Abstract motivic homotopy theory

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1 Introduction

The initial motivation for this work came from the story of the field with one element. This story has two aspects, which are two sides of the same medal.

On the one hand some phenomena in algebraic geometry are independent of the chosen base ring and even do not need any ring structure for their description. Examples are some intersection theoretic counting formulas or the shape of the chamber of the building associated to an algebraic group. These phenomena seem to come from a deeper base, a place where they can exist even before they manifest themselves in the world of algebraic geometry. This leads to the question which objects are *defined over* $\mathbb{F}_1$. Seen from this angle, the world of $\mathbb{F}_1$-geometry is a spartanic place with few inhabitants.

On the other hand many phenomena in arithmetic point to the fact that the understanding of number theory requires some more ingredients than just ring theory, to capture the infinite places of number fields, for example give meaning to the phrase "ring of integers of the real numbers". Here the question is which objects *admit a morphism to* $\mathrm{Spec}(\mathbb{F}_1)$, which should be a very modest requirement. Seen from this angle, the world of $\mathbb{F}_1$-geometry is a rich and varied place that allows everybody in.

Over $\mathbb{F}_1$, everything is allowed, but not much can be done.

In this work we show a few things that *can* be done. All the authors who put forth tentative definitions of $\mathbb{F}_1$-geometry agree that spectra of monoid rings should be defined over $\mathbb{F}_1$. In particular the multiplicative group scheme $\mathbb{G}_m = \mathrm{Spec}(\mathbb{Z}[x, x^{-1}])$ and the affine line $\mathbb{A}^1 = \mathrm{Spec}(\mathbb{Z}[x])$ should be base changed from $\mathbb{F}_1$, hence also projective spaces. Following the usual setup of motivic homotopy theory one would construct the category of motivic spaces by starting with a category of smooth schemes over $\mathbb{F}_1$, passing to the cocompletion, i.e. space valued presheaves, localizing by an appropriate topology and contracting $\mathbb{A}^1$. The topology should be chosen such that the colimit diagrams defining projective spaces remain colimit diagrams after being embedded into motivic spaces. Thus one may as well construct the projective spaces in motivic spaces, and there, $\mathbb{A}^1$ being equivalent to a point, they arise as colimits of diagrams which only involve products of $\mathbb{G}_m$ and the terminal object. With projective spaces available, one can build $\mathbb{P}^\infty$, then pass to the stabilization, invert the Bott element to obtain a K-theory spectrum $\mathbb{P}^\infty[\beta^{-1}]$, show that it splits rationally, and pick one summand to get a rational Eilenberg-MacLane spectrum. This shows that the K-theory spectrum and the rational Eilenberg-MacLane spectrum descend to $\mathbb{F}_1$.

Indeed, they descend much further. We will show that we can perform the above constructions in any good enough, namely cartesian closed and presentable, ∞-category with a group object $\mathbb{G}$. There is an initial such category with a universal group object, and this is the deepest base to which our constructions descend. Many other geometric settings from this deep base, so that our results apply equally to, for example, derived algebraic geometry, log geometry, or $\mathbb{Z}/2$-equivariant geomtry. Their interest in each case has to be determined individually.

Overview over the work: We place ourselves in a cartesian closed, presentable ∞-category $\mathcal{X}$ with a grouplike commutative algebra $\mathbb{G}$ therein. We construct punctured affine spaces $\mathbb{A}^n \setminus 0$ as the colimit in the category of $\mathbb{G}$-modules of a cubical diagram which has the n-fold product of $\mathbb{G}$ at its initial corner and whose morphisms consist of all possible partial projections. There are natural embeddings $\mathbb{A}^n \setminus 0 \to \mathbb{A}^{n+1} \setminus 0$ and we show that their colimit $\mathbb{A}^\infty \setminus 0$ is contractible (Theorem 3.1.16).

We define the projective space $\mathbb{P}^n$ as the quotient of $\mathbb{A}^{n+1} \setminus 0$ by the $\mathbb{G}$-action and can conclude from Theorem 3.1.16 that $\mathbb{P}^\infty$ is $B\mathbb{G}$ and thus itself a commutative algebra. This gives us a "Bott multiplication map" $\beta \colon \mathbb{P}^1 \times \mathbb{P}^\infty \to \mathbb{P}^\infty \times \mathbb{P}^\infty \to \mathbb{P}^\infty$.

Next we pass first to the pointed category $\mathcal{X}_*$ and then to its *monoidal* stabilization with respect to the endofunctor $\mathbb{P}^1 \wedge -$. There one can define Bott inverted infinite projective space $\mathbb{P}^\infty[\beta^{-1}]$ as the colimit of the diagram $\Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^\infty_+ \to \Sigma^{-1}_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^\infty_+ \to \Sigma^{-2}_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^\infty_+ \to \dots$. The spectrum $\mathbb{P}^\infty[\beta^{-1}]$ has a natural orientation which allows to compute the value of the cohomology that it represents on projective spaces (Theorem 3.2.25), in particular on $\mathbb{P}^\infty$. Since $\mathbb{P}^\infty[\beta^{-1}]$ is a colimit of copies of $\mathbb{P}^\infty$, this give some access to endomorphisms of the spectrum $\mathbb{P}^\infty[\beta^{-1}]$

We construct Adams operations on the rationalization $\mathbb{P}^\infty[\beta^{-1}]_\mathbb{Q}$ (Corollary 3.3.2), and show that the spectrum has direct summands which are eigenspaces for these operations. This can be done by applying results of Riou, which we summarize in Section 3.3.3. The 0th summand is what Riou calls the Beilinson spectrum, and is a first version of the rational motivic Eilenberg-MacLane spectrum.

We further show (Theorem 3.4.8) that the Adams eigenspaces are shifted copies of the positive rational sphere (the "Morel spectrum", a second, but equivalent avatar of the rational Eilenberg-MacLane spectrum) and that the $\mathbb{P}^\infty[\beta^{-1}]_\mathbb{Q}$ splits completely into these (i.e. that there is no further summand other than those of the previous paragraph). We call the category of algebras over the positive rational sphere the Morel motives.

In large parts we are able to follow well-known strategies from topology for showing the above. There are two main technical bottle necks that need to be passed in order for the rest to go through smoothly.

The first is the proof of the contractibility of $\mathbb{A}^\infty \setminus 0$. The standard argument from usual or motivic homotopy theory is to give an explicit homotopy between the identity and a constant map on $\mathbb{A}^\infty \setminus 0$, see Remark 3.1.14. This does not work in our general setting, and it fails already in the concrete setting of monoid schemes. Indeed, giving this concrete homotopy requires both an addition and a fixed point for the $\mathbb{G}$-action on $\mathbb{A}^1$. We give a different proof based on Proposition 3.1.11 which rests on an analysis of how to subdivide colimits of hypercube diagrams, like the one defining $\mathbb{A}^n \setminus 0$.

The second are the analogs of two results of Morel, Propositions 3.1.40 and 3.1.42, about the diagonal map of $\mathbb{P}^n$ into its n-fold, resp. $n+1$-fold, smash product. Both are needed for the computation of the oriented cohomology of $\mathbb{P}^n$, and the second also for the construction of the Adams operations and the splitting of $\mathbb{P}^\infty[\beta^{-1}]_\mathbb{Q}$. Here the geometric arguments of Morel, using the glueing of hyperplanes in projective space, can be transformed into a combinatorial argument, which is done in Section 2.3.

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2 Some ∞-categorical technicalities

Here we gather a few technical results on ∞-categories to be used in the later sections.

On notation and terminology:

- Partially ordered sets will be identified with their associated categories.
- N denotes the nerve functor.
- By $[n]$ we denote the ordered set $\{1, \dots, n\}$
- The homotopy category of an ∞-category C is denoted by $\mathrm{Ho}\, C$.
- We denote terminal objects by 1 and the essentially unique map $x \to 1$ by !.
- When we say "commutative square in an ∞-category $\mathcal{X}$" and depict it by a diagram with names on the arrows we mean a map $\Delta^1 \times \Delta^1 \to \mathcal{X}$ whose restrictions to certain 1-simplices are the morphisms with the said names. Likewise, for other types of diagrams we means map from the nerves of the index category to the target ∞-category.
- When we write a composition "$f \circ g$" or two consecutive arrows in a diagram we mean a filler of an appropriate horn. When we say "f factors through g" we mean that there is a 2-simplex with f and g as faces in the appropriate places.
- If $\mathcal{X}$ is an ∞-category with finite products, it gets cartesian monoidal structure, i.e. a cocartesian fibration $\mathcal{X}^{\times} \to N(\mathrm{FinSet}_*)$. This fibration corresponds to a functor $N(\mathrm{FinSet}_*) \to \mathcal{C}at_{\infty}$. We denote by $\times \colon \mathcal{X}^n \to \mathcal{X}$ (or sometimes $\times_n$) the image under this functor of the active morphism $\{1, \dots, n, *\} \to \{1, *\}$ sending $1, \dots, n$ to 1.

2.1 A criterion for a map to be constant

Lemma 2.1.1. *Suppose that in an ∞-category $\mathcal{X}$ we have morphisms $f \colon x \to y$, $g \colon z \to y$ and $p \colon 1 \to x$ and a commutative diagram*

$$\begin{tikzcd} z \times x \arrow[r, "pr_1"] \arrow[d, "pr_2"'] & z \arrow[d, "g"] \\ x \arrow[r, "f"'] & y \end{tikzcd}$$

Then g factors through the terminal object, i.e. there is a morphism $q \colon 1 \to y$ such that $g \simeq q \circ (z \xrightarrow{!} 1)$.

Proof. The following diagram commutes in the homotopy category $\mathrm{Ho}\, \mathcal{X}$:

$$\begin{tikzcd} z \times x \arrow[r, "id \times !"] \arrow[d] & z \times 1 \arrow[r, "id \times p"] & z \times x \arrow[d, "\mathrm{pr}_2"] \\ 1 \arrow[rr, "p"'] & & x \end{tikzcd}$$

Hence so does

$$\begin{tikzcd} z \cong z \times 1 \arrow[r, "id\times p"] \arrow[dr] & z \times x \arrow[r, "id\times !"] \arrow[d] & z \times 1 \arrow[r, "id\times p"] & z \times x \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"] & z \arrow[d, "g"] \\ & 1 \arrow[rr, "p"'] & & x \arrow[r, "f"'] & y \end{tikzcd}$$

The upper horizontal arrows compose to id_z, hence the upper horizontal arrows followed by the right hand vertical arrow compose to g, hence the left diagonal and lower horizontal arrows also compose to g, showing that g factors through 1 in $\mathrm{Ho}\,\mathcal{X}$. Equivalently g is homotopic in $\mathcal{X}$ to a morphism factoring through 1. $\square$

2.2 Colimit pasting for hypercubes

In [Lur09, §4.2.3] Lurie gives a general procedure for decomposing the colimit of a diagram d into a colimit of several colimits of diagrams mapping to d. Here we record some special cases which we will need later.

Suppose that $\mathcal{X}$ is an ∞-category in which all colimits occurring below exist, $K \to \mathcal{X}$ a map of simplicial sets, J an ordered set, $F\colon J \to Sub(K)$ a functor from J to the collection of simplicial subsets of K and inclusions. From this data one can, as in [Lur09, Not. 4.2.3.1] and [Lur09, Prop. 4.2.3.4] construct a map $d\colon N(J) \to \mathcal{X}$ such that $d(j) \simeq \mathrm{colim}(F(j) \to K \to \mathcal{X})$ and such that the morphisms and higher simplices are those coming from the universal property of the involved colimits.

Proposition 2.2.1 (Lurie)**.** *Suppose that for every simplex σ of K the ordered subset $J_\sigma := \{j \in J \mid \sigma \in F(j)\}$ has contractible nerve. Then*

$$\mathrm{colim}(d\colon N(J) \to \mathcal{X}) \simeq \mathrm{colim}(F\colon K \to \mathcal{X})$$

Proof. The hypothesis says that the criterion from [Lur09, Rem. 4.2.3.9] is satisfied, whence we can apply [Lur09, Cor. 4.2.3.10], which implies the claim. $\square$

Let P_n be the ordered set of non-empty subsets of $\{1, \ldots, n\}$ and inclusions, and $K_n := N(P_n^{op})$ the nerve of its opposite category. A cocone over K_n is a hypercube, i.e. $K_n \star \Delta^0 \simeq N(\wp(\{1, \ldots, n\})) \simeq (\Delta^1)^n$, which accounts for the name of this section.

Let $A \coprod B = \{1, \ldots, n\}$ be a partition of $\{1, \ldots, n\}$. We consider the following full simplicial subsets of K_n:

K_n^a, with set of vertices $\{S \subseteq \{1, \ldots, n\} \mid S \cap A \neq \emptyset\}$,
K_n^b, with set of vertices $\{S \subseteq \{1, \ldots, n\} \mid S \cap B \neq \emptyset\}$,
$K_n^0 := K_n^a \cap K_n^b$.

Then we have inclusions of simplicial sets

$$\begin{tikzcd} K_n^0 \arrow[r] \arrow[d] & K_n^b \arrow[d] \\ K_n^a \arrow[r] & K_n \end{tikzcd}$$

Given an ∞-category $\mathcal{X}$ and a diagram $d\colon K \to \mathcal{X}$, we also have its subdiagrams $d|_{K_n^0}$, $d|_{K_n^a}$, $d|_{K_n^b}$, hence compatible maps $\operatorname{colim} d|_{K_n^a} \to \operatorname{colim} d$, $\operatorname{colim} d|_{K_n^0} \to \operatorname{colim} d$, $\operatorname{colim} d|_{K_n^b} \to \operatorname{colim} d$ (see e.g. [Cra09, Prop. 2.29] for the functoriality of finite colimits in the indexing simplicial sets) and hence a map $\operatorname{colim}(\operatorname{colim} d|_{K_n^a} \leftarrow \operatorname{colim} d|_{K_n^0} \to \operatorname{colim} d|_{K_n^b}) \to \operatorname{colim} d$.

Lemma 2.2.2. *The map* $\operatorname{colim}(\operatorname{colim} d|_{K_n^a} \leftarrow \operatorname{colim} d|_{K_n^0} \to \operatorname{colim} d|_{K_n^b}) \to \operatorname{colim} d$ *is an equivalence.*

Proof. Denote by J the index category of a pushout datum, i.e. $J := (a \leftarrow 0 \to b)$. For a simplex σ of K the category J_σ from [Lur09, Notation 4.2.3.7] is in our case simply the ordered subset of J whose vertices are those $x \in \{a, 0, b\}$ for which $\sigma \in K_x$. Since every simplex of K occurs in K_n^a or K_n^b, and then also in $K_n^0 = K_n^a \cap K_n^b$, the only subsets which can occur this way are (a), (b) and $(a \leftarrow 0 \to b)$, all of which have contractible nerves. Now we can apply Prop. 2.2.1. □

Denote by $\widetilde{K}_n^a$ (resp. $\widetilde{K}_n^b$) the full simplicial subset of K with vertices $\{S \subseteq \{1, \dots, n\} \mid S \subseteq A, S \neq \emptyset\}$ (resp. $\{S \subseteq \{1, \dots, n\} \mid S \subseteq B, S \neq \emptyset\}$). Clearly we have $\widetilde{K}_n^a \subseteq K_n^a$ (resp. $\widetilde{K}_n^b \subseteq K_n^b$).

