$ are called *(left) homotopic* if there is a map $H : X \wedge I \to Y$ for some cylinder object $X \wedge I$ of X such that the diagram



commutes. In this case one writes $f \sim g$.

A morphism f is called a *homotopy equivalence* if there exists a morphisms g such that $fg \sim \mathrm{id}$ and $gf \sim \mathrm{id}$.

Remark A.1.5. Since the existence of functorial factorizations is part of our definition of a model structure, there exists both a cofibrant and a fibrant replacement functor and a functorial cylinder object in any model category. The left homotopy relation is not an equivalence relation in general but this is the case if X is cofibrant and Y is fibrant and one may then define the set $\hom(X, Y)/_\sim$. Moreover, in this case it does not matter which cylinder object is used to determine whether two maps are homotopic (confer [Hov99, Proposition 1.2.5]).

Theorem A.1.6 (Quillen, [Hir03, Theorem 8.3.5, 8.3.10], [Hov99, Theorem 1.2.10])**.** Let $\mathcal{C}$ be a model category. Then the *homotopy category* $\mathrm{Ho}\mathcal{C}$ of $\mathcal{C}$ exists. It has, fixing the cofibrant Q and the fibrant replacement functor R induced by the functorial factorization, the same objects as $\mathcal{C}$ and morphisms

$$\hom_{\mathrm{Ho}\mathcal{C}}(X, Y) = \hom_{\mathcal{C}}(RQX, RQY)/_\sim$$

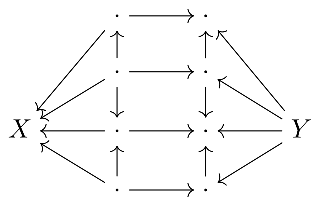
denoted by $[X, Y]$ with composition $[g] \circ [f] = [g \circ f]$ and identities $[\mathrm{id}]$. The canonical full functor $[-] : \mathcal{C} \to \mathrm{Ho}\mathcal{C}$ sends a map to an isomorphism if and only if the map is a weak equivalence. In particular, if $f \sim g$ then $[f] = [g]$ which holds even if f and g do not have a cofibrant domain and fibrant codomain. If X' is cofibrant and weakly equivalent to X and if Y' is fibrant and weakly equivalent to Y, then there is an isomorphism $\hom_{\mathcal{C}}(RQX, RQY)/_\sim \cong \hom_{\mathcal{C}}(X', Y')/_\sim$.

Remark A.1.7. For a pair $(\mathcal{C}, we)$ consisting of a category $\mathcal{C}$ together with a subcategory we, one is interested in forming a category $\mathcal{C}[we^{-1}]$ inverting the morphisms in we. The previous Theorem A.1.6 guarantees the existence of such a category for a pair $(\mathcal{C}, we)$ obtained from a model category. A more general setting is considered in [DHKS04] where the authors focus on pairs $(\mathcal{C}, we)$ called a *homotopical category* if they are fulfilling some axioms [DHKS04, 31.3]. In particular, a model category defines a homotopical category ignoring its cofibrations and fibrations and therefore different model structures on the same category and with the same weak equivalences determine the same homotopical category.

Remark A.1.8. The set of maps $\hom_{\mathrm{Ho}\mathcal{C}}(X,Y)$ may be interpreted as the set of equivalence classes of zigzags

$$X \xleftarrow{\sim} \cdot \to \cdot \xleftarrow{\sim} Y$$

where the equivalence relation is defined by saying that in any commutative diagram



consisting of weak equivalences except for the horizontal maps in the middle column, the top and the bottom row are equivalent [DHKS04, Proposition 11.2].

Definition A.1.9 ([Hov99, Definition 1.3.1])**.** An adjunction $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ is called a *Quillen adjunction* if the left adjoint F preserves cofibrations and acyclic cofibrations and the right adjoint G preserves fibrations and acyclic fibrations. A Quillen adjunction defines an adjunction $\mathcal{L}F : \mathrm{Ho}\mathcal{C} \rightleftarrows \mathrm{Ho}\mathcal{D} : \mathcal{R}G$ the *derived adjunction* where the *left derived functor* $\mathcal{L}F$ is given by $FQ(-)$ (on representants) and the *right derived functor* $\mathcal{R}G$ is defined by $GR(-)$ (on representants). The adjunction (F,G) is called a *Quillen equivalence* if its derived adjunction $(\mathcal{L}F, \mathcal{R}G)$ is an equivalence of categories.

Remark A.1.10. It follows from Theorem A.1.6, that one may use any cofibrant replacement functor to construct the left derived functor of a left Quillen functor. The dual statement holds for right derived functors as well. The expression *on representants* in context with $\mathcal{L}F$ means, that for a morphism in the homotopy category one first has to choose a representant f in $\mathcal{C}$ in order to apply the cofibrant replacement functor $Q : \mathcal{C} \to \mathcal{C}$ and afterwards F. The theory then guarantees, that $FQ(f)$ is equal to $FQ(f')$ in the homotopy category of $\mathcal{D}$ for any other choice f' of a representant.

Definition A.1.11 ([AR94])**.** Let $\mathcal{C}$ be a category and let κ be a *regular cardinal*, which is an infinite cardinal that cannot be written as a union indexed by a strictly smaller cardinal of strictly smaller cardinals.

A diagram $D : \mathcal{I} \to \mathcal{C}$ is called *κ-directed* if $\mathcal{I}$ is a κ-directed partially ordered set, this is a partially ordered set such that every subset of cardinality strictly smaller then κ has an upper bound. A *directed* diagram is a κ-directed diagram for some κ.

A *κ-directed colimit* is a colimit of a κ-directed diagram and a *directed colimit* is a colimit of a directed diagram.

An object T of $\mathcal{C}$ is called *κ-presentable* if the functor $\hom(T,-) : \mathcal{C} \to \mathcal{S}\mathrm{et}$ preserves κ-directed colimits.

An object T of $\mathcal{C}$ is called *small* if it is κ-presentable for some κ and it is called *finitely presentable* if it is $|\mathbb{N}|$-presentable.

A category $\mathcal{C}$ is called *accessible* if it contains a set of κ-presentable objects such that every object of $\mathcal{C}$ is a κ-directed colimit of those objects.

The category $\mathcal{C}$ is called *locally presentable* if it is cocomplete and accessible.

Remark A.1.12. By [AR94, Theorem 1.5], a category has all filtered colimits, if and only if it has all directed colimits. A functor from $\mathcal{C}$ preserves all filtered colimits if and only if it preserves all directed colimits. A κ-presentable object is always κ'-presentable for larger regular cardinals $\kappa' > \kappa$ and hence it is usually a stronger property to be finitely presentable than just to be small. An object T is finitely presentable if and only if the functor $\hom(T, -)$ commutes with all directed (or equivalently filtered) colimits.

Definition A.1.13 ([Hov99, Definition 2.1.17])**.** A model category $\mathcal{C}$ is called $(\mathcal{I}, \mathcal{J})$-*cofibrantly generated* if for sets of morphisms $\mathcal{I} = \{A \to B\}$ and $\mathcal{J} = \{C \to D\}$ the following properties are fulfilled:

(**Smallness**) The A (respectively the C) are small with respect to diagrams consisting only of $\mathcal{I}$-cell complexes (respectively $\mathcal{J}$-cell complexes) where a *relative $\mathcal{I}$-cell complex* is a morphism $X_0 \to \operatorname{colim}_{\alpha<\lambda} X_\alpha$ from the initial object of a sequential diagram into its colimit where each step of the diagram is given by a pushout

$$\begin{array}{ccc} A & \longrightarrow & X_\alpha \\ \downarrow & & \downarrow \\ B & \longrightarrow & X_{\alpha+1} \end{array}$$

and $\operatorname{colim}_{\beta<\alpha} X_\beta = X_\alpha$ when α is a limit ordinal.

(**Cofibrations**) Cofibrations are exactly retracts of relative $\mathcal{I}$-cell complexes.

(**Acyclic cofibrations**) Acyclic cofibrations are exactly retracts of relative $\mathcal{J}$-cell complexes.

A model category is called *combinatorial* if it is locally presentable and cofibrantly generated [Lur09, Definition A.2.6.1].

Definition A.1.14 ([DRØ03a, Definition 3.4])**.** An $(\mathcal{I}, \mathcal{J})$-cofibrantly generated model category $\mathcal{C}$ is called

- *finitely generated* if the domains and codomains of $\mathcal{I}$ and $\mathcal{J}$ are all finitely presentable,
- *weakly finitely generated* if the domains and codomains of $\mathcal{I}$ are all finitely presentable, the domains of $\mathcal{J}$ are small and there is a set $\mathcal{J}'$ of generating acyclic cofibrations with finitely presentable domains and codomains such that a morphism $f : X \to Y$ into a fibrant object is a fibration if and only if f has the RLP with respect to all maps in $\mathcal{J}'$.

Remark A.1.15. It is described in [Hov99, Section 7.4], that the usual model structures on simplicial sets and chain complexes are finitely generated but the usual model structure on topological spaces is not because one has to restrict to diagrams of cofibrations.

Lemma A.1.16 ([DRØ03a, Lemma 3.5])**.** Let $\mathcal{C}$ be a weakly finitely generated model category. Then the classes of

- weak equivalences,
- acyclic fibrations,
- fibrations with fibrant codomain,
- fibrant objects

are closed under filtered colimits each.