Lemma 2.2.3. *There is an equivalence*

$$\operatorname{colim}(\operatorname{colim} d|_{\widetilde{K}_n^a} \leftarrow \operatorname{colim} d|_{K_n^0} \to \operatorname{colim} d|_{\widetilde{K}_n^b}) \to \operatorname{colim} d$$

Proof. This follows from Lemma 2.2.2 via the observation that there is a weak equivalence $\operatorname{colim} d|_{K_n^b} \to \operatorname{colim} d|_{\widetilde{K}_n^b}$ induced by the map $p\colon K_n^b \to \widetilde{K}_n^b$, $N(S \mapsto S \cap B)$: To see that the induced map on colimits is an equivalence, note that the inclusion $i\colon \widetilde{K}_n^b \subseteq K_n^b$ is cofinal by [Lur09, Thm. 4.1.3.1] since for every vertex $S \in K_n^b$ the undercategory $S/\widetilde{K}_n^b \subseteq S/K_n^b$ is contractible, having the initial object $S \setminus A$. Further, we have maps $\operatorname{colim} d|_{\widetilde{K}_n^b} \to \operatorname{colim} d|_{K_n^b} \to \operatorname{colim} d|_{\widetilde{K}_n^b}$ of which the first (induced by i) and the composite (since $p \circ i = id_{\widetilde{K}_n^b}$) are equivalences, hence so is the second. Analogously we have $\operatorname{colim} d|_{K_n^a} \simeq \operatorname{colim} d|_{\widetilde{K}_n^a}$. □

Corollary 2.2.4. *In the special case* $A = \{n\}$*,* $B = \{1, \dots, n-1\}$ *we have an equivalence* $\operatorname{colim}(d(\{n\} \leftarrow \operatorname{colim} d|_{K_n^0} \to \operatorname{colim} d|_{\widetilde{K}_n^b}) \to \operatorname{colim} d$

Proof. In this case we have K_n^a, with set of vertices $\{S \subseteq \{1, \dots, n\} \mid n \in S\}$, K_n^b, with set of vertices $\{S \subseteq \{1, \dots, n\} \mid \emptyset \neq S \neq \{n\}\}$, $K_n^0 := K_n^a \cap K_n^b$.

We have $\operatorname{colim} d|_{K_n^a} \simeq d(\{n\})$ because $\{n\}$ is a terminal object of K_n^a. The claim then follows from Lemma 2.2.3 □

We give a further analysis of the object $\operatorname{colim} d|_{K_n^0}$ occurring in the previous results. For this consider the ordered set $J := (\wp(B) \setminus \{\emptyset\}, \subseteq)^{op}$.

We have a diagram $g\colon J \to \mathrm{sSet}/K_n^0$ sending $T \subseteq B$ to the inclusion $i_T\colon K_n^T \hookrightarrow K_n^0$ where K_n^T is the full simplicial subset with vertices $\{S \subseteq \{1, \dots, n\} \mid S \cap A \neq \emptyset,\ S \cap B \supseteq T\}$ and sending inclusions $T \subseteq T' \subseteq B$ to the corresponding inclusions $K_n^{T'} \hookrightarrow K_n^T$. As in the beginning of the section we obtain a diagram $e\colon \operatorname{colim}(N(J) \to \mathcal{X})$ with $e(T) \simeq \operatorname{colim} K_n^T$.

Lemma 2.2.5. $\operatorname{colim}(e\colon N(J) \to \mathcal{X}) \simeq \operatorname{colim} d|_{K_n^0}$

Proof. To apply Prop. 2.2.1, for a simplex $\sigma \in K_n^0$ we consider the ordered subset $J_\sigma \subseteq J$ consisting of those $T \subseteq B$ for which $\sigma \in K_n^T$.

A k-simplex $\sigma \in (K_n^0)_k$ is a chain of k inclusions of subsets of $\{1, \dots, n\}$, all of which have nonempty intersection with both A and B. A simplex is in K_n^T if and only if all of its vertices are in K_n^T if and only if the smallest of the subsets occurring in the chain is a vertex of K_n^T if and only if this smallest subset contains T. Thus, denoting by V_σ the smallest of the subsets in the chain corresponding to the simplex σ, we have that σ is contained precisely in those K_n^T with $V_\sigma \cap B \supseteq T$, i.e. $J_\sigma = \{T \subseteq B \mid T \subseteq V_\sigma \cap B\}$. We see that J_σ has the initial object $V_\sigma \cap B$ and hence is contractible. □

We further have the full simplicial subsets $\widetilde{K}_n^T \subseteq K_n^T$ with vertices $\{S \subseteq \{1, \dots, n\} \mid S \cap A \neq \emptyset,\ S \cap B = T\}$. Note that $\widetilde{K}_n^T \simeq N(\wp(A) \setminus \emptyset)$. The $\widetilde{K}_n^T$ fit themselves into a diagram $J \to \mathrm{sSet}/K_n^0$ mapping T to the inclusion $\widetilde{K}_n^T \subseteq K_n^T \subseteq K_n^0$, which, again by the construction of [Lur09, Not. 4.2.3.1] and [Lur09, Not. 4.2.3.4], yields a diagram $\widetilde{e}\colon N(J) \to \mathcal{X}$ with $\widetilde{e}(T) = \operatorname{colim} \widetilde{K}_n^T$.

Lemma 2.2.6. $\operatorname{colim}(\widetilde{e}\colon N(J) \to \mathcal{X}) \simeq \operatorname{colim} d|_{K_n^0}$

Proof. We construct an equivalence of diagrams $\widetilde{e} \simeq e$. From this we obtain $\operatorname{colim} \widetilde{e} \simeq \operatorname{colim} e$ and then the claim follows from Lemma 2.2.5.

Let M be the ordered set $(\{T \in \wp(\{1, \dots, n\}) \mid T \cap A \neq \emptyset \neq T \cap B\}, \subseteq)^{op}$ (so that $K_n^0 = N(M)$). We perform the construction of [Lur09, Not. 4.2.3.1] and [Lur09, Not. 4.2.3.4] with our given map of simplicial sets $K_n^0 \to \mathcal{X}$, the ordered set $M \times \Delta^1$, and the diagram $G\colon M \times \Delta^1 \to Sub(K)$ given on objects by

$$G(T, i) := \begin{cases} \widetilde{K}_n^T, & \text{if } i = 0 \\ K_n^T, & \text{if } i = 1 \end{cases}$$

and on morphisms by $G(T \subseteq T', id_0) := \widetilde{K}_n^{T'} \subseteq \widetilde{K}_n^T$, $G(T \subseteq T', id_0) := K_n^{T'} \subseteq K_n^T$, $G(id, (0 \to 1)) := \widetilde{K}_n^T \subseteq K_n^T$. This gives us a diagram $N(M) \times \Delta^1 \simeq N(M \times \Delta^1) \to \mathcal{X}$ with the following features:

1. Its restriction to $N(M) \times \{0\}$ is $\widetilde{e}$.

2. Its restriction to $N(M) \times \{1\}$ is e.

3. The 1-simplices of $N(M) \times \Delta^1$ which come from the morphisms of the form $G(id, (0 \to 1))$ get sent to the canonical morphism $\widetilde{e}(T) = \operatorname{colim} \widetilde{K}_n^T \to \operatorname{colim} K_n^T = e(T)$

Thus the diagram $N(M) \times \Delta^1 \to \mathcal{X}$, resp. its adjoint $\Delta^1 \to \mathcal{X}^{N(M)}$, corresponds to a map of diagrams $\widetilde{e} \to e$. To show that this map is an equivalence of diagrams, it suffices to show that it is an objectwise equivalence, i.e. that $\operatorname{colim} \widetilde{K}_n^T \to \operatorname{colim} K_n^T$ is an equivalence for every T.

This is true because inclusion $i\colon \widetilde{K}_n^T \hookrightarrow K_n^T$ is a cofinal map by [Lur09, Thm. 4.1.3.1], since for every vertex $U \in K_n^T$ the under-∞-category $(\widetilde{K}_n^T)_{U/} \subseteq (K_n^T)_{U/}$ is contractible, having the initial object $U \cap (A \cup T)$ (informally: there is a shortest way of getting from a $U \in K_n^T$ into $\widetilde{K}_n^T$, namely by cutting down the B-part of U to become exactly T and any other way leads through this one). □

Of course analogous results to those of this section can be obtained for finer partitions of more general posets, with similar reasoning but at the cost of more combinatorial bureaucracy.

2.3 A pullback calculation

We keep the notation K_n for a hypercube with the terminal object removed from Section 2.2. Accordingly K_n^{op} is a hypercube with the initial object removed. Let $\mathcal{X}$ be an ∞-category with finite products.

Lemma 2.3.1. *Let $f_i\colon a_i \to b_i$, $i = 1, \dots, n$, be morphisms in $\mathcal{X}$. Then the diagram $l\colon (K_n^{op})^{\triangleleft} \simeq (\Delta^1)^n \xrightarrow{f_1 \times \ldots \times f_n} \mathcal{X}^n \xrightarrow{\times} \mathcal{X}$ is a limit diagram. The restriction to any face of the hypercube is a limit diagram as well.*

Proof. The corresponding claim for 1-categories is true as can be seen by checking the universal property. The first claim of the lemma can then be seen by checking the universal property on mapping spaces. For these we can pass to a model, note that the product of a fibration with an identity map is a fibration again and employ the 1-catgorical fact.

For the second claim note that the faces of the hypercube are diagrams of the same kind for some lower n. $\square$

The hypercube category $(\Delta^1)^n \simeq N(\wp(\{1, \dots, n\}), \subseteq^{op})$ can be filtered by the full simplicial subsets whose vertices are the sets with at most i elements, $0 \le i \le n$. For these we introduce a notation $K^{\le i} := N(\{S \subseteq \{1, \dots, n\} \mid |S| \le i\})$

The next lemma says that hypercubes with enough limit faces are Kan extensions of their lowest two layers.

Lemma 2.3.2. *Let $d\colon (\Delta^1)^n \to \mathcal{X}$ be a hypercube diagram such that the restriction $(K_i)^{\triangleleft} \simeq (\Delta^1)^i \to \mathcal{X}$ to every face of dimension $i \ge 2$ which contains the terminal vertex of the hypercube, is a limit cone. Then d is a right Kan extension of $d|_{K^{\le 1}}\colon K^{\le 1} \hookrightarrow (\Delta^1)^n \to \mathcal{X}$ along $K^{\le 1} \hookrightarrow (\Delta^1)^n$.*

Proof. For every $i \in \{1, \dots, n-1\}$ we consider the right Kan extension of $d|_{K^{\le i}}$ along $K^{\le i} \hookrightarrow K^{\le i+1}$. If this Kan extension exists, by [Lur09, Def. 4.3.2.2] the value of $Ran_{K^{\le i} \to K^{\le i+1}} d|_{K^{\le i}}$ at an $(i+1)$-element set S is the limit of the diagram $d|_{K_{S/}}$ i.e. of the restriction of $d|_{K^{\le i}}$ to the full simplicial subset of $K^{\le i}$ whose vertices are the proper subsets of S. Together with S itself, this is a face of the hypercube and hence restricting d to this yields a limit diagram, by hypothesis.

Thus we obtain $d|_{K^{\le i+1}} \simeq Ran_{K^{\le i} \to K^{\le i+1}} d|_{K^{\le i}}$ for every $i \in \{1, \dots, n-1\}$ (and in particular the Kan extensions in question do exist).

Starting from $d|_{K^{\le 1}}$ we can construct right Kan extensions all the way up till $K^{\le n} = K_n$ and get back the diagram d. An iterated application of [Lur09, Prop. 4.3.2.8] says that this sequence of Kan extensions exhibits d as a Kan extension of $d|_{K^{\le 1}}$. $\square$

In fact, hypercubes with enough limit faces are Kan extensions of any subdiagram containing the lowest two layers:

Corollary 2.3.3. *Let $d\colon (\Delta^1)^n \to \mathcal{X}$ be a hypercube diagram satisfying the hypotheses of Lemma 2.3.2. Let K be a full simplicial subset of $(\Delta^1)^n$ containing $K^{\leq 1}$.*
Then $d \simeq Ran_{K\hookrightarrow(\Delta^1)^n}(d|_K)$.

Proof. We show that $d|_K$ is itself a right Kan extension of $d|_{K^{\leq 1}}$. Indeed,

$$d|_K \simeq (Ran_{K^{\leq 1}\hookrightarrow(\Delta^1)^n}(d|_{K^{\leq 1}}))|_K \simeq (Ran_{K\hookrightarrow(\Delta^1)^n}(Ran_{K^{\leq 1}\hookrightarrow K}(d|_{K^{\leq 1}})))|_K \simeq Ran_{K^{\leq 1}\hookrightarrow K}(d|_{K^{\leq 1}})$$

where for the first equivalence we substitute d with $Ran_{K^{\leq 1}\hookrightarrow(\Delta^1)^n}(d|_{K^{\leq 1}})$ using Lemma 2.3.2, for the second equivalence express a Kan extension as two consecutive Kan extensions [Lur09, Prop. 4.3.2.8], and for the third equivalence use that Kan extending from K and then restricting back to K is equivalent to the identity (since we are extending along inclusions of full subcategories [Lur09, 4.3.2.16]).

Now the claim follows from a sequence of similar moves:
$Ran_{K\hookrightarrow(\Delta^1)^n}(d|_K) \simeq (Ran_{K\hookrightarrow(\Delta^1)^n}(Ran_{K^{\leq 1}\hookrightarrow K}(d|_{K^{\leq 1}}))) \simeq Ran_{K^{\leq 1}\hookrightarrow(\Delta^1)^n}(d|_{K^{\leq 1}}) \simeq d$ □

A simplicial set K with a terminal object $k \in K$ can be written as $K \cong L^{\triangleright}$ where L is the full simplicial subset of K containing all vertices except k. A diagram $d\colon K \to \mathcal{X}$ is the same as a diagram from L into the slice category $\mathcal{X}_{/d(k)}$ and we shall denote this corresponding diagram by $\widetilde{d}\colon L \to \mathcal{X}_{/d(k)}$.

Lemma 2.3.4. *Let $K \cong L^{\triangleright}$ be a simplicial set with a terminal vertex $k \in K$. Then $d\colon K^{\triangleleft} \to \mathcal{X}$ is a limit diagram in $\mathcal{X}$ if and only if $\widetilde{d}\colon L^{\triangleleft} \to \mathcal{X}_{/d(k)}$ is a limit diagram.*

Proof. Morphisms in $(\mathcal{X}_{/d(k)})^{L^{\triangleleft}}$ whose restriction to L is $\widetilde{d}|_L$ are equivalent to morphims in $\mathcal{X}^{K^{\triangleleft}}$ whose restriction to K is $d|_K$. Hence the universal property that the cone $\widetilde{d}\colon L^{\triangleleft} \to \mathcal{X}_{/d(k)}$ is terminal, is equivalent to the cone $d\colon K^{\triangleleft} \to \mathcal{X}$ being terminal. □

The restriction to subsets with zero or one elements in Lemma 2.3.2 cooresponds to restriction to subsets with one element after passing to the slice category. Being right Kan extended from these subsets are corresponding properties:

Lemma 2.3.5. *Let $l\colon (\Delta^1)^n \to \mathcal{X}$ be a diagram and $\widetilde{l}\colon L^{\triangleleft} \to \mathcal{X}_{/l(\emptyset)}$ the corresponding diagram to the slice category. Then l is the right Kan extension of $l|_{K^{\leq 1}}$ along $K^{\leq 1} \hookrightarrow (\Delta^1)^n$ (as in Lemma 2.3.2) if and only if $\widetilde{l}\colon L^{\triangleleft} \to \mathcal{X}_{/l(\emptyset)}$ is the right Kan extension of $\widetilde{l}|_{(L^{\triangleleft})^{\leq 1}}$ along $(L^{\triangleleft})^{\leq 1} \hookrightarrow L^{\triangleleft}$.*

Proof. Both $(\Delta^1)^n$, resp. L, are filtered by $K^{\leq 1}$, resp. $K^{\leq 1} \cap L$. Being a right Kan extension is in both cases equivalent to the property that values at the next layer are limits of the diagram given by the objects of the lower layers which receive maps from it. This property holding in the slice category $\mathcal{X}_{/l(\emptyset)}$ is equivalent to it holding in $\mathcal{X}$ by Lemma 2.3.4. □

We can now consider the limit diagram

$$l\colon (K_n^{op})^{\triangleleft} \simeq (\Delta^1)^n \xrightarrow{f_1\times\ldots\times f_n} \mathcal{X}^n \xrightarrow{\times} \mathcal{X}$$

as a diagram in the slice category over its value at terminal object of K_n, i.e over $l(\emptyset) \simeq b_1 \times \ldots \times b_n$, i.e. as the diagram $\widetilde{l}\colon L^{\triangleleft} \to \mathcal{X}_{/b_1\times\ldots\times b_n}$.

Lemma 2.3.6. *The diagram $\widetilde{l}\colon L^{\triangleleft} \to \mathcal{X}_{/b_1\times\ldots\times b_n}$ is a limit diagram. The same is true for the restrictions of $\widetilde{l}$ arising from the faces of the hypercube as in Lemma 2.3.1.*

Proof. This follows from Lemma 2.3.4 and Lemma 2.3.1. □

Let now $\mathcal{X}$ be a cartesian closed, presentable ∞-category and $f_i\colon a_i \to x$, $i = 1, \ldots, n$, morphisms, now all with the same codomain. As before we have the limit diagram $l\colon (K_n^{op})^{\triangleleft} \to \mathcal{X}$ and the limit diagram $h\colon L^{\triangleleft} \to \mathcal{X}_{/x^n}$.

Since $\mathcal{X}_{/x^n}$ is cocomplete, there is a colimit diagram $\bar{h}\colon (L^{\triangleleft})^{\triangleright} \to \mathcal{X}_{/x^n}$ with the colimit object $\operatorname{colim}(h) \to x^n \in \mathcal{X}_{/x^n}$. Since the forgetful functor $\mathcal{X}_{/x^n} \to \mathcal{X}$ (given by composing with $!\colon x^n \to 1$) is a left adjoint, it preserves colimits, i.e. colimits in slice categories are the underlying colimits in $\mathcal{X}$.

Now consider the pullback along the diagonal $diag\colon x \to x^n$. Since $\mathcal{X}$ is cartesian closed, the pullback functor $diag^*\colon \mathcal{X}_{/x^n} \to \mathcal{X}_{/x}$ preserves both colimits and limits. Hence $diag^*(\operatorname{colim}(h)) \simeq \operatorname{colim} diag^*(h)$ and the occurring diagram

$$diag^*(h)\colon L^{\triangleleft} \xrightarrow{h} \mathcal{X}_{/x^n} \xrightarrow{diag^*} \mathcal{X}_{/x}$$

is still a limit diagram whose restriction to every face is also a limit diagram.

Lemma 2.3.7. *There is an equivalence*

$$diag^*(x \times \ldots \times a_i \times \ldots \times x \xrightarrow{id\times\ldots\times f_i\times\ldots\times id} x^n) \simeq (a_i \xrightarrow{f_i} x).$$

Proof. The corresponding claim for 1-categories is true as can be seen by checking the universal property. The claim can then be seen by passing to mapping spaces, modeling spaces with a right proper model category, e.g. Quillen's standard one on simplicial sets, and noting that the product of a fibration with an identity map is a fibration again. □

Lemma 2.3.7 gives the values of the diagram $diag^*(h)\colon L^{\triangleleft} \to \mathcal{X}_{/x}$ on the vertices given by one element subsets of $\{1, \ldots, n\}$. Starting from this, one can compute the values at the two element subsets by taking pullbacks (since we know that all faces of the cut off hypercube $L^{\triangleleft}$ are sent to limit diagrams), e.g.

$$diag^*(h)(\{i, j\}) \simeq (a_i \times_x a_j \to x).$$

Continuing in this way, raising the cardinality of the considered subsets, one can reconstruct the whole diagram $diag^*(h)$. This is just the iterated right Kan extension process from the proof of Lemma 2.3.2.

Special case 1: We identify K_{n+1} with nerve of the opposite of the set of nonempty subsets of $\{0, \ldots, n\}$. Let us now consider the special case that x is itself given as colimit of a diagram $v\colon K_{n+1} \to \mathcal{X}$. Then we have morphisms $h_i\colon a_i := v(\{i\}) \to \operatorname{colim}(v) = x$ and can use them to define

$$f_i := (id \times \ldots \times h_i \times \ldots \times id)\colon x \times \ldots \times a_i \times \ldots \times x \to x^n \ (i = 1, \ldots, n)$$

Note that we made no use of the morphism h_0. With these f_i we can construct a diagram $h\colon (\Delta^1)^n \to \mathcal{X}$ as before and the corresponding diagram $\widetilde{h}\colon K_n \to \mathcal{X}_{/x^n}$ going into the slice category over x^n. Pulling back along the diagonal $diag\colon x \to x^n$ we obtain a diagram $diag^*(\widetilde{h})\colon K_n \to \mathcal{X}_{/x}$ as just described. This diagram $diag^*(\widetilde{h})$ corresponds to a hypercube diagram $r\colon (\Delta^1)^n \to \mathcal{X}$ with $r(\emptyset) = x$, i.e. we have $\widetilde{r} = diag^*(h)$.

Lemma 2.3.8. *Let $r\colon (\Delta^1)^n \to \mathcal{X}$ be the diagram such that $\widetilde{r} = diag^*(h)$. Then r is a right Kan extension of $r|_{K^{\leq 1}}$*

Proof. It is enough to notice that r satisfies the hypotheses of Lemma 2.3.2, i.e. that every restriction of r to face of $(\Delta)^1$ which contains the vertex $\emptyset$ is a limit diagram. Indeed, $diag^*$ preserves limits, in particular those limit subdiagrams of $\widetilde{h}$ which arise as restrictions of h to faces of the hypercube, i.e. of the form $\widetilde{h|_{K_i}}$. These subdiagrams being limit diagrams in $\mathcal{X}_{/x}$ means exactly that the corresponding faces of the hypercube diagram r are limit diagrams, by Lemma 2.3.4. □

Thus r is a right Kan extension of $r|_{((\Delta^1)^n)^{\leq 1}}$ and by Lemma 2.3.5 $diag^*(\widetilde{h})$ is a right Kan extension of $diag^*(\widetilde{h})|_{K_n^{\leq 1}}$.

We have a second diagram $w\colon K_n \to \mathcal{X}_{/x}$ given by restricting $v\colon K_{n+1} \to \mathcal{X}$ along the map $K_n \hookrightarrow K_{n+1}$, $S \mapsto S$ which identifies K_n as the full simplicial subset of K_{n+1} with vertices $\{S \subseteq \{1, \ldots, n\} \mid S \neq \emptyset, 0 \notin S\}$.

We will end this section is with constructing a map of diagrams $w \to diag^*(\widetilde{h})$: By [Lur09, Prop. 4.3.3.7] to give a map of diagrams $w \to diag^*(\widetilde{h})$ is the same as giving a morphism $w|_{((\Delta^1)^n)^{\leq 1}} \to diag^*(\widetilde{h})|_{K_n^{\leq 1}}$, i.e. a map between the discrete subdiagrams obtained by restricting w, resp. $diag^*(h)$, to the vertices of K_n which are given by one element subsets. As the this morphism we choose the identity maps $a_i \to a_i$.

The map of diagrams then induces a map of colimits $\operatorname{colim}(w) \to \operatorname{colim}(diag^*(h))$, which will be used in the proof of Prop. 3.1.40.

Special case 2: As a second special case suppose again that x is given as colimit of a diagram $v\colon K_{n+1} \to \mathcal{X}$. Again we have morphisms $h_i\colon a_i := v(\{i\}) \to \operatorname{colim}(v) = x$, $i = 0, \ldots, n$ and can use them to define

$$f_i := (id \times \ldots \times h_i \times \ldots \times id)\colon x \times \ldots \times a_i \times \ldots \times x \to x^n \ (i = 0, \ldots, n)$$

Note that this time we *do* use the morphism h_0. With these f_i we can again construct a diagram $h\colon (\Delta^1)^{n+1} \to \mathcal{X}$ with the corresponding diagram $\widetilde{h}\colon K_n \to \mathcal{X}_{/x^{n+1}}$ this time going into the slice category over x^{n+1}. Pulling back along the diagonal $diag\colon x \to x^n$ we obtain a diagram $e := diag^*(\widetilde{h})\colon K_{n+1} \to \mathcal{X}_{/x}$ as just described.

Analogously to special case 1 e is a right Kan extension of its restriction to $K_{\leq 1}$. Since on $K_{\leq 1}$ the diagrams e and v coincide by construction, we have the identity map $id\colon v|_{K_{\leq 1}} \to e|_{K_{\leq 1}}$ which corresponds to a map of diagrams $v \to e$, and gives rise to a map of colimits $x \simeq \operatorname{colim} v \to \operatorname{colim} e$.

2.4 A formula for smash products

Let C be a monoidal ∞-category having all finite colimits, and whose tensor product preserves colimits in each variable.In this section we construct a monoidal structure (the smash product) on the category of pointed objects $C_* := C_{1/}$, derive a formula for it and show that that the left adjoint $C \to C_*,\ x \mapsto (1 \to x \coprod 1)$ is monoidal.

In [GGN15, Thm 5.1] it is established, that if C is a closed monoidal presentable ∞-category there is a *unique* tensor structure on C_* making the above functor $C \to C_*$ monoidal. Hence in the standard cases our structure coincides with the usual ones.

As an intermediate category in the construction we use the arrow category C^{Δ^1}. First note that the arrow category $\Delta^1 = N(0 \to 1)$ has a monoidal structure, given by the categorical product (concretely: $0 \times 0 = 0 \times 1 = 1 \times 0 = 0$ and $1 \times 1 = 1$). From [Gla13, Def. 2.8, Prop. 2.11] we obtain a monoidal structure on C^{Δ^1}, the Day convolution product.

Looking at [Gla13, Lemma 2.4] and the construction of the Day convolution monoidal structure [Gla13, Def. 2.8], one sees that the Day convolution monoidal structure is, as in 1-category theory, constructed by left Kan extensions. E.g. the Day convolution product of two functors $f, g\colon \Delta^1 \to C$ is given by $Lan_{(\Delta^1\times\Delta^1 \xrightarrow{\times} \Delta^1)}(\Delta^1 \times \Delta^1 \xrightarrow{f\times g} C \times C \xrightarrow{\otimes} C)$

In our case this means that the Day convolution product is exactly the pushout product, i.e. the product of $f, g\colon \Delta^1 \to C$ is the dotted arrow given by by the pushout property of the square in the following diagram:

$$\begin{tikzcd}
f(0)\otimes g(0) \arrow[r, "id\otimes g(\to)"] \arrow[d, "f(\to)\otimes id"'] & f(0)\otimes g(1) \arrow[d] \arrow[ddr, "f(\to)\otimes id"] & \\
f(1)\otimes g(0) \arrow[r] \arrow[drr, "id\otimes g(\to)"'] & P \arrow[dr, dotted] & \\
 & & f(1)\otimes g(1)
\end{tikzcd}$$

There is an inclusion functor $C_* \to C^{\Delta^1}$, $(1 \to x) \mapsto (1 \to x)$. This has a left adjoint given by taking the cofiber: $\mathrm{cof}\colon C^{\Delta^1} \to C_*$, $(x \to y) \mapsto (1 \to y/x)$. Clearly, the corresponding monad is idempotent, exhibiting C_* as a reflexive subcategory of C^{Δ^1}.

With this reflexive subcategory we obtain notions of local object and local equivalence: Local objects are those arrows whose domain is a terminal object and local equivalences are all maps inducing equivalences on their cofibers, e.g. all pushout squares.

Lemma 2.4.1. *Tensoring a local equivalence in C^{Δ^1} with any object in C^{Δ^1} results in a local equivalence.*

Proof. Let

$$\begin{tikzcd}
x \arrow[r] \arrow[d] & y \arrow[d] \\
r \arrow[r] & y \coprod_x r
\end{tikzcd}$$

be a pushout square, i.e. a local equivalence.

The pushout product with an object $w \to z \in C^{\Delta^1}$ is given by the rightmost rectangle in the following diagram:

$$\begin{tikzcd}
w \otimes x \arrow[rr] \arrow[dr] \arrow[dd] & & w \otimes y \arrow[dr] \arrow[drr] \arrow[dd] & & \\
& z \otimes x \arrow[rr] \arrow[rrr, bend right] \arrow[dd] & & P \arrow[r, dotted] \arrow[dd] & z \otimes y \arrow[dd] \\
w \otimes r \arrow[rr] \arrow[dr] & & w \otimes y \coprod_x r \arrow[dr] \arrow[drr] & & \\
& z \otimes r \arrow[rr] \arrow[rrr, bend right] & & P' \arrow[r, dotted] & z \otimes (y \coprod_x r)
\end{tikzcd}$$

Here the top and bottom squares are pushout squares, i.e. $P \simeq w \otimes y \coprod_{w \otimes x} z \otimes x$ and $P' \simeq w \otimes (y \coprod_x r) \coprod_{w \otimes r} z \otimes r$

We need to show that the rightmost rectangle is a local equivalence, i.e that both vertical arrows have the same cofiber. For this consider the following diagram (the back and bottom sides of the above cube):

$$\begin{tikzcd}
w \otimes x \arrow[r] \arrow[d] & w \otimes r \arrow[r] \arrow[d] & z \otimes r \arrow[d] \\
w \otimes y \arrow[r] & w \otimes (y \coprod_x r) \arrow[r] & P'
\end{tikzcd}$$

The left hand square is a pushout square because tensoring with w preserves pushout squares. The right hand square is a pushout square by definition. Hence, by [Lur09, Lemma 4.4.2.1], the outer rectangle is a pushout diagram.

This outer rectangle is also the outer rectangle in the following diagram (the upper and front sides of the above cube):

$$\begin{tikzcd}
w \otimes x \arrow[r] \arrow[d] & z \otimes x \arrow[r] \arrow[d] & z \otimes r \arrow[d] \\
w \otimes y \arrow[r] & P \arrow[r] & P'
\end{tikzcd}$$

Here the left and the outer rectangles are pushouts, hence, by [Lur09, Lemma 4.4.2.1], so is the right hand square.

Finally, in the diagram

$$\begin{tikzcd}
z \otimes x \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & z \otimes y \arrow[d] \\
z \otimes r \arrow[r] & P' \arrow[r] & z \otimes (y \coprod_x r)
\end{tikzcd}$$

the outer and the left rectangles are pushout squares, hence, by [Lur09, Lemma 4.4.2.1], so is the right one. □

Proposition 2.4.2. *C_* has an induced monoidal structure, given by $a \wedge b := \mathrm{cof}(i(a) \otimes i(b))$, and the functor $\mathrm{cof}\colon C^{\Delta^1} \to C_*$ is monoidal.*

Proof. This follows from Lemma 2.4.1 since reflexive subcategories of monoidal categories whose tensor product preserves local equivalences with local target in each variable inherit a monoidal structure. □

The monoidal structure on C_* is called the smash product.