Theorem A.1.17 ([Hir03, Theorem 11.6.1], [Lur09, Proposition A.2.8.2])**.** Let $\mathcal{C}$ be a cofibrantly generated model category and $\mathcal{I}$ an essentially small category. Then there is a cofibrantly generated model structure on the functor category $\mathcal{F}\mathrm{un}(\mathcal{I},\mathcal{C})$ called the *projective model structure* with weak equivalences and fibrations defined objectwise. It is combinatorial, if $\mathcal{C}$ is combinatorial. Moreover, if $\mathcal{C}$ is a combinatorial model category, there is a combinatorial model structure on the functor category $\mathcal{F}\mathrm{un}(\mathcal{I},\mathcal{C})$ called the *injective model structure* with weak equivalences and cofibrations defined objectwise. The identities

$$\mathrm{id} : \mathcal{F}\mathrm{un}(\mathcal{I},\mathcal{C})_{proj} \rightleftarrows \mathcal{F}\mathrm{un}(\mathcal{I},\mathcal{C})_{inj} : \mathrm{id}$$

are a Quillen equivalence.

Definition A.1.18 ([Lur09, Remark A.2.8.8])**.** Let $\mathcal{C}$ be a cofibrantly generated model category and $\mathcal{I}$ a small category, then $\mathrm{colim} : \mathcal{F}\mathrm{un}(\mathcal{I},\mathcal{C})_{proj} \rightleftarrows \mathcal{C} : \Delta$ is a Quillen adjunction and the left adjoint of the associated derived adjunction is called the *homotopy colimit.* If $\mathcal{C}$ is combinatorial then $\Delta : \mathcal{C} \rightleftarrows \mathcal{F}\mathrm{un}(\mathcal{I},\mathcal{C})_{inj} : \lim$ is a Quillen adjunction and the right adjoint of the associated derived adjunction is called the *homotopy limit.*

When the symbol hocolim is used, we mean the specific functor

$$\begin{array}{rcl} \mathrm{hocolim} : \mathcal{F}\mathrm{un}(\mathcal{I},\mathcal{C}) & \to & \mathcal{C} \\ D & \mapsto & \mathrm{colim}(Q(D)) \end{array}$$

where Q is the cofibrant replacement functor of $\mathcal{F}\mathrm{un}(\mathcal{I},\mathcal{C})_{proj}$ obtained from the functorial factorization. The functor $\mathrm{holim} : \mathcal{F}\mathrm{un}(\mathcal{I},\mathcal{C}) \to \mathcal{C}$ is defined dually.

Remark A.1.19. The homotopy colimit and the homotopy limit are not the colimit and the limit in the homotopy category. As we discuss below, this is only true for coproducts and products. Moreover, they do not depend on the homotopy category only but it suffices to consider the underlying homotopical category of a model category (confer [DHKS04] or [Shu09, Proposition 4.8]). In particular, all statements concerning homotopy colimits and homotopy limits in one model category are also valid in a Quillen equivalent model category as they define the same (saturated) homotopical category [Shu09, Definition 3.11]. One should note, that the homotopy colimit functor $\mathcal{L}\mathrm{colim} : \mathrm{Ho}\mathcal{F}\mathrm{un}(\mathcal{I},\mathcal{C})_{proj} \to \mathrm{Ho}\mathcal{C}$ can only be applied to diagrams in $\mathcal{C}$ and not necessarily to diagrams in $\mathrm{Ho}\mathcal{C}$.

Definition A.1.20 ([Hir03, Definition 13.1.1])**.** A model category is called *left proper* if every pushout of a weak equivalence along a cofibration is again a weak equivalence, *right proper* if every pullback of a weak equivalence along a fibration is again a weak equivalence and *proper* if it is left proper and right proper.

Remark A.1.21. To calculate the homotopy colimit of a certain diagram in a model category, one replaces that diagram cofibrantly in the projective model structure. This cofibrant replacement is usually not given by just replacing all the arrows and objects cofibrantly. Nevertheless, this is true for certain diagrams. Particularly, to calculate a *homotopy pushout* (or dually a *homotopy pullback*)

$$\begin{array}{ccc} A & \xrightarrow{f} & C \\ {\scriptstyle g}\downarrow & & \vdots \\ B & \cdots\rightarrow & \mathrm{hocolim} \end{array} \qquad \begin{array}{ccc} \mathrm{holim} & \cdots\rightarrow & C \\ \vdots & & \downarrow{\scriptstyle p} \\ B & \xrightarrow{q} & A \end{array}$$

in a left proper model category $\mathcal{C}$, it suffices to replace f or g with a cofibration (or dually p or q with a fibration if $\mathcal{C}$ is right proper), take the ordinary pushout (or pullback) in $\mathcal{C}$ and consider the resulting diagram in the homotopy category (confer [Hir03, Corollary 13.3.8]).

If the object B is the terminal object of $\mathcal{C}$, the above homotopy pushout (or homotopy pullback) is called the *homotopy cofiber* of the map f (or the *homotopy fiber* of the map p) [Hir03, Definition 13.4.3]. Whenever we speak of the object hocofib(f) in a left proper model category $\mathcal{C}$ (or of the object hofib(f) in a right proper model category $\mathcal{C}$), we mean the categorical pushout of the diagram $*'' \leftarrow A \to B$ where the cofibration $A \rightarrowtail *''$ is obtained by the functorial factorization of $\mathcal{C}$ (or the categorical pullback of the diagram $*' \to A \leftarrow C$ where the fibration $*' \to A$ is obtained by the functorial factorization of $\mathcal{C}$). In particular, one has an actual diagram $A \to B \to \mathrm{hocofib}(f)$ in $\mathcal{C}$ (or an actual diagram $\mathrm{hofib}(f) \to C \to A$ in $\mathcal{C}$).

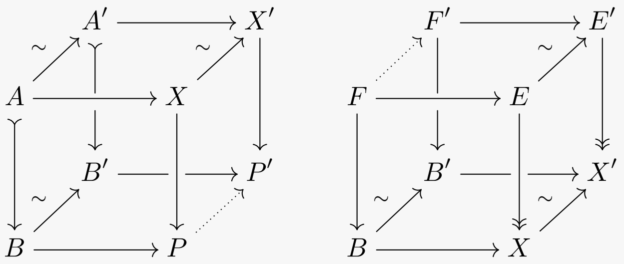
A diagram $D : \mathcal{I} \to \mathcal{C}$ is a *homotopy colimit diagram* (or *homotopy limit diagram*) if the canonical map $\mathrm{hocolim}D \to \mathrm{colim}D$ (or if the canonical map $\mathrm{lim}D \to \mathrm{holim}D$) is a weak equivalence. The two upper diagrams are diagrams in the homotopy category $\mathrm{Ho}\mathcal{C}$ only since the dotted arrows need not to be represented by morphisms in $\mathcal{C}$.

Left Quillen functors $F : \mathcal{C} \to \mathcal{D}$ preserve homotopy colimits (and dually, right Quillen functors preserve homotopy limits) as the diagram

$$\begin{array}{ccc} \mathcal{F}\mathrm{un}(\mathcal{I}, \mathcal{C})_{proj} & \xrightarrow{\mathrm{colim}} & \mathcal{C} \\ \downarrow{\scriptstyle F_*} & & \downarrow{\scriptstyle F} \\ \mathcal{F}\mathrm{un}(\mathcal{I}, \mathcal{D})_{proj} & \xrightarrow{\mathrm{colim}} & \mathcal{D} \end{array}$$

of left Quillen functors commutes and derivation goes along with composition in model categories [Hov99, Theorem 1.3.7].

Lemma A.1.22 (Glueing/Coglueing, [Hir03, Proposition 13.5.4, 13.3.9])**.** Consider the commutative cubes



where the front and the back are pushout squares in the first cube and pullback squares in the second cube and where the dotted arrows are the induced maps respectively. If $\mathcal{C}$ is left proper, then the induced map of the left cube is a weak equivalence. If $\mathcal{C}$ is right proper, then the induced map of the right cube is a weak equivalence.

A.2. Monoidal model categories

Definition A.2.1 ([Bor94b, Section 6.1])**.** A *symmetric monoidal category* $(\mathcal{C}, \wedge, \mathbb{I})$ consists of a category $\mathcal{C}$ together with a functor

$$- \wedge - : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$

and an object $\mathbb{I} \in \mathcal{C}$, natural associativity isomorphisms $(X \wedge Y) \wedge Z \to X \wedge (Y \wedge Z)$, natural unit isomorphisms $(\mathbb{I} \wedge X) \to X$ as well as natural symmetry isomorphisms $X \wedge Y \to Y \wedge X$ subject to certain coherence laws.

A *closed symmetric monoidal category* is a symmetric monoidal category $(\mathcal{C}, \wedge, \mathbb{I})$ together with a functor

$$\underline{\mathrm{hom}}(-,-) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$$

and an adjunction

$$- \wedge X : \mathcal{C} \rightleftarrows \mathcal{C} : \underline{\mathrm{hom}}(X, -).$$

Definition A.2.2 ([Hov99, Definition 4.2.6])**.** A *monoidal model structure* on a closed symmetric monoidal category $(\mathcal{C}, \wedge, \mathbb{I})$ is a model structure on $\mathcal{C}$ such that the following properties are fulfilled:

(**Pushout product**) Let $i : A \rightarrowtail B$ and $j : C \rightarrowtail D$ be cofibrations in $\mathcal{C}$ then their *pushout product*

$$i \,\square\, j : (B \wedge C) \textstyle\coprod_{(A \wedge C)} (A \wedge D) \to B \wedge D$$

is a cofibration and an acyclic cofibration if i or j is a weak equivalence.
(**Unit**) A unit property which is automatically fulfilled for a cofibrant unit $\mathbb{I}$.

Remark A.2.3. For a symmetric monoidal model category and a cofibrant object X the adjunction $(- \wedge X, \underline{\mathrm{hom}}(X, -))$ is a Quillen adjunction [Hov99, Remark 4.2.3]. The usual model structure on the closed symmetric monoidal category $(\mathrm{s}\mathcal{S}\mathrm{et}, \times, \Delta^0)$ of simplicial sets is monoidal [Hov99, Proposition 4.2.8].