From the monoidality of the cofiber functor we obtain a formula with quotients and several smash factors. Let K_n be the nerve of the poset of non-full subsets of $\{1, \ldots, n\}$ and inclusions. Given $f_1\colon u_1 \to a_1, \ldots, f_n\colon u_n \to a_n \in C^{\Delta^1}$, we have a diagram $h\colon K_n \to C$ which is defined as follows:

$$h\colon K_n \hookrightarrow N(\wp(\{1, \ldots, n\})) \simeq (\Delta^1)^n \xrightarrow{f_1 \times \ldots \times f_n} C^n \xrightarrow{\otimes} C$$

More concretely, the diagram h is a hypercube in C with one corner missing, whose objects are domains or codomains of the given morphims: For $S \subsetneq \{1, \ldots, n\}$ we have $h(S) := \bigotimes_{i=1}^n x_S(i)$ where $x_S(i) := u_i$ if $i \notin S$ and $x_S(i) := a_i$ if $i \in S$.

Corollary 2.4.3. *Let $u_1 \to a_1, \ldots, u_n \to a_n \in C^{\Delta^1}$. The smash product satisfies $a_1/u_1 \wedge \ldots \wedge a_n/u_n \simeq (a_1 \otimes \ldots \otimes a_n)/\operatorname{colim}(h)$*

Proof. The Day convolution product of $f_1, \ldots, f_n$ is the map $\operatorname{colim}(h) \to a_1 \otimes \ldots \otimes a_n$. Applying the cofiber functor yields the right hand side from the claim. By Prop. 2.4.2 the result is equivalent to what one gets by applying the cofiber functor to each f_i and then taking the product in C_*, which gives the left hand side from the claim. □

A useful instance of this in motivic homotopy theory is for example the case $u_i \to a_i = \mathbb{G} \to \mathbb{A}^1$ which gives $(\mathbb{A}^1/\mathbb{G})^{\wedge n} \simeq \mathbb{A}^n/(\mathbb{A}^n \setminus \{0\})$.

We still note the usual formula for binary smash products.

Corollary 2.4.4. *The smash product satisfies $a \wedge b \simeq a \otimes b/\operatorname{colim}(a \otimes 1 \leftarrow 1 \otimes 1 \to 1 \otimes b)$*

Proposition 2.4.5. *If we have $0 \otimes x \simeq 0$ for 0 an initial object and x any object, then the functor $I\colon C \to C^{\Delta^1}$, $x \mapsto (0 \to x)$ is monoidal.*

Proof. The monoidal structure on C^{Δ^1} is given by the pushout product. The pushout product of $I(x) = (0 \to x)$ and $I(y) = (0 \to y)$ is an arrow with domain the pushout of a diagram with the three initial objects $0 \otimes 0 \simeq 0$, $x \otimes 0 \simeq 0$, $0 \otimes y \simeq 0$, i.e. it is an initial object itself. The codomain is $x \otimes y$. Hence the pushout product is $0 \to x \otimes y \simeq I(x \otimes y)$. □

2.5 $\mathbb{G}$-modules

Let $\mathcal{X}$ be a cartesian closed, presentable ∞-category. By [Lur11, Prop. 2.4.5.1(4)] $\mathcal{X}$ is a monoidal ∞-category, whence we have a notion of commutative algebra object and of modules over such an object. In the following we denote by A a commutative algebra in $\mathcal{X}$ and by $\mathrm{Mod}_A^{\mathbb{E}_\infty}(\mathcal{X})$ the associated ∞-category of A-modules. We collect some results from [Lur11].

Proposition 2.5.1. $\mathrm{Mod}_A^{\mathbb{E}_\infty}(\mathcal{X})$ *is a presentable symmetric monoidal ∞-category.*

Proof. Commutative algebras are algebras for the operad $\mathbb{E}_\infty$, which is coherent in the sense of [Lur11, Def. 3.3.1.9] by [Lur11, Ex. 3.3.1.12]. Hence we can apply [Lur11, Thm. 3.4.4.2] which asserts the claim. □

Proposition 2.5.2. *Limits and colimits in $\mathrm{Mod}_A^{\mathbb{E}_\infty}(\mathcal{X})$ are formed underlying in $\mathcal{X}$, i.e. they commute with the forgetful functor $\theta\colon \mathrm{Mod}_A^{\mathbb{E}_\infty}(\mathcal{X}) \to \mathcal{X}$.*

Proof. For limits the statement is [Lur11, Cor. 3.4.3.2]. For colimits the relevant statement is [Lur11, Cor. 4.2.3.3], which is about left modules for associative algebras, together with the fact [Lur11, Cor. 4.5.1.6] that for modules for a commutative algebra are equivalent to left modules for the underlying associative algebra. □

Proposition 2.5.3. (i) *A morphism of commutative algebras $A \to B$ induces a forgetful functor $\theta_{(A\to B)}\colon \mathrm{Mod}_B^{\mathbb{E}_\infty}(\mathcal{X}) \to\colon \mathrm{Mod}_A^{\mathbb{E}_\infty}(\mathcal{X})$. This functor has a left and a right adjoint. The left adjoint ist given by the relative tensor product [Lur11, §4.5.2], i.e. it sends M to $B \times_A M$ which obtains its B-module structure by the multiplication of B.*

(ii) *For two composable morphisms of commutative algebras $A \to B \to C$ we have equivalences $C \times_B (B \times_A M) \simeq C \times_A M$ and $\theta_{(A\to B)}(\theta_{(B\to C)}(M)) \simeq \theta_{(A\to C)}(M)$, where the left hand sides are given by any composition $A \to C$ of the given morphisms.*

Proof. (i) The forgetful functor $\theta_{(A\to B)}$ is constructed in [Lur11, §3.4.3]. Its left adjoint is constructed as described in the statement in [Lur11, 4.6.2.17].

(ii) The functoriality of the functors from item (i) with respect to commutative algebra morphisms is encoded in the fact that there is a total category of modules $LMod(\mathcal{X})$, that the left adjoints $(B \times_A -)$ (resp. right adjoints $\theta_{(A\to B)}$) are the cocartesian (resp. cartesian) morphisms asociated to a cocartesian (resp. cartesian) fibration $LMod(\mathcal{X}) \to CAlg(\mathcal{X})$, see [Lur11, Lem. 4.5.3.6] (resp. [Lur11, 4.2.3.2]). □

Proposition 2.5.4. *The forgetful functor θ detects colimits: Suppose $p\colon K \to \mathrm{Mod}_A^{\mathbb{E}_\infty}(\mathcal{X})$ is a diagram and $\bar{p}\colon K \to \mathrm{Mod}_A^{\mathbb{E}_\infty}(\mathcal{X})$ an extension of p to a cocone. Then $\bar{p}$ is a colimit diagram if and only if $\theta \circ \bar{p}$ is a colimit diagram.*

Proof. This is a consequence of [Lur11, Cor. 4.2.3.5], whose assumptions are satisfied because we assumed $\mathcal{X}$ to be cartesian closed and presentable. □

A terminal object $1 \in \mathcal{X}$, being the unit for the cartesian monoidal structure, is the underlying object of an obvious trivial commutative algebra (it is "trivial" in the sense of [Lur11, Def. 3.2.1.7]). For these we have the following result:

Proposition 2.5.5 ([Lur11, Prop. 3.4.2.1])**.** *If 1 is the trivial commutative algebra, there is an equivalence $\theta\colon \mathrm{Mod}_1^{\mathbb{E}_\infty}(\mathcal{X}) \to \mathcal{X}$.*

The previous proposition can be combined with Prop. 2.5.2 to give another proof of Prop. 2.5.2.

Since a trivial commutative algebra is an initial object in the category of commutative algebras ([Lur11, Prop. 3.2.1.8]), we have from Prop 2.5.3 for any commutative algebra A a morphism of algebras $1 \to A$, which induces the forgetful functor $\theta\colon \mathrm{Mod}_1^{\mathbb{E}_\infty}(\mathcal{X}) \to \mathcal{X}$

Proposition 2.5.6. *Let A be a commutative algebra. There is a free-forgetful adjunction $(A \times -)\colon \mathcal{X} \to \mathrm{Mod}_A^{\mathbb{E}_\infty}(\mathcal{X}) : \theta$. In particular the free A-modules are of the form $A \times M$ for $M \simeq \mathrm{Triv}(\theta(M))$ a module with trivial action.*

Proof. Combine Prop. 2.5.3 and Prop. 2.5.5. □

For any commutative algebra A the essentially unique map $A \to 1$ is the underlying map of an algebra map. This gives us another adjunction for every commutative algebra A:

Definition 2.5.7. *Let A be a commutative algebra. We denote the adjoint functors associated to the morphism $A \to 1$ by $(-/A)\colon \mathrm{Mod}_A^{\mathbb{E}_\infty}(\mathcal{X}) \to \mathcal{X} : \mathrm{Triv}$ and call $(-/A)$ the quotient functor and* Triv *the trivial action functor.*

Lemma 2.5.8. *We have the following formulas:*

(a) $(A \times X)/A \simeq X$

(b) $\theta(\mathrm{Triv}(X)) \simeq X$

Proof. (a) The functor $(A \times -)/A\colon \mathcal{X} \to \mathcal{X}$ is the left adjoint functor associated (by Prop. 2.5.3(i)) to the consecutive maps of commutative algebras $1 \to A \to 1$. Any composite of the consecutive maps is equivalent to te identity, hence so is the induced functor by Prop. 2.5.3(ii).

(b) As in (a) but with the right adjoints. □

Proposition 2.5.9. *The forgetful functor $\theta\colon \mathrm{Mod}_A^{\mathbb{E}_\infty}(\mathcal{X}) \to \mathcal{X}$ reflects equivalences.*

Let X be an A-module. The identity $\theta(X) \to \theta(X)$ corrsponds, under the adjunction $((A \times -) \dashv \theta)$, to a map $A \times \mathrm{Triv}(\theta(X)) \to X$ of A-modules. We also have the projection $A \times \mathrm{Triv}(\theta(X)) \xrightarrow{pr} A$. Together these two maps induce a map into the product A-module $A \times \theta(X) \to A \times X$.

Proposition 2.5.10. *Let A be a grouplike commutative algebra. Then the map $A \times \mathrm{Triv}(\theta(X)) \to A \times X$ is an equivalence.*

Proof. We check that the map from the claim is an isomorphism in the homotopy category $\mathrm{Ho}\,\mathrm{Mod}_A^{\mathbb{E}_\infty}(\mathcal{X})$, or equivalently by Prop. 2.5.9, in the homotopy category $\mathrm{Ho}\,\mathcal{X}$.

Denoting by $m\colon A \times X \to X$ the morphism obtained from the A-module structure on X. Then the morphism of the claim is given by $A \times \mathrm{Triv}(\theta(X)) \xrightarrow{(pr_1, m)} A \times X$ (intuitively it can be described as $(a, x) \mapsto (a, a \cdot x)$).

Since A is grouplike, in the homotopy category there is an inverse morphism $i\colon A \to A$ fitting into the usual diagrams from the definition of group objects. Using this we can produce a two-sided inverse of (pr_1, m) in the homotopy category, namely $(pr_1, m \circ (i \times id_X))\colon A \times X \to A \times \mathrm{Triv}(\theta(X))$. □

Proposition 2.5.11. *Let A be a grouplike commutative algebra and X be an A-module. Then we have an equivalence $(A \times X)/A \simeq \theta(X)$ in $\mathcal{X}$.*

Proof. By Prop. 2.5.10 we can consider $A \times \mathrm{Triv}(\theta(X))$ instead of $A \times X$. But $(A \times \mathrm{Triv}(\theta(X)))/A$ is the result of applying to $\theta(X)$ first the functor $(A \times -)$, which is the left adjoint associated to the morphism of commutative algebras $1 \to A$ as in Prop. 2.5.3(i), and then the functor $(-)/A$, which is the left adjoint associated to the morphism of commutative algebras $A \to 1$. By Prop. 2.5.3(ii) the composition of these functors is the functor associated to the identity $1 \to 1$ and hence itself the identity. □

2.6 Powers in commutative monoids

Let M be a commutative algebra object in an ∞-category $\mathcal{X}$ with cartesian monoidal structure, i.e. a section s of the cocartesian fibration $\mathcal{X}^\times \to N(\mathrm{FinSet}_*)$. Then in $\mathcal{X}$ we have k-fold product maps $\mu_k \colon M^k \to M$ obtained by applying the section s to the active map $[k] \to [1]$ in FinSet_* and factoring it into a cocartesian lift and a map over $[1]$. Precomposing with the k-fold diagonal we obtain the k-th power operation $M \xrightarrow{\Delta} M^k \xrightarrow{\mu_k} M$. One can ask whether this is, as in 1-categorical algebra, a morphism of commutative algebras. This is not completely obvious because, in a 1-category, showing the compatibility with the products requires some shuffling of coordinates, which can become a nontrivial operation in ∞-categories. However, we can show that it remains true in ∞-categories.

Since the morphism in question involves a diagonal we need to place ourselves in a setting for *cartesian* commutative algebra, which allows an easy treatment of diagonals. This is provided by Cranch in his thesis [Cra09] and the preprint [Cra11], as well as [GGN15] and some passages of [Lur09].

The general setup of cartesian algebra is to encode types of algebraic structures in categories with finite products, such that there is an object of which every other object is a finite power. Such categories are called *Lawvere theories* or *algebraic theories*. A Lawvere theory is always given by the opposite category of its free algebras generated by finite sets. A model of an algebraic theory $\mathbb{T}$ in some target category $\mathcal{X}$ is a product preserving functor $\mathbb{T} \to \mathcal{X}$. We denote by $\mathrm{Fun}_{pp}(\mathbb{T}, \mathcal{X})$ the full subcategory of the functor category $\mathcal{X}^{\mathbb{T}}$ whose objects are the product preserving functors.

We now sketch some basic ideas of Cranch's approach. As a substitute for the category FinSet_* occurring in Lurie's setup for general algebra in symmetric monoidal ∞-categories, Cranch considers the $(2,1)$-category (i.e. category enriched in 1-groupoids) 2Span [Cra11, Section 4.2] whose objects are finite sets, morphisms are spans and a 2-morphism between two spans is a bijection between the middle objects making the two triangles commute. Composition of 1-morphisms is given by placing two spans side by side and taking the pullback of the inner morphisms. There is a nerve functor N from $(2,1)$-categories to ∞-categories, defined by taking the nerve of the Hom-groupoids and then applying the usual simplicial nerve of [Lur09, Def. 1.1.5.5] to the resulting simplicial category.

A cartesian monoidal structure on an ∞-category $\mathcal{X}$ is then modelled by a cocartesian fibration $\mathcal{X}^\times \to N(\mathrm{2Span})$, a commutative monoid by a section $m \colon N(\mathrm{2Span}) \to \mathcal{X}$ of this fibration that maps collapsing morphisms (the analoga of the inert morphisms from [Lur11], see [Cra09, Section 4.9]) to collapsing morphisms and a morphism of commutative monoids by a morphisms of functors without further conditions.

The basic idea is, as in [Lur11], that the functor m corresponding to a monoid M sends a finite set $[r]$ to M^r (more accurately to a sequence $(M,\dots,M)$ of r copies of M, living in the fiber over $[r]$ of $\mathcal{X}^{\times}$). A span $[r] \leftarrow [s] \rightarrow [t]$ can be decomposed into its left and its right half. The role of the right half is, as in symmetric monoidal algebra [Lur11], to model the permutation of factors and the multiplication through automorphisms, resp. active morphisms of finite sets. The role of the left half is to model diagonals and projections: The morphism $M^r \rightarrow M^s$ corresponding to a map $([r] \leftarrow [s] \colon f)$ sends the ith factor of M^r to $M^{|f^{-1}(i)|}$ via the $|f^{-1}(i)|$-fold diagonal (where 0-fold diagonal means the unique map to the terminal object $1 = M^0$ and 1-fold diagonal means the identity map). Thus the k-th power map is modelled by the span $[1] \leftarrow [k] \rightarrow [1]$ where both arrows are the unique morphisms. Cranch shows in [Cra11, Prop. 5.3] that $N(\text{2Span})$ is equivalent to the ∞-category $\mathbb{T}_{CMon}$ of free $\mathbb{E}_\infty$-algebras over finite sets, and in [Cra11, Thm. 5.7]) that commutative monoids in his sense are equivalent to commutative monoids in Lurie's sense.

Given a commutative monoid $g\colon N(\text{2Span}) \rightarrow \mathcal{X}^{\times}$, we want to produce, for every $k \in \mathbb{N}$, a morphism of commutative monoids whose underlying morphism is the above k-th power map, i.e. a map of functors $h^k\colon \Delta^1 \times N(\text{2Span}) \rightarrow \mathcal{X}$ such that $h^k|_{\{0\}\times N(\text{2Span})} \simeq h^k|_{\{1\}\times N(\text{2Span})} \simeq g$ and $h^k|_{\Delta^1\times\{[1]\}} \simeq (M \xrightarrow{\Delta} M^k \xrightarrow{\mu_k} M)$. We will actually do a bit more: It is also true in 1-category theory that the kth and the lth power maps compose to give the $(k \cdot l)$-th power map (in particular they commute). We will establish this fact, too.

To this end we consider a functor $F\colon \text{2Span}\times\text{2Span} \rightarrow \text{2Span}$ given by taking the product of finite sets and maps between them. Explicitly it is given as follows:

1. On objects: $F(X, X') := X \times X'$.
2. For every pair of objects $x := (X, X'), y := (Y, Y')$ we have a functor

$$F_{x,y}\colon (\text{2Span} \times \text{2Span})((X, X'), (Y, Y')) \rightarrow \text{2Span}(X \times X', Y \times Y'),$$

defined

- On objects: $F_{x,y}(X \xleftarrow{f} S \xrightarrow{g} Y,\ X' \xleftarrow{f'} S' \xrightarrow{g'} Y') := X \times X' \xleftarrow{f\times f'} S \times S \xrightarrow{g\times g'} Y \times Y'$
- On morphisms: If $i\colon S \rightarrow T$ is a morphism from the span $X \xleftarrow{f_1} S \xrightarrow{g_1} Y$ to the span $X \xleftarrow{f_2} T \xrightarrow{g_2} Y$, and $j\colon S \rightarrow T$ is a morphism from the span $X' \xleftarrow{f_1'} S' \xrightarrow{g_1'} Y'$ to the span $X' \xleftarrow{f_2'} T' \xrightarrow{g_2'} Y'$, we define $F_{X,X',Y,Y'}(i, j) := i \times j$.

3. To three objects $x := (X, X'), y := (Y, Y'), z := (Z, Z')$ in $Ob\,\text{2Span} \times \text{2Span}$ natural isomorphisms $\gamma_{x,y,z}\colon c_{X\times X',Y\times Y',Z\times Z'} \circ (F_{x,y} \times F_{y,z}) \Rightarrow F_{x,z} \circ c_{x,y,z}$ (where $c_{ijk}\colon Hom(i, j) \times Hom(j, k) \rightarrow Hom(i, k)$ denotes the composition functors). Objectwise these are isomorphisms between the upper and the lower span in the following diagram, where the upper span arises by first applying F then composing, and the lower one by first composing, then applying F:

$$
\begin{tikzcd}
 & & (S \times S') \times_{Y \times Y'} (T \times T') \arrow[dl] \arrow[dr] & & \\
 & S \times S' \arrow[dl, "f \times f'"'] \arrow[dr, "g \times g'"] & & T \times T' \arrow[dl, "h \times h'"'] \arrow[dr, "k \times k'"] & \\
X \times X' & & Y \times Y' & & Z \times Z' \\
 & & (S \times_Y T) \times (S' \times_{Y'} T') \arrow[ull] \arrow[urr] & &
\end{tikzcd}
$$

We now define $\gamma_{r,t,x}$ to be the obvious isomorphism

$$
\begin{aligned}
(S \times S') \times_{Y \times Y'} (T \times T') &\cong \{(s, s', t, t') \in S \times S' \times T \times T' \mid (g \times h)(s, t) = (g' \times h')(s', t')\} \\
&= \{(s, s', t, t') \in S \times S' \times T \times T' \mid g(s) = h(t), g'(s') = h'(t')\} \cong (S \times_Y T) \times (S' \times_{Y'} T')
\end{aligned}
$$

coming from the fact that the upper and the lower object represent the same functor. This latter fact also implies naturality.

Lemma 2.6.1. *With the above definition F is a bifunctor.*

Proof. Identities in each level are preserved strictly, so the usual coherences for identities of a pseudofunctor are not an issue. It is clear that the $F_{x,y}$ are functors, i.e. preserve compositions. It remains to check the compatibility condition that, for morphisms $f := (X \leftarrow S \rightarrow Y, X' \leftarrow S' \rightarrow Y'), g := (Y \leftarrow T \rightarrow Z, Y' \leftarrow T' \rightarrow Z'), h := (Z \leftarrow U \rightarrow W, Z' \leftarrow U' \rightarrow W')$ of 2Span $\times$ 2Span, the following diagram commutes (where as before we abbreviate $x := (X, X'), y := (Y, Y'), z := (Z, Z'), w := (W, W')$).

$$
\begin{tikzcd}
 & F(h) \circ (F(g) \circ F(f)) \arrow[r, "F(h) \circ \gamma_{x,y,z}(f,g)"] & F(h) \circ F(g \circ f) \arrow[dr, "\gamma_{x,z,w}(h,g \circ f)"] & \\
(F(h) \circ F(g)) \circ F(f) \arrow[ur, "\cong"] \arrow[dr, "\gamma_{x,y,z}(g,h) \circ F(f)"'] & & & F(h \circ (g \circ f)) \\
 & F(h \circ g) \circ F(f) \arrow[r, "\gamma_{y,z,w}(f,h \circ g)"'] & F((h \circ g) \circ f) \arrow[ur, "\cong"'] &
\end{tikzcd}
$$

This, i.e. the equality of the upper and the lower path, follows from the fact that both give the isomorphism between the left and the right object which arises from from both representing the same functor. □

Proposition 2.6.2. *The bifunctor F preserves products in each variable.*

Proof. By [Cra11, Prop. 4.6] products in 2Span are given by coproducts of finite sets. Since in FinSet coproducts distribute over products, the claim follows from the definition of F via products in FinSet. □

Thus for every commutative monoid in a cartesian closed category we get by adjunction another (but in fact equivalent) commutative monoid structure on that monoid:

Corollary 2.6.3. *There is a functor $CMon(\mathcal{X}) \simeq Fun_{pp}(N(2Span), \mathcal{X}) \xrightarrow{-\circ N(F)} Fun_{pp}(N(2Span) \times N(2Span), \mathcal{X}) \simeq Fun_{pp}(N(2Span), Fun_{pp}(N(2Span), \mathcal{X})) \simeq CMon(CMon(\mathcal{X}))$.*

The functor of this last corollary should in fact be an equivalence, since commutative monoid objects in commutative monoid objects are just commutative monoid objects. This latter statement is [Lur11, Thm 5.1.2.2] in the context of operads (which is equivalent to that of algebraic theories by [Cra11, Prop. 5.3]). In the context of presentable categories see also [GGN15, Thm. 4.6]; CMon(−) is a smashing localization. However, while it clarifies the general situation, we do not explicitly need this fact here.

Power operations for monoids: Let us now switch to the notation $\mathbb{T}_{CMon} := N(2\mathrm{Span})$. To obtain the power operations for a commutative monoid we now simply precompose the functor $\mathbb{T}_{CMon} \to \mathrm{Fun}_{pp}(\mathbb{T}_{CMon}, \mathcal{X})$ with the inclusion of the full subcategory $\mathbf{\Psi}_{\mathbb{N}} \hookrightarrow \mathbb{T}_{CMon}$ whose only object is the one element set. This only object is mapped to the underlying object of the given monoid und thus the datum of the composite functor consists of a map of mapping spaces $map_{\mathbb{T}_{CMon}}([1],[1]) \to map_{\mathrm{CMon}}(M, M)$, endowing M with monoid endomorphisms. By the standard construction of algebraic theories as the opposite category of the free models generated by finite sets, or by the explicit construction above, we know that $map_{\mathbb{T}_{CMon}}([1],[1])$ is the free commutative monoid $\coprod_{n\in\mathbb{N}} B\Sigma_n$. One sees that the components of the mapping space are in bijection with the natural numbers, where the nth component corresponds to the operation $x \mapsto x^n$.

Power operations for groups: We denote $\mathbb{T}_{\mathrm{CGrp}}$ the algebraic theory of commutative groups objects. Now suppose that the given commutative monoid is in fact a group object, i.e. that its classifying map factors through the algebraic theory of groups: $\mathbb{T}_{\mathrm{CMon}} \to \mathbb{T}_{\mathrm{CGrp}} \to \mathcal{X}$. Then again we can precompose with $N(F)$ obtaining a product preserving functor which factors through group objects: $\mathbb{T}_{\mathrm{CMon}} \to \mathrm{Fun}_{pp}(\mathbb{T}_{\mathrm{CGrp}}, \mathcal{X}) \to \mathrm{Fun}_{pp}(\mathbb{T}_{\mathrm{CMon}}, \mathcal{X})$. But a monoid object in the ∞-category of group objects is automatically a group object: indeed, it is enough to run the classical Eckmann-Hilton argument in the homotopy category since whether a monoid is group object can be detected in the homotopy category.

Thus for a group object G in $\mathcal{X}$, we obtain a functor $\mathbb{T}_{\mathrm{CGrp}} \to \mathrm{Fun}_{pp}(\mathbb{T}_{\mathrm{CGrp}}, \mathcal{X})$, which we can again restrict to its full subcategory $\mathbf{\Psi}$ containing only th object [1]. Again this restriction amounts to a map of mapping spaces $map_{\mathbb{T}_{\mathrm{CGrp}}}([1],[1]) \to map_{\mathrm{CGrp}}(G, G)$. And again, the mapping space is given by the free commutative group, which is in turn obtained by applying $\Omega^\infty\Sigma^\infty$ to the free monoid. Thus $map_{\mathbb{T}_{\mathrm{CGrp}}}([1],[1]) \simeq QS^0$, the infinite loop space of the sphere spectrum. One sees that the components of the mapping space are in bijection with the integers, where the nth component corresponds to the operation $x \mapsto x^n$ (now $n \in \mathbb{Z}$).

Remark 2.6.4. Later on we will consider Ψ as an operad via the embedding of categories into operads. In these terms, we have seen that given a commutative group object, we can see it as a (CMon ⊗ $\mathbf{\Psi}$)-algebra, where CMon denotes the operad for commutative monoids, $\mathbf{\Psi}$) denotes the operad above, and ⊗ denotes the tensor product of operads. We pass here from algebraic theories to operads, because we wish to transport this structure along monoidal funcors which go to non-cartesian monoidal ∞-categories. ◊

3 Abstract Motivic Homotopy Theory

Let $\mathcal{X}$ be a cartesian closed, presentable ∞-cat with a grouplike commutative algebra $\mathbb{G}$.

The example to have in mind is that where Sm_S is some category of smooth "schemes" (where by "scheme" one may also understand things like log schemes, monoid schemes or spectral schemes), and $\mathcal{X} := (\mathrm{sSet}^{Sm_S^{op}})_{loc}$ is the localization of presheaves on Sm_S by an appropriate topology and enforcing the condition that the projections $X \times \mathbb{A}^1 \to X$ be weak equivalences. For this class of examples, on which we will occasionally comment on the way, one has a good model, given by some Bousfield localization of simplicial presheaves on Sm_S.

3.1 Basic unstable objects and calculations

3.1.1 Punctured affine spaces

The punctured affine spaces $\mathbb{A}^n \setminus 0$ will be certain $\mathbb{G}$-modules. We shall use the abbreviation $\mathbb{G}\text{-Mod} := \mathrm{Mod}_{\mathbb{G}}^{\mathbb{E}_\infty}(\mathcal{X})$ for the category of $\mathbb{G}$-modules. We will notationally not distinguish $\mathbb{G}$-modules from their underlying objects, i.e. we will leave the forgetful functor $\theta\colon \mathbb{G}\text{-Mod} \to \mathcal{X}$ from section 2.5 out of the notation.

We have a functor $f\colon \Delta^1 \to \mathbb{G}\text{-Mod}$ corresponding to the (essentially unique) map $\mathbb{G} \to 1$ from $\mathbb{G}$ to the terminal $\mathbb{G}$-module. This gives us a hypercube diagram $(\Delta^1)^n \xrightarrow{f^n} (\mathbb{G}\text{-Mod})^n \xrightarrow{\times} \mathbb{G}\text{-Mod}$, where the second map is the product functor coming from the cartesian monoidal structure of $\mathbb{G}$-Mod [1]. The hypercube $(\Delta^1)^n$ can be identified with the nerve of the set of subsets of $\{1, \ldots, n\}$ ordered by the opposite of the inclusion relation (of course the inclusion relation itself would serve as well, but it will shorten notation in the things to come to take the opposite).

We denote by P_n the ordered set of nonempty subsets of the finite set $\{1, \ldots, n\}$ and by $K_n := N(P_n^{op})$ the nerve of its opposite category. We have the inclusion $K_n = N(\wp(\{1, \ldots, n\} \setminus \{\emptyset\})^{op}) \hookrightarrow N(\wp(\{1, \ldots, n\})^{op}) \simeq (\Delta^1)^n$ and we denote by d_n the composition

$$K_n \hookrightarrow (\Delta^1)^n \xrightarrow{f^n} (\mathbb{G}\text{-Mod})^n \xrightarrow{\times} \mathbb{G}\text{-Mod}$$

Definition 3.1.1. *The n-dimensional punctured affine plane ist defined to be the $\mathbb{G}$-module $\mathbb{A}^n \setminus 0 := \operatorname{colim} d_n$.*

Remark 3.1.2. By the description of the Day convolution monoidal structure on $\mathcal{X}^{\Delta^1}$ of section 2.4, the object $\mathbb{A}^n \setminus 0$ is the domain of the n-fold product of the arrow $\mathbb{G} \to 1$. $\diamond$

Remark 3.1.3. We give a more concrete description of the diagram d_n.