Definition A.2.4 ([Bor94b, Section 6.2])**.** A *(lax) monoidal functor* $F : \mathcal{C} \to \mathcal{D}$ between symmetric monoidal closed categories $(\mathcal{C}, \wedge_{\mathcal{C}}, \mathbb{I}_{\mathcal{C}})$ and $(\mathcal{D}, \wedge_{\mathcal{D}}, \mathbb{I}_{\mathcal{D}})$ consists of a functor $F : \mathcal{C} \to \mathcal{D}$, natural morphisms

$$t_F : F(X) \wedge_{\mathcal{D}} F(Y) \to F(X \wedge_{\mathcal{C}} Y)$$

and a morphism $e_F : \mathbb{I}_{\mathcal{D}} \to F(\mathbb{I}_{\mathcal{C}})$ subject to certain coherence laws. A monoidal functor is called *strong* if all its structure maps t and e are isomorphisms.

A *monoidal natural transformation* $\eta : F \to G$ of two monoidal functors F and G from $\mathcal{C}$ to $\mathcal{D}$ is a natural transformation $\eta : F \to G$ such that all the diagrams

$$\begin{array}{ccc} F(X) \wedge_{\mathcal{D}} F(Y) & \xrightarrow{\eta_X \wedge \eta_Y} & G(X) \wedge_{\mathcal{D}} G(Y) \\ \downarrow{\scriptstyle t_F} & & \downarrow{\scriptstyle t_G} \\ F(X \wedge_{\mathcal{C}} Y) & \xrightarrow{\eta_{X \wedge Y}} & G(X \wedge_{\mathcal{C}} Y) \end{array} \qquad \begin{array}{ccc} & \mathbb{I}_{\mathcal{D}} & \\ {\scriptstyle e_F} \swarrow & & \searrow {\scriptstyle e_G} \\ F(\mathbb{I}_{\mathcal{C}}) & \xrightarrow{\eta} & G(\mathbb{I}_{\mathcal{C}}) \end{array}$$

commute.

A *monoidal adjunction* is an adjunction $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ of two monoidal functors such that its unit $\eta : \mathrm{id} \to GF$ and its counit $\epsilon : FG \to \mathrm{id}$ are monoidal natural transformations.

Lemma A.2.5. Let $\mathcal{C}$ and $\mathcal{D}$ be symmetric monoidal closed categories and

$$F : \mathcal{C} \rightleftarrows \mathcal{D} : G$$

an adjunction of categories between monoidal functors. This adjunction is monoidal if and only if F is a strong monoidal functor.

Proof. The adjunction being monoidal is that there are monoidal transformations $\eta : \mathrm{id} \to GF$ and $\epsilon : FG \to \mathrm{id}$ such that the diagrams

$$\begin{array}{ccc} F & \xrightarrow{F\eta} & FGF \\ & \searrow^{=} & \downarrow{\scriptstyle \epsilon F} \\ & & F \end{array} \qquad \begin{array}{ccc} G & \xrightarrow{\eta G} & GFG \\ & \searrow^{=} & \downarrow{\scriptstyle G\epsilon} \\ & & G \end{array}$$

commute. The map

$$F(X \wedge Y) \xrightarrow{F(\eta\wedge\eta)} F(GF(X) \wedge GF(Y)) \xrightarrow{F(t_R)} FG(F(X) \wedge F(Y)) \xrightarrow{\epsilon(\dots)} F(X) \wedge F(Y)$$

is inverse to the structure map $t_F : F(X) \wedge F(Y) \to F(X \wedge Y)$ since the diagram

$$\begin{array}{ccccccc} FX \wedge FY & \xrightarrow{F\eta\wedge F\eta} & FGFX \wedge FGFY & & & \xrightarrow{\epsilon F \wedge \epsilon F} & FX \wedge FY \\ \downarrow{\scriptstyle t_F} & & \downarrow{\scriptstyle t_F} & \searrow^{t_{FG}} & & & \\ F(X \wedge Y) & \xrightarrow{F(\eta\wedge\eta)} & F(GFX \wedge GFY) & \xrightarrow{Ft_G} & FG(FX \wedge FY) & \xrightarrow{\epsilon(\dots)} & FX \wedge FY \\ & \searrow_{F\eta} & & \searrow^{Ft_{GF}} & \downarrow{\scriptstyle FGt_F} & & \downarrow{\scriptstyle t_F} \\ & & & & FGF(X \wedge Y) & \xrightarrow{\epsilon F} & F(X \wedge Y) \end{array}$$

commutes. We show that the inverse of the morphism $e_F : \mathbb{I}_{\mathcal{D}} \to F(\mathbb{I}_C)$ is given by the adjoint $\epsilon F(e_G) : F(\mathbb{I}_{\mathcal{C}}) \to \mathbb{I}_{\mathcal{D}}$ of the structure map $e_G : \mathbb{I}_{\mathcal{C}} \to G(\mathbb{I}_{\mathcal{D}})$ of G. The equality $\epsilon F(e_G)e_F = \mathrm{id}$ holds since $\epsilon : FG \to \mathrm{id}$ is a monoidal natural transformation and hence the diagram

$$\begin{array}{ccc} \mathbb{I}_{\mathcal{D}} & & \\ \downarrow{\scriptstyle F(e_G)e_F} & \searrow^{=} & \\ FG(\mathbb{I}_{\mathcal{D}}) & \xrightarrow{\epsilon} & \mathbb{I}_D \end{array}$$

commutes. The equality $e_F \epsilon F(e_G) = \mathrm{id}$ holds by considering the two adjoint situations

$$\begin{array}{ccc} F(\mathbb{I}_{\mathcal{C}}) & \xrightarrow{\epsilon F(e_G)} & \mathbb{I}_{\mathcal{D}} \\ & \searrow^{=} & \downarrow{\scriptstyle e_F} \\ & & F(\mathbb{I}_{\mathcal{C}}) \end{array} \qquad \begin{array}{ccc} \mathbb{I}_{\mathcal{C}} & \xrightarrow{e_G} & G(\mathbb{I}_{\mathcal{D}}) \\ & \searrow_{\eta} & \downarrow{\scriptstyle G(e_F)} \\ & & GF(\mathbb{I}_{\mathcal{C}}) \end{array}$$

of which the right one commutes since since η is a monoidal natural transformation.

For the other implication let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be an adjunction of categories such that F is strong monoidal. Define the structure maps of G as

$$G(X) \wedge G(Y) \xrightarrow{\eta} GF(G(X) \wedge G(Y)) \xrightarrow{G(t_F)} G(FG(X) \wedge FG(Y)) \xrightarrow{G(\epsilon\wedge\epsilon)} G(X \wedge Y)$$

and

$$\mathbb{I}_{\mathcal{C}} \xrightarrow{\eta} GF(\mathbb{I}_{\mathcal{C}}) \xrightarrow{G(e_F^{-1})} G(\mathbb{I}_{\mathcal{D}}).$$

Then one checks that all the coherence diagrams commute. □

Definition A.2.6 ([Hov99, Definition 4.2.16])**.** A *(strong) monoidal Quillen adjunction* is a Quillen adjunction

$$F : \mathcal{C} \rightleftarrows \mathcal{D} : G$$

which is also a monoidal adjunction such that a certain unit property holds which is obsolete if $\mathbb{I}_{\mathcal{C}}$ is cofibrant.

Theorem A.2.7 ([Hov99, Theorem 4.3.2, 4.3.3])**.** Let $\mathcal{C}$ be a symmetric monoidal model category. Then the associated homotopy category $\mathrm{Ho}\mathcal{C}$ is a symmetric monoidal closed category $(\mathrm{Ho}\mathcal{C}, \wedge^{\mathcal{L}}, Q\mathbb{I}_{\mathcal{C}})$ with involved functors (on representants)

$$Q(-) \wedge Q(-) : \mathrm{Ho}\mathcal{C} \quad \times \mathrm{Ho}\mathcal{C} \to \mathrm{Ho}\mathcal{C}$$
$$\underline{\mathrm{hom}}(Q(-), R(-)) : \mathrm{Ho}\mathcal{C}^{\mathrm{op}} \times \mathrm{Ho}\mathcal{C} \to \mathrm{Ho}\mathcal{C}.$$

Moreover, if $\mathcal{D}$ is another symmetric monoidal model category, the derived adjunction

$$\mathcal{L}F : \mathrm{Ho}\mathcal{C} \rightleftarrows \mathrm{Ho}\mathcal{D} : \mathcal{R}G$$

of a monoidal Quillen adjunction $F : \mathcal{C} \rightleftarrows D : G$ is again monoidal.

A.3. Simplicial model categories

Definition A.3.1 ([Hov99, Definition 4.1.6])**.** Let $\mathcal{S}$ be a symmetric monoidal category. A *(right) $\mathcal{S}$-module* consists of a category $\mathcal{C}$ together with a functor

$$- \otimes - : \mathcal{C} \times \mathcal{S} \to \mathcal{C}$$

natural associativity isomorphisms $(X \otimes A) \otimes B \to X \otimes (A \otimes_{\mathcal{S}} B)$ and a natural unit isomorphism $X \otimes \mathbb{I}_{\mathcal{S}} \to X$ subject to certain coherence laws.

A *closed $\mathcal{S}$-module* is a S-module $\mathcal{C}$ together with functors

$$\mathcal{S}(-,-) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{S}$$
$$(-)^{(-)} : \mathcal{S}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$$

and adjunctions

(1)
$$\begin{aligned} X \otimes - : \mathcal{S} &\rightleftarrows \mathcal{C} : \mathcal{S}(X,-) \\ - \otimes A : \mathcal{C} &\rightleftarrows \mathcal{C} : (-)^A \\ \mathcal{S}(-,Y) : \mathcal{C}^{\mathrm{op}} &\rightleftarrows \mathcal{S} : Y^{(-)}. \end{aligned}$$

Definition A.3.2 ([Hov99, Definition 4.2.18])**.** Let $\mathcal{S}$ be a symmetric monoidal model category. A *$\mathcal{S}$-model structure* on a closed $\mathcal{S}$-module $\mathcal{C}$ is a model structure on $\mathcal{C}$ such that the following properties are fulfilled:

(**Pushout product**) Let $i : X \rightarrowtail Y$ be a cofibration in $\mathcal{C}$ and $j : A \rightarrowtail B$ be a cofibration in $\mathcal{S}$ then their pushout product

$$i \square j : (Y \otimes A) \coprod_{(X \otimes A)} (X \otimes B) \to Y \otimes B$$

is a cofibration in $\mathcal{C}$ and an acyclic cofibration if i or j is a weak equivalence.

(**Unit**) A unit property which is automatically fulfilled for a cofibrant unit $\mathbb{I}_{\mathcal{S}}$.

An s$\mathcal{S}$et-model category is called a *simplicial model category.*

Remark A.3.3. For a symmetric monoidal model category and cofibrant $X \in \mathcal{C}$, cofibrant $A \in \mathcal{S}$ and fibrant $Y \in \mathcal{C}$ the adjunctions (1) are Quillen adjunctions [Hov99, Remark 4.2.3]. The *simplicial mapping spaces* $\mathrm{s}\mathcal{S}\mathrm{et}(X,Y)$ of a simplicial model category are necessarily of the form $\mathrm{hom}_{\mathcal{C}}(X \otimes \Delta^{(\text{-})}, Y)$ [Hir03, Proposition 9.1.9].

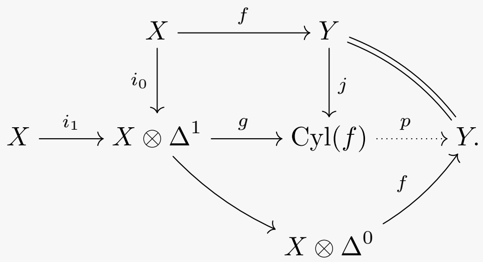
Theorem A.3.4 ([Hov99, Proposition 4.3.1])**.** Let $\mathcal{C}$ be an $\mathcal{S}$-model category for a symmetric monoidal model category $\mathcal{S}$. Then the associated homotopy category $\mathrm{Ho}\mathcal{C}$ is a closed $\mathrm{Ho}\mathcal{S}$-module with involved derived functors (on representants)

$$\begin{aligned} Q(-)\otimes Q(-) &: \mathrm{Ho}\mathcal{C} \quad \times \mathrm{Ho}\mathcal{S} \to \mathrm{Ho}\mathcal{C} \\ \mathcal{S}(Q(-),R(-)) &: \mathrm{Ho}\mathcal{C}^{\mathrm{op}} \times \mathrm{Ho}\mathcal{C} \to \mathrm{Ho}\mathcal{S} \\ R(-)^{Q(-)} &: \mathrm{Ho}\mathcal{S}^{\mathrm{op}} \times \mathrm{Ho}\mathcal{C} \to \mathrm{Ho}\mathcal{C}. \end{aligned}$$

Remark A.3.5. Using the theory of framings it is possible to equip the homotopy category of any model category and not only a simplicial one naturally with the structure of a closed $\mathrm{Hos}\mathcal{S}\mathrm{et}$-module [Hov99, Chapter 5].

Lemma A.3.6. Let $\mathcal{C}$ be a simplicial model category.

- For a cofibrant object X and a fibrant object Y of $\mathcal{C}$ there is a natural isomorphism of sets $\pi_0\mathrm{s}\mathcal{S}\mathrm{et}(X,Y)\cong[X,Y]$.
- The object $X\otimes\Delta^1$ is a cylinder object for a cofibrant object X.
- The *simplicial mapping cylinder* $\mathrm{Cyl}(f)$ of a map $f:X\to Y$ in $\mathcal{C}$ is defined as the pushout of the bottom inclusion i_0 along f in the diagram

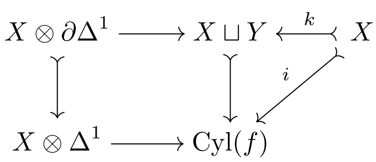


  If X and Y are cofibrant, there is a factorization $X\rightarrowtail\mathrm{Cyl}(f)\xrightarrow{\sim}Y$ of f as a cofibration $i=gi_1$ followed by a homotopy equivalence p and moreover, j is an acyclic cofibration. If Y is the terminal object, this construction gives the contractible *cone* $X\rightarrowtail CX$ *on* X.
- A map $f:X\to Y$ in $\mathcal{C}$ is a weak equivalence if and only if the induced map $\mathrm{s}\mathcal{S}\mathrm{et}(QY,Z)\to\mathrm{s}\mathcal{S}\mathrm{et}(QX,Z)$ is a weak equivalence for all fibrant $Z\in\mathcal{C}$.

Proof. The first statement follows from the natural isomorphism

$$\pi_0\mathrm{s}\mathcal{S}\mathrm{et}(X,Y)\cong[\Delta^0,\mathrm{s}\mathcal{S}\mathrm{et}(X,Y)]\cong[\Delta^0\otimes X,Y]\cong[X,Y].$$

The second statement is [GJ99, Lemma II.3.5]. For the third statement, we observe that i is a cofibration by writing the above pushout diagram slightly differently as



where the left square is again a pushout and cofibrations are stable under pushouts. Since Y is cofibrant, k is a cofibration. The map p is a homotopy equivalence since $Y\otimes\Delta^1$ is a cylinder object. The fourth statement follows from [Hir03, Theorem 8.3.10]. □

Definition A.3.7. Let $\mathcal{C}$ be a simplicial model category and M a class of morphisms in $\mathcal{C}$. A fibrant object Z of $\mathcal{C}$ is called *M-local*, if the map $\mathrm{s}\mathcal{S}\mathrm{et}(QY, Z) \to \mathrm{s}\mathcal{S}\mathrm{et}(QX, Z)$ is a weak equivalence of simplicial sets for every morphism $f : X \to Y$ in M.

A *M-local equivalence* is a morphism $f : X \to Y$ such that for every M-local object Z the map $\mathrm{s}\mathcal{S}\mathrm{et}(QY, Z) \to \mathrm{s}\mathcal{S}\mathrm{et}(QX, Z)$ is a weak equivalence.

Theorem A.3.8 ([Lur09, Proposition A.3.7.3])**.** For a left proper combinatorial simplicial model category $\mathcal{C}$ and a set of morphisms M, the *left Bousfield localization* $L_M\mathcal{C}$ of $\mathcal{C}$ at M exists. This is a left proper combinatorial simplicial model structure on the category $\mathcal{C}$ with the same cofibrations, exactly the M-local objects as fibrant objects and exactly the M-local equivalences as weak equivalences. One has a Quillen equivalence $\mathrm{id} : \mathcal{C} \rightleftarrows L_M\mathcal{C} : \mathrm{id}$.

Remark A.3.9. A Bousfield localization of a monoidal model category need not be monoidal again but it is for instance a sufficient condition that $A \wedge -$ preserves local acyclic cofibrations for all domains and codomains A in a set $\mathcal{I}$ of generating cofibrations [Hor06, Lemma 1.10].

For a Bousfield localization of a left proper combinatorial simplicial model category $\mathcal{C}$ at a set of morphisms M, there is the following criterion for identifying M-local objects by [Hir03, Lemma 3.3.11]: A fibrant object Z of $\mathcal{C}$ is *M-local* if and only if the morphism $Z \to *$ has the RLP with respect to the morphisms

$$f' \,\square\, (\partial\Delta^n \to \Delta^n) = (X' \otimes \Delta^n) \textstyle\coprod_{(X' \otimes \partial\Delta^n)} (Y' \otimes \partial\Delta^n) \to Y' \otimes \Delta^n$$

for all $n \geq 0$ and all morphisms $f : X \to Y$ in M to which one associates a cofibration $f' : X' \to Y'$ between cofibrant objects fitting into a diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \sim\uparrow & & \uparrow\sim \\ X' & \overset{f'}{\rightarrowtail} & Y'. \end{array}$$

Lemma A.3.10. Let $L_M\mathcal{C}$ be a Bousfield localization of $\mathcal{C}$ and $f : X \to Y$ a map between local fibrant objects. Then f is a weak equivalence (respectively a fibration) in $L_M\mathcal{C}$ if and only if it is a weak equivalence (respectively a fibration) in $\mathcal{C}$.

If $\mathcal{C}$ is left proper and if

$$\begin{array}{ccc} X & \xrightarrow{\sim} & X' \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & Y & \end{array}$$

is a diagram with properties indicated by the arrows in the model category $\mathcal{C}$, then p is a fibration in $L_M\mathcal{C}$ if and only if q is so. In particular, $L_M\mathcal{C}$-fibrancy for $\mathcal{C}$-fibrant objects is invariant under weak equivalence in $\mathcal{C}$.

Proof. By the Quillen adjunction of (1) and [Hir03, Theorem 17.7.7], a map $f : X \to Y$ between fibrant objects in a simplicial model category is a weak equivalence if and only if the morphism $\mathrm{s}\mathcal{S}\mathrm{et}(A, f)$ is a weak equivalence of simplicial sets for every cofibrant object A of $\mathcal{C}$. Since $\mathcal{C}$ and $L_M\mathcal{C}$ share the same cofibrant objects, a weak equivalence $f : X \to Y$ in $L_M\mathcal{C}$ between fibrant objects is also a weak equivalence in $\mathcal{C}$.