For a nonempty subset $S \subseteq \{1, \ldots, n\}$ we have

$$d_n(S) = \prod_{i=1\ldots n} \mathbb{G}^{\chi_S(i)} \simeq X^{|S|},$$

[1] More precisely the product functor is the map $\Delta^1 \to Cat_\infty$ corresponding to the categorical fibration $E \to \Delta^1$ which is obtained by taking the categorical fibration $\mathcal{X}^\times \to N(\mathrm{Fin}_*)$ given by the cartesian monoidal structure on $\mathcal{X}$ and pulling back along the map $\Delta^1 \to N(\mathrm{Fin}_*)$ which chooses the active morphism $\{*, 1, \ldots, n\} \to \{*, 1\}$ sending $1, \ldots, n$ to 1.

where χ_S denotes the characteristic function of $S \subset \{1, \dots, n\}$. Thus for example $d_4(\{1,3\}) \simeq \mathbb{G} \times 1 \times \mathbb{G} \times 1 \simeq \mathbb{G}^2$.

For an inclusion $j \colon S \to T$ we define

$$p_j(i) := \begin{cases} pr \colon \mathbb{G} \to 1, & \text{if } \chi_S(i) = 0 \text{ and } \chi_T(i) = 1 \\ id_1, & \text{if } \chi_S(i) = \chi_T(i) = 0 \\ id_{\mathbb{G}}, & \text{if } \chi_S(i) = \chi_T(i) = 1 \end{cases}$$

On 1-simplices d_n is then given by $d_n(j) = \prod_{i=1}^n p_j(i) \colon d_n(T) \to d_n(S)$, i.e. for every element of T we have a $\mathbb{G}$-factor in the domain, for every element of S we have a $\mathbb{G}$-factor in the codomain, and those factors which occur in the domain but not the codomain get projected away. $\diamond$

Remark 3.1.4. In the case where $\mathcal{X}$ is modelled by a cartesian Bousfield localization of the injective model structure on $\mathrm{sSet}^{Sm_S^{op}}$, with Sm_S some category of "schemes", one typically has injections $i \colon \mathbb{G} \hookrightarrow \mathbb{A}^1$. Replacing the occurrences of $1 = \mathbb{G}^0$ in the definition of the diagram d_n by A^1, and the maps $\mathbb{G} \to 1$ by $i \colon \mathbb{G} \hookrightarrow \mathbb{A}^1$, we obtain a diagram d_n' in the model category. Taking its 1-categorical colimit we obtain what is typically called "$\mathbb{A}^n \setminus 0$" in geometry, e.g. in usual algebraic geometry the Zariski sheafification of the functor $Rings \to Sets \hookrightarrow \mathrm{sSet}$ which associates to a ring the set of n-tuples of elements of which at least one is a unit. Since in these settings we have injections, i.e. cofibrations, $\mathbb{A}^n \setminus 0 \hookrightarrow \mathbb{A}^n$, and products of injections are injections, the diagram d_n' can immediately be seen to be Reedy cofibrant and hence a Reedy cofibrant replacement of (the lift to the model category of) the diagram d_n. Thus the usual geometric objects "$\mathbb{A}^n \setminus 0$" are indeed representatives of the $\mathbb{A}^n \setminus 0$ we just constructed in the ∞-category $\mathcal{X}$. $\diamond$

Remark 3.1.5. In our general setting we could present $\mathrm{Mod}_{\mathbb{G}}^{\mathbb{E}_\infty}(\mathcal{X})$ by a model category and define $\mathbb{A}^1$ to be the object occurring in the middle of a factorization $\mathbb{G} \hookrightarrow \mathbb{A}^1 \to 1$ into a cofibration followed by a weak equivalence. It is possible to do this in such a way that the diagram d_n' of Remark 3.1.4 is Reedy cofibrant.

If we choose to start with the *injective* model structure $\mathrm{sSet}^{C^{op}}$ and then localize appropriately, we obtain a model in which cofibrations are monomorphisms and products of cofibrations are again cofibrations. Then the pushout product morphisms that we need to check for Reedy cofibrancy sit in diagrams of the form

$$\begin{tikzcd}
A \times A \arrow[r, tail] \arrow[d, tail] & A \times B \arrow[d, tail] \arrow[ddr] & \\
B \times A \arrow[r, tail] \arrow[drr, tail] & P \arrow[dr, dotted] & \\
& & B \times B
\end{tikzcd}$$

From the pointwise and simplicially levelwise computation of colimits in $\mathrm{sSet}^{C^{op}}$ we can deduce that the dotted arrow is also a monomorphism, hence a cofibration. $\diamond$

We have two inclusions $P_n \hookrightarrow P_{n+1}$: The first one, i_l identifies P_n with the ordered set of those subsets of $\{1, \dots, n+1\}$ which do not contain $n+1$. The second one, i_r sends $S \subseteq \{1, \dots, n\}$ to the set $S' \subseteq \{1, \dots, n+1\}$ obtained from S by adding 1 to each element. This gives us two maps of simplicial sets $N(P_n^{op}) \to N(P_{n+1}^{op})$ inducing maps between the colimits $\operatorname{colim} d_n \to \operatorname{colim} d_{n+1}$ (see [Cra09, Prop. 2.29] for the functoriality of finite colimits in the indexing simplicial sets)..

Definition 3.1.6. *We denote the map induced by i_l by $z_n \colon \mathbb{A}^n \setminus 0 \to \mathbb{A}^{n+1} \setminus 0$ and the map induced by i_r by $z'_n \colon \mathbb{A}^n \setminus 0 \to \mathbb{A}^{n+1} \setminus 0$. Composing these maps we obtain $l_n := z_{2n-1} \circ \ldots \circ z_n \colon \mathbb{A}^n \setminus 0 \to \mathbb{A}^{2n} \setminus 0$ and $r_n := z'_{2n-1} \circ \ldots \circ z'_n \colon \mathbb{A}^n \setminus 0 \to \mathbb{A}^{2n} \setminus 0$*

Intuitively (and in the case of schemes: in fact) the map z_n (resp. z'_n) inserts a given sequence of coordinates of a point of $\mathbb{A}^n \setminus 0$ as the left (resp. right) bit of the sequence of coordinates of a point of $\mathbb{A}^{n+1} \setminus 0$. The last coordinate can be left undetermined in homotopy theory (and in the case of schemes: can be set to zero).

We have a diagram

$$\begin{tikzcd} \mathbb{A}^n \setminus 0 \times \mathbb{G} \arrow[r, "pr_2"] \arrow[d, "pr_1"'] & \mathbb{G} \arrow[d] \\ \mathbb{A}^n \setminus 0 \arrow[r, "z_n"] & \mathbb{A}^{n+1} \setminus 0 \end{tikzcd}$$

where the right vertical arrow is the canonical arrow $d_{n+1}(\{n+1\}) \to \operatorname{colim} d_{n+1}$.

Proposition 3.1.7. *The above diagram is a pushout diagram.*

Proof. This is an instance of Corollary 2.2.4: The lower left and upper right corner are exactly as stated there. For the upper left corner note that the subdiagram K^0_{n+1} of Corollary 2.2.4 is the full simplicial subset of K_{n+1} with vertices those $S \subseteq \{1, \dots, n+1\}$ containing $n+1$ and at least one other element.

The objects in the image of the diagram $d_{n+1}|_{K^0_{n+1}}$ are of the form $X \times \mathbb{G}$ and the morphisms of the form $f \times id_{\mathbb{G}}$, where X, resp. f, runs over the objects, resp. morphims, occurring in the diagram d_n. Since taking product with $\mathbb{G}$ preserves colimits, we have $\operatorname{colim} d_{n+1}|_{K^0_{n+1}} \simeq \mathbb{G} \times \mathbb{A}^n \setminus 0$ □

Remark 3.1.8. Of course by symmetry the above proposition holds if one replaces z_n with z'_n. We will need both versions. ◊

Remark 3.1.9. In the situation of Remark 3.1.4 we have maps $inc_n \colon \mathbb{A}^n \setminus 0 \to \mathbb{A}^n$, given by the n-fold product of the morphism $\mathbb{G} \to \mathbb{A}^1$ in the Day convolution structure on the category of morphisms. The above diagram then turns into

$$\begin{tikzcd} \mathbb{A}^n \setminus 0 \times \mathbb{G} \arrow[r, "inc_n \times id"] \arrow[d, "id \times inc_1"'] & \mathbb{A}^n \times \mathbb{G} \arrow[d] \\ \mathbb{A}^n \setminus 0 \times \mathbb{A}^1 \arrow[r, "z_n"] & \mathbb{A}^{n+1} \setminus 0 \end{tikzcd}$$

In the case where $\mathcal{X}$ is modelled by a cartesian Bousfield localization of the injective model structure on $\mathrm{sSet}^{Sm_S^{op}}$, with Sm_S some category of schemes (this time actual schemes, not "schemes"!) the map z_n can be taken to be the one adding a 0 coordinate at the extra $\mathbb{A}^1$-factor. Since 0 is a $\mathbb{G}$-fixed point in the $\mathbb{G}$-module $\mathbb{A}^1$, this is indeed a map of $\mathbb{G}$-modules.

In more general situations such a fixed point is not always available, e.g. in the case of monoid geometry, when we do not require monoids to have a zero element. This explains a bit of the complications which follow now. ◊

Remark 3.1.10. We get pushouts like in the previous proposition for every partition $A \coprod B = \{1, \ldots, 2n\}$, not just $A := \{1, \ldots, n\} \coprod \{n+1, \ldots, 2n\} =: B$ (for the set $\{1, \ldots, 2n\}$). The proof almost no changes. This is not necessary for the contractibility of $\mathbb{A}^\infty \setminus 0$ but it might be useful for the analysis of the Σ_n-action on $\mathbb{A}^n \setminus 0$. ◊

Proposition 3.1.11. *There is a pushout square*

$$\begin{tikzcd}
\mathbb{A}^n \setminus 0 \times \mathbb{A}^n \setminus 0 \arrow[r, "pr_2"] \arrow[d, "pr_1"'] & \mathbb{A}^n \setminus 0 \arrow[d, "r_n"] \\
\mathbb{A}^n \setminus 0 \arrow[r, "l_n"] & \mathbb{A}^{2n} \setminus 0
\end{tikzcd}$$

Proof. We employ Lemma 2.2.3 for the partition $A := \{1, \ldots, n\}$, $B := \{n+1, \ldots, 2n\}$ of the set $\{1, \ldots, 2n\}$.

Maintaining the notation of section 2.2 we obtain the following full simplicial subsets of $K_{2n} = N(P_{2n}^{op})$, (which indexes the diagram whose colimit is $\mathbb{A}^{2n} \setminus 0$):

K_{2n}^a with vertices $\{S \subseteq \{1, \ldots, 2n\} \mid S \cap A \neq \emptyset\}$, $\widetilde{K}_n^a$ with vertices $\{S \mid \emptyset \neq S \subseteq A\}$,
K_{2n}^b with vertices $\{S \subseteq \{1, \ldots, 2n\} \mid S \cap B \neq \emptyset\}$, $\widetilde{K}_{2n}^b$ with vertices $\{S \mid \emptyset \neq S \subseteq B\}$,
$K_{2n}^0 = K_{2n}^a \cap K_{2n}^b$ with vertices $\{S \mid S \cap A \neq \emptyset \neq S \cap B\}$.

We have isomorphisms $i_a \colon K_n \to \widetilde{K}_{2n}^a$ induced by the inclusion $\{1, \ldots, n\} \subseteq \{1, \ldots, 2n\}$ and $i_b \colon K_n \to \widetilde{K}_{2n}^b$ induced by the map $\{1, \ldots, n\} \subseteq \{1, \ldots, 2n\},\ i \mapsto i + n$.

Consider the following diagram of simplicial sets:

$$\begin{tikzcd}
\widetilde{K}_{2n}^a \arrow[r, hook] & K_{2n} \arrow[r, hook] & (\Delta^1)^{2n} \arrow[r, "f^{2n}"] & \mathcal{X}^{2n} \arrow[r, "\times"] \arrow[dr, phantom, "\Downarrow"] & \mathcal{X} \\
 & & (\Delta^1)^n \times \{1\}^n \arrow[u, hook] & \mathcal{X}^n \times \{*\}^n \arrow[u, "id^n \times 1^n"] & \\
 & K_n \arrow[r, hook] \arrow[uul, "i_a", "\cong"'] & (\Delta^1)^n \arrow[r, "f^n"] \arrow[u, "\cong"] & \mathcal{X}^n \arrow[u, "\cong"] \arrow[uur, "\times"'] &
\end{tikzcd}$$

The upper vertical arrows fill up the second half of the factors in the target $(\Delta^1)^{2n}$, resp. $\mathcal{X}^{2n}$ with the respective terminal objects. The right and middle areas commute. The double arrow in the left triangle is meant to indicate the natural isomorphism of the two functors $\mathcal{X}^n \to \mathcal{X}$ lifting the active morphism $\{*, 1, \ldots, n\} \to \{*, 1\}$, resp. the composition of morphisms $\{*, 1, \ldots, n\} \hookrightarrow \{*, 1, \ldots, 2n\} \to \{*, 1\}$ of FinSet_* in the opfibration $\mathcal{X}^\times \to N(\mathrm{FinSet}_*)$ giving the cartesian monoidal structure on $\mathcal{X}$. The left and upper arrows compose to give the diagram $d_{2n}|_{\widetilde{K}_{2n}^a}$ and the lower and right arrows compose to give the diagram d_n.

Altogether we get an equivalence of diagrams $d_{2n}|_{\widetilde{K}^a_{2n}} \simeq d_n$ and similarly $d_{2n}|_{\widetilde{K}^b_{2n}} \simeq d_n$[2]. Hence $\operatorname{colim} d_{2n}|_{\widetilde{K}^a_{2n}} \simeq \operatorname{colim} d_n \simeq \mathbb{A}^n \setminus 0$ and similarly $\operatorname{colim} d_{2n}|_{\widetilde{K}^b_{2n}} \simeq \mathbb{A}^n \setminus 0$, which accounts for the upper right and lower left corners of the diagram in the claim.