The statement about fibrations is [Hir03, Proposition 3.3.16] and for the latter statement it is referred to [Hir03, Proposition 3.4.6]. □

Remark A.3.11. Focusing again on homotopy pushouts (or homotopy pullbacks) as considered in Remark A.1.21, one is tempted to ask, whether the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & C \\ {\scriptstyle g}\downarrow & & \vdots \\ B & \cdots\to & \text{hocolim} \end{array}$$

is an universal cocone in a certain sense. Let $\mathcal{C}$ be a combinatorial and simplicial model category and let $D : \mathcal{I} \to \mathcal{C}$ be a diagram. The coend construction defines two left Quillen functors

$$\begin{aligned}(-) \otimes_{\mathcal{I}} (-) = \int^{i\in\mathcal{I}} (-)(i) \otimes (-)(i) : \mathcal{F}\mathrm{un}(\mathcal{I}, \mathcal{C})_{proj} \times \mathcal{F}\mathrm{un}(\mathcal{I}^{\mathrm{op}}, \mathrm{s}\mathcal{S}\mathrm{et})_{inj} &\to \mathcal{C} \\ \mathcal{F}\mathrm{un}(\mathcal{I}, \mathcal{C})_{inj} \times \mathcal{F}\mathrm{un}(\mathcal{I}^{\mathrm{op}}, \mathrm{s}\mathcal{S}\mathrm{et})_{proj} &\to \mathcal{C}\end{aligned}$$

by [Gam10, Theorem 3.2, 3.3]. This expresses the homotopy colimit of $\mathcal{D}$ as

$$\begin{aligned}\mathrm{colim}\ \tilde{Q}\mathcal{D} &\cong \int^{i\in\mathcal{I}} (\tilde{Q}D)(i) \otimes (\Delta^0)(i) \\ &\cong \int^{i\in\mathcal{I}} QD(i) \otimes (\tilde{Q}*)(i) \\ &\cong \int^{i\in\mathcal{I}} QD(i) \otimes B(i \downarrow \mathcal{I})^{\mathrm{op}}\end{aligned}$$

where Q is a cofibrant replacement functor in $\mathcal{C}$, where $\tilde{Q}$ is a cofibrant replacement functor for the projective structure and where $*$ denotes the terminal diagram. The last equality is implied by calculating an explicit cofibrant replacement of the terminal diagram [Hir03, Proposition 14.8.9]. From this observation one obtains again (confer Remark A.1.21) that left Quillen functors preserve homotopy colimits as they commute with colimits and preserve cofibrant objects.

As shown for instance in [Shu09, Theorem 9.1], this may be rewritten further as

$$\begin{aligned}\mathrm{hocolim} D &\sim |B_{(-)}(*, \mathcal{I}, QD)| \\ &\cong |\coprod_{i_0 \to \ldots \to i_{(-)}} QD(i_0)| \\ &\cong \int^{n\in\Delta} \left(\coprod_{i_0\to\ldots\to i_n} QD(i_0) \right) \otimes \Delta^n.\end{aligned}$$

The right notation of cocones as described in [CP97] can be deduced from this observations as follows: A categorical colimit of a diagram D is characterized by the isomorphism $\hom(\mathrm{colim} D, T) \cong \hom(D, \Delta T)$. For the homotopy colimit in a simplicial model category and a fibrant object T, we have

$$\mathrm{s}\mathcal{S}\mathrm{et}(\mathrm{hocolim} D, T) \sim \int_{n\in\Delta^{\mathrm{op}}} \mathrm{s}\mathcal{S}\mathrm{et}(\coprod_{i_0\to\ldots\to i_n} QD(i_0) \otimes \Delta^n, T).$$

The latter simplicial set is the space of *coherent cocones* $Coh(D, \Delta T)$.

A zero-simplex of this simplicial set is giving for each string of composable morphisms $i_0 \to ... \to i_n$ in $\mathcal{I}$ a morphism $QD(i_0) \otimes \Delta^n \to T$ which may be interpreted as a higher homotopy. For special diagrams like $\cdot \leftarrow \cdot \rightarrow \cdot$ one gets the description of a homotopy pushout as in [Mat76] for example.

Theorem A.3.12 ([Hir05, Theorem 1.5], [Sch11, Proposition 2.1.37, 2.1.38])**.** Let $\mathcal{C}$ be an $(\mathcal{I}, \mathcal{J})$-cofibrantly generated, proper, combinatorial and simplicial model category. Let X be an object of $\mathcal{C}$ and $\mathcal{I}_X$ (respectively $\mathcal{J}_X$) be the set of maps $(E \to X) \to (D \to X)$ of the over category $\mathcal{C}/X$ where $E \to D$ is a map of $\mathcal{I}$ (respectively of $\mathcal{J}$). Then, there is an $(\mathcal{I}_X, \mathcal{J}_X)$-cofibrantly generated, proper, combinatorial and simplicial model structure on $\mathcal{C}/X$, called the *over model structure*, where a map $(E \to X) \to (D \to X)$ is a weak equivalence (respectively a cofibration, respectively a fibration) if and only if $E \to D$ is a weak equivalence (respectively cofibration, respectively fibration) of $\mathcal{C}$.

A.4. Pointed model categories

Definition A.4.1. A category is called *pointed* if it has a zero object.

The *wedge product* $X \vee Y$ of two objects in a bicomplete pointed category $\mathcal{C}_*$ is given by the pushout of the diagram $X \leftarrow * \rightarrow Y$. This construction defines a functor

$$- \vee - : \mathcal{C}_* \times \mathcal{C}_* \to \mathcal{C}_*.$$

The *smash product* $X \wedge Y$ of two objects in a pointed monoidal category $\mathcal{C}_*$ whose unit is the terminal object is given by the quotient $X \otimes Y/X \vee Y$. This construction defines a functor

$$- \wedge - : \mathcal{C}_* \times \mathcal{C}_* \to \mathcal{C}_*$$

with $X \wedge * \cong *$.

Remark A.4.2. The wedge product is the coproduct in $\mathcal{C}_*$ but the smash product is not necessarily the product in $\mathcal{C}_*$ if $\otimes$ is the product in $\mathcal{C}$. For a bicomplete category $\mathcal{C}$ with terminal object $*$, the *under category* $*{\downarrow}\mathcal{C}$ is denoted by $\mathcal{C}_*$ and it is a bicomplete pointed category with zero object $* \to *$ for which we write $(*, *)$ or just $*$. There is an adjunction

$$(-)_* : \mathcal{C} \rightleftarrows \mathcal{C}_* : (-)_-$$

given by $X_* = (X \sqcup *, *)$ and $(X, x)_- = X$. If $\mathcal{C}$ is already pointed, this is an equivalence of categories.

Lemma A.4.3 ([Hov99, Proposition 1.1.8, 1.3.5, 1.3.17, Lemma 2.1.21], [Hir05, Theorem 2.8])**.** The following statements are true for a model category $\mathcal{C}$.

- There is a model structure on $\mathcal{C}_*$ where cofibrations, fibrations and weak equivalences are preserved and detected by the right adjoint $(-)_-$. The morphism sets in its homotopy category are denoted by $\langle -, - \rangle$.
- If $\mathcal{C}$ is $(\mathcal{I}, \mathcal{J})$-cofibrantly generated then $\mathcal{C}_*$ is $(\mathcal{I}_*, \mathcal{J}_*)$-cofibrantly generated.
- Left properness, right properness and combinatoriality of $\mathcal{C}$ lifts also to the model structure on $\mathcal{C}_*$.
- A Quillen adjunction $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ induces a Quillen adjunction on the pointed model categories $F_* : \mathcal{C}_* \rightleftarrows \mathcal{D}_* : G_*$ with $F_*(X_*) \cong F(X)_*$

naturally and where $F_*(X,x)$ is defined by the pushout

$$\begin{array}{ccc} F(*_{\mathcal{C}}) & \xrightarrow{Fx} & F(X) \\ \downarrow & & \downarrow \\ *_{\mathcal{D}} & \longrightarrow & F_*(X,x). \end{array}$$

- The property of a Quillen adjunction to be a Quillen equivalence lifts also to the pointed model structures if the terminal object $*$ of $\mathcal{C}$ is cofibrant and the left adjoint F preserves the terminal object.

Remark A.4.4. A symmetric monoidal structure $(\mathcal{C}, \times, *)$ does not necessarily induce a symmetric monoidal structure $(\mathcal{C}_*, \wedge, *_*)$ since for example the smash product of topological spaces is not associative.

Lemma A.4.5 ([Hov99, Proposition 4.2.9])**.** Let $(\mathcal{C}, \otimes, *)$ be a symmetric monoidal model category such that the unit $*$ is cofibrant and the terminal object. Then the model structure on $(\mathcal{C}_*, \wedge, *_*)$ is symmetric monoidal. Moreover, $(\_{}_*, \_)$ is a strong monoidal Quillen adjunction, so $(X \wedge Y)_* \cong X_* \otimes Y_*$.

Remark A.4.6. Let $\mathcal{C}$ be a left proper symmetric monoidal model category such that the unit $*$ is cofibrant and the terminal object. Sometimes the smash product

$$- \wedge - : \mathrm{Ho}\mathcal{C}_* \times \mathrm{Ho}\mathcal{C}_* \to \mathrm{Ho}\mathcal{C}_*$$

is defined via the homotopy cofiber square

$$\begin{array}{ccc} X \vee Y & \longrightarrow & X \otimes Y \\ \downarrow & & \downarrow \\ * & \longrightarrow & X \wedge Y. \end{array}$$

On the other hand, the last lemma and Theorem A.2.7 equip $\mathrm{Ho}\mathcal{C}_*$ with a closed monoidal structure with respect to the derived smash product $- \wedge^{\mathcal{L}} -$. These two constructions coincide since the map $X \vee Y \rightarrowtail X \otimes Y$ is a cofibration by the pushout product property in $\mathcal{C}$ for cofibrant X and Y and hence the homotopy pushout is actually a pushout in this case.

Lemma A.4.7 ([Hov99, Section 6.5., Proposition 4.2.19])**.** Let $\mathcal{S}$ be a symmetric monoidal model category with a cofibrant terminal object as its unit, and $\mathcal{C}$ an $\mathcal{S}$-model category. Then $\mathcal{C}_*$ is an $\mathcal{S}_*$-model category with monoidal structure

$$- \wedge - : \mathcal{C}_* \times \mathcal{S}_* \to \mathcal{C}_*.$$

defined as the quotient $X \otimes A / X \vee A$ where $X \vee A$ is the pushout of the diagram $* \otimes A \leftarrow * \otimes \mathbb{I} \to X \otimes \mathbb{I}$. The pullback $\mathcal{S}_*(X,Y)$ of $\mathbb{I} \to \mathcal{S}(*,Y) \leftarrow \mathcal{S}(X,Y)$ based by the zero map defines the functor $\mathcal{S}_*(-,-)$ and $(-)^{(-)}$ is given analogously. An $\mathcal{S}_*$-model category $\mathcal{C} \neq \emptyset$ is necessarily pointed and also an $\mathcal{S}$-model category by setting:

$$\begin{aligned} X \otimes A &= X \wedge A_* \\ \mathcal{S}(X,Y) &= \mathcal{S}_*(X,Y)_{\_} \\ X^A &= X^{A_*}. \end{aligned}$$

Remark A.4.8. By the above lemma, a pointed $\mathcal{S}$-model category can be considered as an $\mathcal{S}_*$-model category and vice versa. A pointed simplicial set is the same as a functor from Δ^{op} to the category $\mathcal{S}\mathrm{et}_*$ of pointed sets. The pointed simplicial mapping spaces $\mathrm{s}\mathcal{S}\mathrm{et}_*(X,Y)$ of an $\mathrm{s}\mathcal{S}\mathrm{et}_*$-model category are necessarily of the form $\hom_{\mathcal{C}_*}(X \wedge \Delta^{(\text{-})}_*, Y)$ since

$$\mathrm{s}\mathcal{S}\mathrm{et}_*(X,Y)_n \cong \hom_{\mathrm{s}\mathcal{S}\mathrm{et}_*}(\Delta^n_*, \mathrm{s}\mathcal{S}\mathrm{et}_*(X,Y)) \cong \hom_{\mathcal{C}_*}(X \wedge \Delta^n_*, Y).$$

Definition A.4.9. Let $\mathcal{C}_*$ be a pointed model category. The *suspension functor* and the *loop functor* which are part of an adjunction

$$\Sigma : \mathrm{Ho}\mathcal{C}_* \rightleftarrows \mathrm{Ho}\mathcal{C}_* : \Omega$$

are defined by the homotopy pushout and homotopy pullback

$$\begin{array}{ccc} X & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & \Sigma X \end{array} \qquad \begin{array}{ccc} \Omega X & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & X \end{array}$$

respectively, or more precisely and as $\mathcal{C}_*$-endofunctors $\Sigma X = \mathrm{hocolim}(* \leftarrow X \rightarrow *)$ and $\Omega X = \mathrm{holim}(* \rightarrow X \leftarrow *)$.

Lemma A.4.10. Let $\mathcal{C}_*$ be a left proper pointed simplicial model category. The adjunction (Σ, Ω) is the derived adjunction of the Quillen adjunction

$$- \wedge S^1 : \mathcal{C}_* \rightleftarrows \mathcal{C}_* : (-)^{S^1}.$$

where S^1 denotes the pointed simplicial set $\Delta^1/\partial\Delta^1$.

Proof. We show only that the left derived functor of $- \wedge S^1$ is the suspension functor since the statement for the right adjoint is dual. Let X be a cofibrant object of $\mathcal{C}_*$. One has the pushout diagram

$$\begin{array}{ccc} X & \rightarrowtail & * \\ \downarrow & & \downarrow \\ X \vee X & \longrightarrow & X \end{array}$$

which is also a homotopy pushout. We have $X \vee X \cong X \wedge \partial\Delta^1_*$. Moreover one has $X \wedge \Delta^1_* \cong X \otimes \Delta^1 \xrightarrow{\sim} X$ since $X \otimes \Delta^1$ is a cylinder object for X. The pointed simplicial set S^1 is the cokernel of the cofibration $\partial\Delta^1_* \rightarrowtail \Delta^1_*$. In a pointed model category, it holds $X \wedge * \cong *$ since $X \wedge -$ is a left adjoint and hence respects in particular the colimit over the empty diagram. Applying the functor $X \wedge -$ to the cofibration diagram defining S^1 yields a pushout diagram

$$\begin{array}{ccc} X \wedge \partial\Delta^1_* & \rightarrowtail & X \wedge \Delta^1_* \\ \downarrow & & \downarrow \\ * & \longrightarrow & X \wedge S^1 \end{array}$$

which is also a homotopy pushout. The combination of the two homotopy pushout diagrams is again a homotopy pushout. This yields the claimed result. □

Lemma A.4.11 ([Hov99, Lemma 6.1.2])**.** Let $\mathcal{C}_*$ be a pointed simplicial model category, X be a cofibrant and Y a fibrant object of $\mathcal{C}_*$. Then there are natural isomorphisms

$$\pi_n \mathrm{s}\mathcal{S}\mathrm{et}_*(X,Y) \cong \langle \Sigma^n X, Y\rangle \cong \langle X, \Omega^n Y\rangle$$

and in particular $\langle \Sigma^n X, Y\rangle$ is a group for $n \geq 1$ which is abelian if $n \geq 2$.

A.5. Stable model categories

Throughout the whole section we fix an $(\mathcal{I},\mathcal{J})$-cofibrantly generated proper, combinatorial, simplicial, monoidal and pointed model category $\mathcal{C}$ whose unit S we assume to be cofibrant. Moreover, we fix a cofibrant object T. One has a Quillen adjunction

$$- \wedge T : \mathcal{C} \rightleftarrows \mathcal{C} : \underline{\mathrm{hom}}(T,-)$$

which will often be abbreviated by (Σ_T, Ω_T) in the sequel. By abuse of notation, we will denote the derived adjunction $\Sigma_T : \mathrm{Ho}\mathcal{C} \rightleftarrows \mathrm{Ho}\mathcal{C} : \Omega_T$ by the same symbols.

Definition A.5.1. The model category $\mathcal{C}$ is called *T-stable* if (Σ_T, Ω_T) is a Quillen equivalence.

Definition A.5.2. A *T-spectrum* X is a sequence $X_0, X_1, \dots$ of (pointed) objects of $\mathcal{C}$ together with structure maps $\sigma_n : \Sigma_T X_n \to X_{n+1}$ for all natural numbers n. A morphism $f : X \to Y$ of spectra is a sequence $f_n : X_n \to Y_n$ of maps such that for all n the diagram

$$\begin{array}{ccc} \Sigma_T X_n & \xrightarrow{\sigma} & X_{n+1} \\ {\scriptstyle \Sigma_T f_n}\downarrow & & \downarrow{\scriptstyle f_{n+1}} \\ \Sigma_T Y_n & \xrightarrow{\sigma} & Y_{n+1} \end{array}$$

is commutative. This defines the pointed category $\mathcal{S}p_T$ or $\mathcal{S}p_T(\mathcal{C})$ of T-spectra.

Remark A.5.3. There is an adjunction

$$\Sigma_T^\infty : \mathcal{C} \rightleftarrows \mathcal{S}p_T : (-)_0$$

where the *suspension spectrum functor* Σ_T^∞ sends an object X of $\mathcal{C}$ to the spectrum $X, \Sigma_T X, \Sigma_T^2 X, \dots$ with the identities as structure maps. The *T-sphere spectrum* $\mathbb{S}_T$ (or T or T^0, by abuse of notation) is the object $\Sigma_T^\infty S^0$ of $\mathcal{S}p_T$ where $S^0 = *_*$ is the point with a disjoint basepoint.

For every natural number m, there is an adjunction

$$\text{(shift to the right)} \qquad [-m] : \mathcal{S}p_T \rightleftarrows \mathcal{S}p_T : [m] \qquad \text{(shift to the left)}$$

of *shift functors* defined by $X[m]_n = X_{m+n}$ where we set $X_{<0} = *$.

Combining these two adjunctions yields

$$\Sigma_T^\infty[-m] : \mathcal{C} \rightleftarrows \mathcal{S}p_T : (-)_m$$

with unit $\Sigma_T^\infty[-m]_m$ the identity on $\mathcal{C}$ for any natural number m. One sometimes writes $F_m X$ instead of $\Sigma_T^\infty X[-m]$ and calls this the *free spectrum* generated by X in level m. Via Σ_T^∞, the category $\mathcal{C}$ is a full subcategory of $\mathcal{S}p_T$. The functor $(-)_m$ is also a left adjoint and hence commutes with colimits [Hov01b, Remark 1.4].