To address the upper left corner, note that we have an isomorphism of ordered sets $\{S \subseteq \{1, \dots, 2n\} | S \cap A \neq \emptyset \neq S \cap B\} \to \{S \subseteq \{1, \dots, n\} | S \cap A \neq \emptyset\} \times \{S \subseteq \{n+1, \dots, 2n\} | S \cap B \neq \emptyset\} \cong L_n \times L_n$ given by $S \mapsto (S \cap A, S \cap B)$ (and with inverse $(S, T) \mapsto S \cup T$). This induces an isomorphism of nerves $K^0_{2n} \cong K_n \times K_n$ fitting into the following diagram of simplicial sets:

$$\begin{tikzcd}
K^0_{2n} \arrow[r, hook] \arrow[dr, "\cong"'] & K_{2n} \arrow[r, hook] & (\Delta^1)^{2n} \arrow[r, "f^{2n}"] & \mathcal{X}^{2n} \arrow[r, "\times_{2n}"] \arrow[dr, phantom, "\Downarrow"] & \mathcal{X} \\
& K_n \times K_n \arrow[r, hook] & (\Delta^1)^n \times (\Delta^1)^n \arrow[r, "f^n \times f^n"'] \arrow[u, "\cong"] & \mathcal{X}^n \times \mathcal{X}^n \arrow[r, "(\times_n)\times(\times_n)"'] \arrow[u, "\cong"] & \mathcal{X} \times \mathcal{X} \arrow[u, "\times"']
\end{tikzcd}$$

Again the left and middle areas commute and the right hand square is bounded by two equivalent functors, the equivalence coming from the cocartesian fibration $\mathcal{X}^\times \to \mathrm{FinSet}_*$ by considering the factorization of the active map $\{*, 1, \dots, 2n\} \to \{*, 1\}$ as $\{*, 1, \dots, 2n\} \to \{*, 1, 2\} \to \{*, 1\}$ where the first map sends $\{1, \dots, n\}$ to $\{1\}$ and $\{n+1, \dots, 2n\}$ to $\{2\}$. Thus we obtain an equivalence of functors $d_{2n}|_{K^0_{2n}} \simeq d_n \times d_n$

Since $\mathcal{X}$ is cartesian closed, hence colimits commute with products in each variable, we have an equivalence of functors as indicated in the following diagram

$$\begin{tikzcd}
1 \arrow[r, "d_n \times d_n"] & \mathcal{X}^{K_n} \times \mathcal{X}^{K_n} \arrow[r] \arrow[d, "\mathrm{colim} \times \mathrm{colim}"'] \arrow[dr, phantom, "\Swarrow"] & (\mathcal{X} \times \mathcal{X})^{K_n \times K_n} \arrow[dr, "(\times:\, \mathcal{X}\times\mathcal{X}\to\mathcal{X})\circ-"] & \\
& \mathcal{X} \times \mathcal{X} \arrow[r, "\times"'] & \mathcal{X} & \mathcal{X}^{K_n \times K_n} \arrow[l, "\mathrm{colim}"']
\end{tikzcd}$$

Altogether we obtain

$$\operatorname{colim} d_{2n}|_{K^0_{2n}} \simeq \operatorname{colim}(d_n \times d_n) \simeq (\operatorname{colim} d_n) \times (\operatorname{colim} d_n) \simeq \mathbb{A}^n \setminus 0 \times \mathbb{A}^n \setminus 0.$$

Finally, we show that the left and upper maps occurring in the diagram are indeed the projections. We do it for the first projection, the other case being analogous. First note that there is an isomorphism of ordered sets $\{S \subseteq \{1, \dots, 2n\} \mid S \cap A \neq \emptyset\} \to \{S \subseteq \{1, \dots, n\} \mid S \neq \emptyset\} \times \wp(\{1, \dots, n\})$ given by $S \mapsto (S \cap A, S \cap B)$. This induces an isomorphism on nerves $K^a_{2n} \cong K_n \times (\Delta^1)^n$.

The map we want to identify is the map induced on colimits by

$$\begin{tikzcd}
S \cup T + n & K^0_{2n} \arrow[rr, hook] \arrow[ddr, "d_{2n}|_{K^0_{2n}}"] & & K^a_{2n} \arrow[ddl, "d_{2n}|_{K^a_{2n}}"'] \arrow[d, "\cong"] & S \arrow[d, mapsto] \\
(S, T) \arrow[u, mapsto] & K_n \times K_n \arrow[u, "\cong"] \arrow[rr, "id\times incl"] \arrow[dr, "\times\circ(d_n\times d_n)"'] & & K_n \times (\Delta^1)^n \arrow[dl, "\times\circ(d_n\times(\times_n\circ f^n))"] & (S \cap A, S \cap B) \\
& & \mathcal{X} & &
\end{tikzcd}$$

[2]What we described here are the "obvious" maps of diagrams $d_{2n}|_{K^a_{2n}} \circ i_a \simeq d_n \simeq d_{2n}|_{K^b_{2n}} \circ i_b$ given by projecting away the additional n factors with terminal objects occurring in $d_{2n}|_{K^a_{2n}}$ (on the $\{n+1, \dots 2n\}$-half) and in $d_{2n}|_{K^a_{2n}}$ (on the $\{1, \dots, n\}$-half). It just took the little extra effort above to make sure these projections can be assembled coherently into a map of diagrams.

Since $\mathcal{X}$ is cartesian closed, we can take the colimits separately in each variable. One sees that the sought map is of the form $id_{\operatorname{colim} d_n} \times h$ for some $h\colon \operatorname{colim} d_n \to \operatorname{colim}(\times_n \circ f^n)$. Since $(\Delta^1)^n$ has a terminal object, the colimit occurring in the codomain is simply given by evaluation at the terminal object. Thus $\operatorname{colim}(\times_n \circ f^n) \simeq (\times_n \circ f^n)(\emptyset) \simeq 1$, and the h must be the essentially unique morphism $h = !\colon \operatorname{colim} d_n \to 1$. But a product of an identity with this morphism is equivalent to the first projection. ◻

Corollary 3.1.12. *The underlying map of $l_n\colon \mathbb{A}^n \setminus 0 \to \mathbb{A}^{2n} \setminus 0$ in $\mathcal{X}$ (i.e. the image of the map l_n in $\mathbb{G}$-Mod under the forgetful functor $\theta\colon \mathbb{G}\text{-Mod} \to \mathcal{X}$) factors through the terminal object.*

Proof. Since the forgetful functor $\theta\colon \mathbb{G}\text{-Mod} \to \mathcal{X}$ preserves limits and colimits by Prop. 2.5.2 we have a diagram as in Prop. 3.1.11 in $\mathcal{X}$. In $\mathcal{X}$ we also have a morphism $1 \to \mathbb{A}^n \setminus 0$, for example any composition of the cocone map $\mathbb{G} \simeq d_n(\{1\}) \to \operatorname{colim} d_n \simeq \mathbb{A}^n \setminus 0$ with the unit map $1 \to \mathbb{G}$. Hence we can apply Lemma 2.1.1. ◻

Remark 3.1.13. Note that for Corollary 3.1.12 we need to pass from $\mathbb{G}$-Mod to $\mathcal{X}$, because Lemma 2.1.1 requires a morphism $1 \to \mathbb{A}^n \setminus 0$. This does in general not exist in $\mathbb{G}$-Mod it would mean that $\mathbb{A}^n \setminus 0$ has a $\mathbb{G}$-fixed point, which is for example not the case in the setting of monoid schemes. ◊

Remark 3.1.14. In usual algebraic geometry the embedding $\mathbb{A}^n \setminus 0 \hookrightarrow \mathbb{A}^{n+1} \setminus 0, (a_1, \dots, a_n) \mapsto (a_1, \dots, a_n, 0)$ is homotopic to a constant map by the homotopy $\mathbb{A}^1 \times \mathbb{A}^n \setminus 0 \to \mathbb{A}^{n+1} \setminus 0, (t, (a_1, \dots, a_n)) \mapsto ((1-t)a_1, \dots, (1-t)a_n, t)$. Note that since Zariski locally always one of t and $(1-t)$ is a unit, this homotopy does indeed take values in $\mathbb{A}^{n+1} \setminus 0$.

This proof is of course much simpler than the one of Corollary 3.1.12, but a similar strategy fails already for monoid schemes: in that setting the affine line $\mathbb{A}^1$ has only one point and is therefore rather useless for parametrizing homotopies. ◊

Definition 3.1.15. *The colimit (in $\mathbb{G}$-Mod) of the diagram*

$$\mathbb{G} \simeq \mathbb{A}^1 \setminus 0 \xrightarrow{z_1} \mathbb{A}^2 \setminus 0 \xrightarrow{z_2} \mathbb{A}^3 \setminus 0 \xrightarrow{z_3} \mathbb{A}^4 \setminus 0 \xrightarrow{z_5} \dots$$

is called $\mathbb{A}^\infty \setminus 0$.

Theorem 3.1.16. *The $\mathbb{G}$-module $\mathbb{A}^\infty \setminus 0$ is contractible, i.e. the unique map of $\mathbb{G}$-modules $\mathbb{A}^\infty \setminus 0 \to 1$ is an equivalence.*

Proof. The forgetful functor $\theta\colon \mathbb{G}\text{-Mod} \to \mathcal{X}$ preserves colimits, hence

$$\theta(\mathbb{A}^\infty \setminus 0) \simeq \operatorname{colim}(\theta(\mathbb{A}^1 \setminus 0) \xrightarrow{\theta(z_1)} \theta(\mathbb{A}^2 \setminus 0) \xrightarrow{\theta(z_2)} \theta(\mathbb{A}^3 \setminus 0) \xrightarrow{\theta(z_3)} \dots).$$

Equivalently, we can pass to the cofinal subdiagram of punctured affine spaces whose dimension is a power of 2:

$$\theta(\mathbb{A}^\infty \setminus 0) \simeq \operatorname{colim}(\theta(\mathbb{A}^1 \setminus 0) \xrightarrow{\theta(z_1)} \theta(\mathbb{A}^2 \setminus 0) \xrightarrow{\theta(l_2)} \theta(\mathbb{A}^4 \setminus 0) \xrightarrow{\theta(l_4)} \theta(\mathbb{A}^8 \setminus 0) \xrightarrow{\theta(l_8)} \dots).$$

By Cor. 3.1.12 we can replace the l_n by equivalent morphisms factoring through the terminal object. This new diagram now has a cofinal subdiagram in which every object is terminal.

Therefore we know that the essentially unique morphism $\theta(\mathbb{A}^\infty \setminus 0) \to 1$ is an equivalence. But this morphism is equivalent to $\theta(! \colon \mathbb{A}^\infty \setminus 0 \to 1)$, and by Prop. 2.5.9 this implies the claim. $\square$

Remark 3.1.17. Considering the diagram

$$\begin{tikzcd}
\mathbb{A}^n \setminus 0 \arrow[dr, "diag"] \arrow[drr, equal] \arrow[ddr, equal] & & \\
& \mathbb{A}^n \setminus 0 \times \mathbb{A}^n \setminus 0 \arrow[r, "pr_2"'] \arrow[d, "pr_1"] & \mathbb{A}^n \setminus 0 \arrow[d, "r_n"] \\
& \mathbb{A}^n \setminus 0 \arrow[r, "l_n"] & \mathbb{A}^{2n} \setminus 0
\end{tikzcd}$$

shows that the maps l_n and r_n both factor through the diagonal and are actually equivalent. One can model this duplication of "the coordinates of a point of $\mathbb{A}^n \setminus 0$" directly using diagonal maps for each coordinate in the defining diagrams of $\mathbb{A}^n \setminus 0$. $\diamond$

Remark 3.1.18. In the case where $\mathcal{X}$ is modelled by a cartesian Bousfield localization of the injective model structure on $\mathrm{sSet}^{Sm_S^{op}}$, with Sm_S some category of "schemes", we need to model the projection $\mathbb{G} \to 1$ by some cofibration $i \colon \mathbb{G} \hookrightarrow \mathbb{A}^1$. In this case one may ask how to model the map l_n directly. If one has a $\mathbb{G}$-fixed point in $\mathbb{A}^1$ (e.g. 0 in usual algebraic geometry), then one can pass from $\mathbb{A}^n \setminus 0$ to $\mathbb{A}^{2n} \setminus 0$ by inserting the given coordinates in the left half and the fixed point in the right half. When no such fixed point is available, one can still model l_n by duplication of the string of coordinates: The diagram of Remark 3.1.17 can in the model be replaced by

$$\begin{tikzcd}
\mathbb{A}^n \setminus 0 \arrow[dr, "diag"] \arrow[drr, equal] \arrow[ddr, equal] & & \\
& \mathbb{A}^n \setminus 0 \times \mathbb{A}^n \setminus 0 \arrow[r, "pr_2"'] \arrow[d, "pr_1"] & \mathbb{A}^n \setminus 0 \times \mathbb{A}^n \arrow[d, "r_n"] \\
& \mathbb{A}^n \times \mathbb{A}^n \setminus 0 \arrow[r, "l_n"] & \mathbb{A}^{2n} \setminus 0
\end{tikzcd}$$

and the diagonal is indeed an equivariant map of $\mathbb{G}$-modules. $\diamond$

We now set out to define maps $(\mathbb{A}^n \setminus 0) \times (\mathbb{A}^m \setminus 0) \to \mathbb{A}^{nm} \setminus 0$ which "multiply every coordinate with every coordinate". After quotienting out the $\mathbb{G}$-action these will become analogs of the Segre embeddings.

Construction 3.1.19 (Multiplication maps)**.** We have the map of ordered sets $max \colon \{0 < 1\} \times \{0 < 1\} \to \{0 < 1\}$ which induces the map $\coprod \colon \Delta^1 \times \Delta^1 \to \Delta^1$ on nerves. It gives us a diagram $\Delta^1 \times \Delta^1 \xrightarrow{\coprod} \Delta^1 \xrightarrow{f} \mathcal{X}$ which can be depicted as

$$\begin{tikzcd}
\mathbb{G} \arrow[r] \arrow[d] & 1 \arrow[d, equal] \\
1 \arrow[r, equal] & 1
\end{tikzcd}$$

Recall that the diagram d_2 was defined as the composition $K_2 \hookrightarrow \Delta^1 \times \Delta^1 \xrightarrow{f \times f} \mathcal{X}^2 \xrightarrow{\times} \mathcal{X}$. We can construct a morphism of diagrams $t \colon d_2 \to (f \circ \coprod)$ i.e. a map $\Delta^1 \to \mathcal{X}^{\Delta^1 \times \Delta^1}$ using the multiplication map $\mu \colon \mathbb{G} \times \mathbb{G} \to \mathbb{G}$: This morphism can be depicted as

$$\begin{tikzcd}
\mathbb{G}^2 \arrow[rr] \arrow[dr, "\mu"] \arrow[dd] & & \mathbb{G} \arrow[dr] \arrow[dd] & \\
& \mathbb{G} \arrow[rr] \arrow[dd] & & 1 \arrow[dd, equal] \\
\mathbb{G} \arrow[rr] \arrow[dr] & & 1 \arrow[dr, equal] & \\
& 1 \arrow[rr, equal] & & 1
\end{tikzcd}$$

To actually define t, note that the inclusion $\{(0,0)\} \hookrightarrow \Delta^1 \times \Delta^1$ induces a functor $\mathcal{X}^{\Delta^1 \times \Delta^1} \to \mathcal{X}^{\{(0,0)\}} \simeq \mathcal{X}$, namely the evaluation at the object $(0,0)$. By [Lur09, Lemma 4.3.2.13] this functor has a pointwise right Kan extension which sends the object $\mathbb{G} \in \mathcal{X}$ to the diagram $\Delta^1 \times \Delta^1 \to \mathcal{X}$ depicted as the front face of the above cube. By the universal property of right Kan extensions, or more precisely by [Lur09, Prop. 4.3.3.7], to give a map between the two diagrams at hand is equivalent to giving a map between their objects at the corner $\{(0,0)\}$. For this map we take the multiplication map $\mu \colon \mathbb{G}^2 \to \mathbb{G}$ (more precisely: if $\mathbb{G}$ is given as section of the cocartesian fibration $\mathcal{X}^{\times} \to N(\mathrm{FinSet}_*)$, then μ is the morphism obtained by factorizing the image of the active morphism $\{1,2,*\} \to \{1,*\}$ into a cocartesian lift of that morphism and a part living over $\{1,*\}$ and taking that second part).