Remark A.5.4. The category $\mathcal{S}p_T$ is isomorphic to the category $\mathcal{F}\mathrm{un}(T\mathcal{S}\mathrm{ph}, \mathcal{C})$ of enriched functors from the $\mathcal{C}$-category $T\mathcal{S}\mathrm{ph}$ to the $\mathcal{C}$-category $\mathcal{C}$. The objects of $T\mathcal{S}\mathrm{ph}$ are the natural numbers and one sets $\mathcal{C}(m,n) = T^{n-m}$ for $n \geq m \geq 0$ and $\mathcal{C}(m,n) = *$ otherwise. To a $\mathcal{C}$-functor $X : T\mathcal{S}\mathrm{ph} \to \mathcal{C}$ one associates the spectrum with $X_n = X(n)$ and structure maps $X_n \wedge T \to X_{n+1}$ adjoint to

$$T \cong \mathcal{C}(n, n+1) \to \mathcal{C}(X(n), X(n+1))$$

as it is described in [DRØ03a, Proposition 2.12]. By [DRØ03a, Lemma 2.2], the category of spectra is thus bicomplete and limits and colimits are calculated objectwise. Every spectrum X may be written as the coend

$$X(-) \cong \int^{m \in \mathbb{N}} \mathcal{C}(m, -) \wedge X_m \cong \int^{m \in \mathbb{N}} F_m X_m$$

[DRØ03a, Corollary 2.3].

Theorem A.5.5. There exists an $(\mathcal{I}_T, \mathcal{J}_T)$-cofibrantly generated model structure on $\mathcal{S}p_T$ called the *(projective) unstable model structure* with weak equivalences and fibrations defined objectwise and with a map $A \to B$ being a (respectively acyclic) cofibration if and only if

- $A_0 \rightarrowtail B_0$ is a (respectively acyclic) cofibration and
- $A_n \vee_{\Sigma_T A_{n-1}} \Sigma_T B_{n-1} \rightarrowtail B_n$ for $n \geq 1$ is is a (respectively acyclic) cofibration.

Its weak equivalences are called *unstable weak equivalences*, its cofibrations *unstable cofibrations* and its fibrations *unstable fibrations*. The generating cofibrations and generating acyclic cofibrations are given by

$$\mathcal{I}_T = \bigcup_{m \in \mathbb{N}} \Sigma_T^\infty(\mathcal{I})[-m] \quad \text{and} \quad \mathcal{J}_T = \bigcup_{m \in \mathbb{N}} \Sigma_T^\infty(\mathcal{J})[-m]$$

respectively. Properness lifts from $\mathcal{C}$ to $\mathcal{S}p_T$ as well as combinatoriality and being a pointed s$\mathcal{S}$et model category with structure maps $- \otimes A$ and $(-)^A$ defined levelwise for any simplicial set A [Hov01b, Theorem 1.13], [DRØ03a, Proposition 2.12].

Remark A.5.6. The adjunctions considered in Remark A.5.3 are Quillen adjunctions with respect to the unstable model structure on $\mathcal{S}p_T$. The homotopy category $\mathrm{Ho}\mathcal{C}$ embeds via Σ_T^∞ into the homotopy category $\mathrm{Ho}\mathcal{S}p_T$ with respect to the unstable model structure.

Definition A.5.7. Let $F : \mathcal{C} \to \mathcal{C}$ be a functor and $\gamma : FX \wedge T \to F(X \wedge T)$ a natural transformation. The pair (F, γ) induces a functor called the *prolongation* of F with respect to γ

$$F : \mathcal{S}p_T \to \mathcal{S}p_T$$

which is by abuse of language also denoted by F and which is given by $(FX)_n = FX_n$ together with the structure map

$$\Sigma_T(FX)_n = \Sigma_T(FX_n) \xrightarrow{\gamma} F(\Sigma_T X_n) \xrightarrow{F(\sigma)} F(X_{n+1}) = (FX)_{n+1}$$

for every natural number n.

Remark A.5.8. The prolongation of a functor depends on the choice of the involved natural transformation. Moreover, the diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\Sigma_T^\infty} & \mathcal{S}p_T \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F} \\ \mathcal{C} & \xrightarrow{\Sigma_T^\infty} & \mathcal{S}p_T \end{array}$$

does usually not commute but one gets a natural transformation $\Sigma_T^\infty F \to F\Sigma_T^\infty$ which is an isomorphism if γ is so. There are at least two canonical possibilities to prolong the functor $\Sigma_T : \mathcal{C} \to \mathcal{C}$ to the category of spectra. One the one hand, one may choose the identity $\Sigma_T\Sigma_T X \to \Sigma_T\Sigma_T X$ as the involved natural transformation and on the other hand, one could pick the natural transformation which twists the T-factors. The first prolonged functor Σ_T^ℓ is called the *fake T-suspension functor* and the second prolonged functor Σ_T, involving the twist, is called the *T-suspension.* We have an adjunction

$$\Sigma_T^\ell : \mathcal{S}p_T \rightleftarrows \mathcal{S}p_T : \Omega_T^\ell$$

where the natural transformation defining the prolonged right adjoint *fake T-loop functor* Ω_T^ℓ is $\Sigma_T\Omega_T X \xrightarrow{\epsilon} X \xrightarrow{\eta} \Omega_T\Sigma_T X$, the composition of the counit and the unit of the adjunction in $\mathcal{C}$. Moreover, there is an adjunction

$$\Sigma_T : \mathcal{S}p_T \rightleftarrows \mathcal{S}p_T : \Omega_T$$

where the natural transformation defining the *T-loop functor* Ω_T is the adjoint of $\Sigma_T\Sigma_T\Omega_T X \xrightarrow{\tau\wedge 1} \Sigma_T\Sigma_T\Omega_T X \xrightarrow{\Sigma_T\epsilon} \Sigma_T X$ with the twist τ. Both adjunctions are Quillen adjunctions with respect to the unstable model structure on spectra since the structure maps do not matter to detect weak equivalences and fibrations. For all natural m, we have $\Sigma_T^\ell\Sigma_T^\infty[-m] \cong \Sigma_T^\infty[-m]\Sigma_T \cong \Sigma_T\Sigma_T^\infty[-m]$ since the right adjoints $(\Omega_T(-))_m$ and $\Omega_T((-)_m)$ coincide.

Definition A.5.9. A spectrum X is called an *Ω_T-spectrum*, if X is fibrant in the unstable model structure and if for all natural n the adjoint $X_n \xrightarrow{\sim} \Omega_T X_{n+1}$ of the structure map is a weak equivalence in $\mathcal{C}$.

Definition A.5.10. The Bousfield localization of the unstable model structure on $\mathcal{S}p_T$ at the set

$$M = \bigcup_{m\geq 0} \bigcup_{\substack{A\in dom\mathcal{I} \\ A\in codom\mathcal{I}}} \{\Sigma_T^\infty\Sigma_T QA[-(m+1)] \xrightarrow{s_m} \Sigma_T^\infty QA[-m]\}$$

where Q denotes the cofibrant replacement functor of $\mathcal{C}$ and where s_n is the adjoint morphism under the adjunction $(\Sigma_T^\infty[-(m+1)],(-)_{m+1})$ of the identity morphism $\Sigma_T QA \to \Sigma_T QA = \Sigma_T^\infty QA[-m]_{m+1}$ exists by Theorem A.3.8 and Theorem A.5.5 and is called the *(projective) stable model structure.* Its weak equivalences are called *stable weak equivalences*, its cofibrations *stable cofibrations* and its fibrations *stable fibrations.*

Remark A.5.11. Since the identity is a Quillen adjunction from the unstable to the stable model structure on $\mathcal{S}p_T$, all the adjunctions of Remark A.5.3 are still Quillen adjunctions for the stable model structure.

Remark A.5.12. The given definition of the stable model structure on T-spectra is very abstract. The following observations make it possible to deal with this stable model structure in a more handy way when $\mathcal{C}$ fulfills some properties summarized in the following Definition A.5.13.

Definition A.5.13. A model category $\mathcal{C}$ (having the properties assumed throughout this section) is called *well stabilizable*, if $\mathcal{C}$ is weakly finitely generated and sequential colimits commute with finite products.

Remark A.5.14. The usual model structures on simplicial sets and chain complexes are well stabilizable but the usual model structure on topological spaces is not.

Definition A.5.15. The *T-stabilization functor* $\Theta_T^\infty : \mathcal{S}p_T \to \mathcal{S}p_T$ and the natural map $X \to \Theta_T^\infty X$ are defined by

$$\Theta_T^\infty X = \operatorname{colim}(X \xrightarrow{\sigma_*} \Omega_T^\ell X[1] \xrightarrow{\Omega_T^\ell \sigma_*[1]} (\Omega_T^\ell)^2 X[2] \xrightarrow{(\Omega_T^\ell)^2\sigma_*[2]} (\Omega_T^\ell)^3 X[3] \to \dots)$$

where $\sigma_* : X \to \Omega_T^\ell X[1]$ is induced by the adjoints of the structure maps of X.

Theorem A.5.16. Let $\mathcal{C}$ be well stabilizable and T be finitely presentable. Then the following statements are true where R denotes a fibrant replacement functor for the unstable model structure on $\mathcal{S}p_T$.

- The stably fibrant objects are exactly the Ω_T-spectra.
- If X is unstably fibrant then $\Theta^\infty X$ is an Ω_T-spectrum.
- If X is unstably fibrant then $X \to \Theta^\infty X$ is a stable weak equivalence.
- A map $f : X \to Y$ of spectra is a stable weak equivalence if and only if the map $\Theta^\infty Rf$ is an unstable weak equivalence.
- The stable model structure on $\mathcal{S}p_T$ is proper.
- A map $f : X \to Y$ of spectra is a stable fibration if and only if it is an unstable fibration and

$$\begin{array}{ccc} X & \longrightarrow & \Theta^\infty RX \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \Theta^\infty Rf} \\ Y & \longrightarrow & \Theta^\infty RY \end{array}$$

is a homotopy pullback in the unstable model structure.

Proof. The first statement is [Hov01b, Theorem 3.4]. Cellularity is used there only to guarantee that Bousfield localizations exist. In view of Theorem A.3.8, cellularity may then be replaced by combinatoriality. The second statement is [Hov01b, Proposition 4.6], the third statement is [Hov01b, Proposition 4.7] and the fourth statement is [Hov01b, Theorem 4.9]. The fifth and sixth statements are [Hov01b, Corollary 4.14]. □

Definition A.5.17. An *interval* $(i_0 \vee i_1 : S \vee S \to I \wedge I, p_S : I \wedge I \to S)$ is a cylinder object for the unit S such that there exists a map $H : I \wedge I \to I$ with the property that $H(1 \wedge i_0) = H(i_0 \wedge 1) = i_0 p$ and $H(1 \wedge i_1) = H(i_1 \wedge 1) = 1$. The cofibrant object T of $\mathcal{C}$ is called *symmetric*, if there exists an unit interval and a homotopy $T \wedge T \wedge T \wedge I \to T \wedge T \wedge T$ from the cyclic permutation to the identity [Hov01b, Definition 10.2].

Remark A.5.18. The simplicial set Δ^1 is an interval in s$\mathcal{S}$et. Left adjoints of monoidal Quillen adjunctions preserve intervals [Hov01b, Section 10].

Theorem A.5.19. Suppose that T is weakly equivalent to a symmetric object and the domains of the generating cofibrations $\mathcal{I}$ of $\mathcal{C}$ are cofibrant. Then the following statements are true with respect to the stable model structure on $\mathcal{S}p_T$.

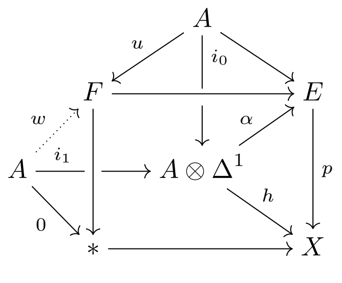
- The adjunction $\Sigma_T^\ell : \mathcal{S}p_T \rightleftarrows \mathcal{S}p_T : \Omega_T^\ell$ is a Quillen equivalence.
- All the adjunctions $[-m] : \mathcal{S}p_T \rightleftarrows \mathcal{S}p_T : [m]$ are Quillen equivalences.
- The adjunction $\Sigma_T : \mathcal{S}p_T \rightleftarrows \mathcal{S}p_T : \Omega_T$ is a Quillen equivalence.
- The derived functors of Σ_T, Σ_T^ℓ and $[1]$ are naturally isomorphic.
- The derived functors of Ω_T, Ω_T^ℓ and $[-1]$ are naturally isomorphic.
- If $\mathcal{C}$ is already stable then $\Sigma_T^\infty : \mathcal{C} \rightleftarrows \mathcal{S}p_T : (-)_0$ is a Quillen equivalence.

Proof. The first two statements are [Hov01b, Theorem 3.9], the third is obtained from [Hov01b, Theorem 10.3]. The isomorphism of the derived functors of Σ_T^ℓ, $[1]$ and Ω_T^ℓ, $[-1]$ respectively is [Hov01b, Theorem 3.9]. The proof of [Hov01b, Theorem 10.3] shows that the derived functors of $\Sigma_T\Sigma_T$ and $\Sigma_T^\ell\Sigma_T^\ell$ are naturally isomorphic and because Σ_T^ℓ is an equivalence of categories, Σ_T is an equivalence as well. The sixth statement is [Hov01b, Theorem 5.1]. □

A.6. Fiber and cofiber sequences

Throughout the whole section, let $\mathcal{C}$ be a cofibrantly generated proper, combinatorial, simplicial pointed model category. Since the notations for cofiber sequences are dual and more frequent in the literature, we restrict ourself to fiber sequences in the following only.

Definition A.6.1. Let $p : E \to X$ be a fibration of fibrant objects with categorical fiber $F = p^{-1}(*)$. The *canonical action* on the fiber $\phi : F \times \Omega X \to F$ is particularly a morphism in Ho$\mathcal{C}$ defined by constructing a morphism $\langle A, F\rangle \times \langle \Sigma A, X\rangle \to \langle A, F\rangle$ for every A as follows: A pair $(\langle u\rangle, \langle h\rangle)$ is mapped to $\langle w\rangle$ where in the diagram



the morphism α is a lift and w is defined by the pullback property. A *pre-fiber sequence* is a diagram

$$F \to E \xrightarrow{p} X$$

in Ho$\mathcal{C}$ together with an action $\varphi : F \times \Omega X \to F$.

A *morphism of pre-fiber sequences* is a diagram

$$\begin{array}{ccccc} F & \longrightarrow & E & \xrightarrow{p} & X \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} \\ F' & \longrightarrow & E' & \xrightarrow{p'} & X' \end{array}$$

in Ho$\mathcal{C}$ such that the induced diagram

$$\begin{array}{ccc} F \times \Omega X & \xrightarrow{\varphi} & F \\ \downarrow{\scriptstyle a\times\Omega c} & & \downarrow{\scriptstyle a} \\ F' \times \Omega X' & \xrightarrow{\varphi'} & F' \end{array}$$

in Ho$\mathcal{C}$ commutes.

A *fiber sequence* is a pre-fiber sequence which is isomorphic to a pre-fiber sequence

$$F \to E \xrightarrow{p} X$$

where p is a fibration of fibrant objects and the action is the canonical one on the fiber $F = p^{-1}(*)$.

Remark A.6.2. There is a *boundary morphism* $\partial : \Omega X \xrightarrow{(0,\mathrm{id})} F \times \Omega X \xrightarrow{\varphi} F$ induced by the action of a fiber sequence. If the canonical map $\Omega X \vee F \to \Omega X \times F$ is a weak equivalence as it is in a stable model category, the action is recovered by the boundary morphism via $\Omega X \times F \cong \Omega X \vee F \xrightarrow{\partial+\mathrm{id}} F$. In this case, a morphism of fiber sequences is exactly a commutative diagram

$$\begin{array}{ccccccc} \Omega X & \xrightarrow{\partial} & F & \longrightarrow & E & \xrightarrow{p} & X \\ \downarrow{\scriptstyle \Omega c} & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} \\ \Omega X' & \xrightarrow{\partial'} & F' & \longrightarrow & E' & \xrightarrow{p'} & X' \end{array}$$

in Ho$\mathcal{C}$.

Lemma A.6.3 ([Hov99, Proposition 6.5.3])**.** For an object A and a fiber sequence $F \to E \to X$, there is a long exact sequence of pointed sets (respectively (respectively abelian) groups)

$$\ldots \to \langle A, \Omega^2 X\rangle \to \langle A, \Omega F\rangle \to \langle A, \Omega E\rangle \to \langle A, \Omega X\rangle \to \langle A, F\rangle \to \langle A, E\rangle \to \langle A, X\rangle.$$

Remark A.6.4. Dually, for a cofiber sequence $A \to B \to C$ and an object X, there is a long exact sequence

$$\ldots \to \langle \Sigma^2 A, X\rangle \to \langle \Sigma C, X\rangle \to \langle \Sigma B, X\rangle \to \langle \Sigma A, X\rangle \to \langle C, X\rangle \to \langle B, X\rangle \to \langle A, X\rangle.$$

Theorem A.6.5. The following assertions are true and the dual statements for cofiber sequences hold as well.

(**Identity**) The diagram $E \xrightarrow{\mathrm{id}} E \to *$ can be given the structure of a fiber sequence.

(**Mapping cocone**) For every morphism $p : E \to X$ there is a fiber sequence

$$F \to E \xrightarrow{p} X.$$

(**Shifting**) If

$$F \xrightarrow{i} E \xrightarrow{p} X$$

is a fiber sequence,

then there is a fiber sequence

$$\Omega X \xrightarrow{\partial} F \xrightarrow{i} E.$$

(**Extensibility**) For every commutative diagram

$$\begin{array}{ccccc} F & \xrightarrow{i} & E & \xrightarrow{p} & X \\ & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} \\ F' & \xrightarrow{i'} & E' & \xrightarrow{p'} & X' \end{array}$$

in Ho$\mathcal{C}$, where the rows are fiber sequences, there exists a (not necessarily unique) fill-in morphism $a : F \to F'$ defining together with b and c a morphism of fiber sequences and particularly making the whole diagram commutative. If b and c are isomorphisms, then so is a.

Proof. The first statement is [Hov99, Lemma 6.3.2]. The mapping cocone statement is [Hov99, Lemma 6.3.3] and the shifting is [Hov99, Proposition 6.3.4]. The statement on extensibility is [Hov99, Proposition 6.3.5] and [Hov99, Proposition 6.5.3]. □

Remark A.6.6. The non-uniqueness of the fill-in morphism in the extensibility statement is closely related to the fact that a homotopy (co)limit is unique up to isomorphism in the homotopy category but not unique up to unique isomorphism although this is true for categorical (co)limits.

The following lemma shows, that the work with fiber sequences is easier in right proper model categories.

Lemma A.6.7. Let $\mathcal{C}$ be right proper. For a morphism $p : E \to X$ in Ho$\mathcal{C}$ there is up to isomorphism of fiber sequences only one fiber sequence

$$\mathrm{hofib} \to E \xrightarrow{p} X.$$

Proof. The extensibility statement of Theorem A.6.5 shows the uniqueness and hence we have only to show that there is a diagram

$$\mathrm{hofib} \to E \xrightarrow{p} X$$

in Ho$\mathcal{C}$ which can be furnished with the structure of a fiber sequence. Factorize a representant in $\mathcal{C}$ of the Ho$\mathcal{C}$-morphism $QE \xrightarrow{\sim} E \xrightarrow{p} X \xrightarrow{\sim} RX$ as a acyclic cofibration followed by a fibration. By Remark A.1.21, the categorical fiber of this fibration is a homotopy fiber of p which uses the assumed right properness. One equips the diagram $\mathrm{hofib} \to E \xrightarrow{p} X$ with the action coming from the canonical action of this fibration. □

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