From the map $t \colon \Delta^1 \times (\Delta^1)^2 \to \mathcal{X}$ that we just constructed we obtain the following map:

$$t^{nm} \colon \Delta^1 \times (\Delta^1)^{nm} \times (\Delta^1)^{nm} \cong \Delta^1 \times ((\Delta^1)^2)^{nm} \xrightarrow{diag^{nm} \times id} (\Delta^1)^{nm} \times ((\Delta^1)^2)^{nm} \cong (\Delta^1 \times (\Delta^1)^2)^{nm} \xrightarrow{t^{nm}} \mathcal{X}^{nm} \qquad \diamond$$

Construction 3.1.20 (Copying Coordinates)**.** For an object X of an ∞-category denote by $diag^n_X$ the n-fold diagonal $X \to X^n$. Given a diagram $g \colon K \to \mathcal{X}$, we have its n-fold product in $\mathcal{X}^K$, given by $g^n \colon K \xrightarrow{diag^n_K} K^n \xrightarrow{\prod_{i=1}^n} \mathcal{X}^n \xrightarrow{\times} \mathcal{X}$, and the n-fold diagonal map $diag^n_g \colon g \to g^n$ which is on each object $k \in K$ given by the n-fold diagonal $diag^n_{g(k)} \colon g(k) \to g^n(k) \simeq g(k)^n$ (since colimits are pointwise by [Lur09, 5.1.2.3]).

Next, consider the following two ways of replicating (morphisms into) an object of an ∞-category: First, there is the map $diag^m_{X^n} \colon X^n \to X^{nm}$, which intuitively takes a sequence of n "elements" of X and produces a string of m copies of that sequence. Second, there is the map $\prod_{i=1}^m diag^n_X \colon X^m \to X^{nm}$ which intuitively takes a sequence of m "elements" of X

and produces a string with n copies of the first element, followed by n copies of the second element, up until n copies of the mth element.

Now given the diagrams $g := \times \circ f^n \in \mathcal{X}^{(\Delta^1)^n}$ and $h := \times \circ f^m \in \mathcal{X}^{(\Delta^1)^m}$, we obtain their product $g \times h \colon (\Delta^1)^n \times (\Delta^1)^m \to \mathcal{X} \times \mathcal{X}$ and a morphism of diagrams given by the product of the above diagonal maps of diagrams $u_{n,m} := diag_g^m \times \prod_{i=1}^m diag_h^n$. ◊

Construction 3.1.21. Given a morphism of K-shaped diagrams, i.e. a map of simplicial sets $h \colon \Delta^1 \times K \to \mathcal{C}$ with $\mathcal{C}$ an ∞-category, then

- from a map $j \colon L \to K$, we obtain a morphism of L-shaped diagrams $\Delta^1 \times L \xrightarrow{id_{\Delta^1} \times j} \Delta^1 \times K \xrightarrow{h} \mathcal{C}$
- from a map $f \colon \mathcal{C} \to \mathcal{D}$, we obtain a morphism of (still K-shaped) diagrams $\Delta^1 \times K \xrightarrow{h} \mathcal{C} \xrightarrow{f} \mathcal{D}$
- from a further morphism of diagrams $g \colon \Delta^1 \times K \to \mathcal{C}$ which is composable with h in the sense that $h|_{\{1\} \times K} = g|_{\{0\} \times K}$, we obtain first a map $\Lambda_1^2 \times K \to \mathcal{C}$ corresponding to a map $\Lambda_1^2 \to \mathcal{C}^K$ which can be extended to a map $\Delta^2 \to \mathcal{C}^K$ (since $\mathcal{C}^K$ is an ∞-category) and then be restricted to the inner face to give a morphism of diagrams $g \circ h \colon \Delta^1 \times K \to \mathcal{C}$.

With these operations we can splice together the morphisms of diagrams obtained in Constructions 3.1.19 and 3.1.20 as in the following diagram:

$$
\begin{tikzcd}
K_n \times K_m \arrow[r] \arrow[d, hook] \arrow[rr, hook, bend left] & K_{nm} \times K_{nm} \arrow[d, hook] & K_{nm} \arrow[d, hook] \\
(\Delta^1)^n \times (\Delta^1)^m \arrow[r, "diag^m \times \prod_{i=1}^m diag^n"] \arrow[d, "f^n \times f^m"'] \arrow[dr, phantom, "u_{n,m} \Nearrow"] & (\Delta^1)^{nm} \times (\Delta^1)^{nm} \arrow[r, "(\coprod)^{nm}"] \arrow[d, "f^{nm} \times f^{nm}"'] \arrow[dr, phantom, "t^{nm} \Nearrow"] & (\Delta^1)^{mn} \arrow[d, "f^{nm}"'] \\
\mathcal{X}^n \times \mathcal{X}^m \arrow[dr, "\times"'] & (\mathcal{X})^{nm} \times (\mathcal{X})^{nm} \arrow[r, "(\times)^{nm}"] \arrow[d, "\times"'] & \mathcal{X}^{nm} \arrow[d, "\times"'] \\
 & \mathcal{X} \times \mathcal{X} \arrow[r, "\times"'] & \mathcal{X}
\end{tikzcd}
$$

Note that the restriction of the morphism $\times \circ f^{nm} \circ (\coprod^{mn}) \circ (diag^m \times \prod_{i=1}^m diag^n)$ along $K_n \times K_m \hookrightarrow (\Delta^1)^n \times (\Delta^1)^m$ factors through K_{nm}.

Altogether we obtain a map of diagrams $\Delta^1 \times K_n \times K_m \to \mathcal{X}$ whose restriction to $\{0\} \times K_n \times K_m$ is $\times \circ (d_n \times d_m)$ and whose restriction to $\{1\} \times K_n \times K_m$ is $K_n \times K_m \to K_{nm} \hookrightarrow (\Delta^1)^{nm} \xrightarrow{f^{nm}} \mathcal{X}^{nm} \xrightarrow{\times} \mathcal{X}$. We finally obtain an induced morphism on colimits $\widetilde{s}_{n,m} \colon \mathbb{A}^n \setminus 0 \times \mathbb{A}^m \setminus 0 \to \mathbb{A}^{nm} \setminus 0$. This is a $\mathbb{G} \times \mathbb{G}$-equivariant morphism, where the action on $\mathbb{A}^{nm} \setminus 0$ is the one obtained from the $\mathbb{G}$-action by precomposing with the multiplication map $\mathbb{G} \times \mathbb{G} \to \mathbb{G}$ (it is the underlying map of an algebra morphism, since $\mathbb{G}$ is commutative). This map will, after quotienting out the $\mathbb{G} \times \mathbb{G}$-actions, yield the Segre embeddings of projective spaces. ◊

3.1.2 Projective spaces

The following definition reflects the description of projective space by homogeneous coordinates:

Definition 3.1.22. *The n-dimensional projective space is defined as the quotient of* $\mathbb{A}^{n+1} \setminus 0$ *by the* $\mathbb{G}$*-action:* $\mathbb{P}^n := (\mathbb{A}^{n+1} \setminus 0)/\mathbb{G}$.

Remark 3.1.23. As in usual geometry we also have a description of $\mathbb{P}^n$ as being patched together by n-dimensional affine planes. For this consider the diagram $q_n\colon K_{n+1} \xrightarrow{d_{n+1}} \mathbb{G}\text{-Mod} \xrightarrow{-/\mathbb{G}} \mathcal{X}$. Using that the functor $(-)/\mathbb{G}$, being a left adjoint, commutes with colimits, we obtain

$$\mathbb{P}^n \simeq (\mathbb{A}^{n+1} \setminus 0)/\mathbb{G} \simeq \operatorname{colim}(d_{n+1})/\mathbb{G} \simeq \operatorname{colim}(((-)/\mathbb{G}) \circ d_{n+1}) \simeq \operatorname{colim} q_n.$$

To match with the usual indexing of homogeneous coordinates we replace the set $\{1, \ldots, n+1\}$ occurring in the definition of K_{n+1} with $\{0, \ldots, n\}$. For $S \subseteq \{0, \ldots, n\}$ we have $d_{n+1}(S) \simeq \mathbb{G}^{|S|}$, hence $q_n(S) \simeq \mathbb{G}^{|S|}/\mathbb{G} \simeq \mathbb{G}^{|S|-1}$ by Prop. 2.5.11. In particular, for the $n + 1$ subsets S with $|S| = 1$ we have terminal objects occurring in the diagram, yielding $n + 1$ points of $\mathbb{P}^n$ (in monoid geometry $\mathbb{P}^n$ does indeed have exactly $n + 1$ points arising from the same combinatorics). Here the point $p_i\colon 1 \simeq q_n(\{i\}) \to \operatorname{colim} q_n \simeq \mathbb{P}^n$ corresponds to the affine plane $\{[a_0 : \ldots : a_n] \mid a_i \neq 0\}$.

In fact, all of the points p_i are equivalent: The indexing simplicial set K_{n+1} of the diagram q_n has the initial object $\{0, \ldots, n\}$ and there the value $q_n(\{0, \ldots, n\}) \simeq \mathbb{G}^n$. Thus in our diagram we have a map $\mathbb{G}^n \to q_n(S)$ for each S, in particular for each copy of the terminal object occurring there. Precomposing this map with the point $1 \to \mathbb{G}^n$ given by the unit map $1 \to \mathbb{G}$ in each factor, gives a map $1 \to 1$ which is equivalent to the identity. Thus all the above points are equivalent to the point $v\colon 1 \to \mathbb{G}^n \simeq q_n(\{0, \ldots, n\}) \to \operatorname{colim}(p_n)$. ◇

Definition 3.1.24. *The maps* $z_n\colon \mathbb{A}^n \setminus 0 \to \mathbb{A}^{n+1} \setminus 0$ *and* $z'_n\colon \mathbb{A}^{n-1} \setminus 0 \to \mathbb{A}^n \setminus 0$ *of Def. 3.1.6 which insert coordinates on the left, resp. on the right, being* $\mathbb{G}$*-equivariant, induce maps betwen projective spaces. We will denote these by* $k_{n-1}\colon \mathbb{P}^{n-1} \to \mathbb{P}^n$*, resp.* $k'_{n-1}\colon \mathbb{P}^{n-1} \to \mathbb{P}^n$.

In the following statements we will abbreviate writing $\mathbb{A}^n \setminus 0$ for $\theta(\mathbb{A}^n \setminus 0)$.

Proposition 3.1.25. *For* $n \geq 1$ *there are pushout diagrams*

$$\begin{tikzcd} \mathbb{A}^n \setminus 0 \arrow[r] \arrow[d] & 1 \arrow[d, "p_n"] \\ \mathbb{P}^{n-1} \arrow[r, "k_{n-1}"] & \mathbb{P}^n \end{tikzcd} \qquad \begin{tikzcd} \mathbb{A}^n \setminus 0 \arrow[r] \arrow[d] & 1 \arrow[d, "p_0"] \\ \mathbb{P}^{n-1} \arrow[r, "k'_{n-1}"] & \mathbb{P}^n \end{tikzcd}$$

Proof. This is obtained from Prop.3.1.7, resp. Remark 3.1.8 using that $(-)/\mathbb{G}$ commutes with colimits and that $(\mathbb{G} \times \mathbb{A}^n \setminus 0)/\mathbb{G} \simeq \mathbb{A}^n \setminus 0$ by Prop. 2.5.11. □

We give an alternative proof of the case $n = 1$ of Prop. 3.1.25:

Proposition 3.1.26. $\mathbb{P}^1 \simeq S^1 \wedge \mathbb{G}$

Proof. We have $\mathbb{P}^1 \simeq \operatorname{colim}(\mathbb{G} \leftarrow \mathbb{G}^2 \to \mathbb{G})/\mathbb{G} \simeq \operatorname{colim}(\mathbb{G}/\mathbb{G} \leftarrow \mathbb{G}^2/\mathbb{G} \to \mathbb{G}/\mathbb{G}) \simeq S^1 \wedge \mathbb{G}^2/\mathbb{G} \simeq S^1 \wedge \mathbb{G}$ using that quotienting out a $\mathbb{G}$-action is a left adjoint, hence commutes with colimits, and Prop. 2.5.11 □

Remark 3.1.27. The map $\mathbb{G} \to 1$ induces a quotient map $\eta\colon \mathbb{A}^2 \setminus 0 \simeq \mathbb{A}^2 \setminus 0 \times_{\mathbb{G}} \mathbb{G} \to \mathbb{A}^2 \setminus 0 \times_{\mathbb{G}} 1 \simeq \mathbb{P}^1$. This map is called the *Hopf map*. We will not make any use of it here, but since it plays an important role in motivic homotopy theory, we wish to remark that it is available in our general setting. ◊

Definition 3.1.28. *The inifinite dimensional projective space is* $\mathbb{P}^\infty := \operatorname{colim}(\mathbb{P}^1 \xrightarrow{k_1} \mathbb{P}^2 \xrightarrow{k_2} \dots)$

Theorem 3.1.29. $\mathbb{P}^\infty \simeq B\mathbb{G}$*,*

Proof.

$$\begin{aligned}
\mathbb{P}^\infty &= \operatorname{colim}(\mathbb{P}^1 \xrightarrow{k_1} \mathbb{P}^2 \xrightarrow{k_2} \dots) \\
&\simeq \operatorname{colim}((\mathbb{A}^2 \setminus 0)/\mathbb{G} \xrightarrow{z_2/\mathbb{G}} (\mathbb{A}^3 \setminus 0)/\mathbb{G} \xrightarrow{z_3/\mathbb{G}} \dots) \\
&\simeq \operatorname{colim}((\mathbb{A}^2 \setminus 0) \xrightarrow{z_2} (\mathbb{A}^3 \setminus 0) \xrightarrow{z_3} \dots)/\mathbb{G} \\
&\simeq \mathbb{A}^\infty \setminus 0/\mathbb{G} \simeq */\mathbb{G} = B\mathbb{G}
\end{aligned}$$

□

Corollary 3.1.30. $\mathbb{P}^\infty$ *is a commutative algebra.*

Proof. For a commutative algebra A, the object BA inherits a commutative algebra structure. □

Proposition 3.1.31. *The commutative algebra structure of Corollary 3.1.30 coincides with the "Segre embedding"* $\mathbb{P}^\infty \times \mathbb{P}^\infty \to \mathbb{P}^\infty$ *obtained by quotienting out the* $(\mathbb{G} \times \mathbb{G})$*-action from the map* $\mathbb{A}^\infty \setminus 0 \times \mathbb{A}^\infty \setminus 0 \to \mathbb{A}^\infty \setminus 0$ *of Construction 3.1.21.*

Proof. Both maps in question arise by quotienting out the $(\mathbb{G} \times \mathbb{G})$-action from maps $\mathbb{A}^\infty \setminus 0 \times \mathbb{A}^\infty \setminus 0 \to \mathbb{A}^\infty \setminus 0$. Since the codomain is terminal, there is just one such map up to equivalence. □

Definition 3.1.32. *The Picard ∞-groupoid of an object $x \in \mathcal{X}$ is defined as* $Pic^{Gpd}(x) := map_{\mathcal{X}}(x, B\mathbb{G})$*. The Picard group of $x \in \mathcal{X}$ is* $Pic x := \pi_0 map_{\mathcal{X}}(x, B\mathbb{G})$*, the Picard* 1*-groupoid is* $\tau_{\leq 1}\pi_0 map_{\mathcal{X}}(x, B\mathbb{G})$*.*

3.1.3 Pointed projective spaces

With the point of Remark 3.1.23 we can regard $\mathbb{P}^n$ as a pointed object, i.e. an object of $\mathcal{X}_*$. Likewise the object $\mathbb{A}^n \setminus 0$ with its point $1 \simeq d_n(\{1, \dots, n\}) \to \operatorname{colim} d_n$. On $\mathcal{X}_*$ we have the monoidal structure given by the smash product and have the formulas from section 2.4 at our disposal.

Lemma 3.1.33. *For every n there is an equivalence* $\mathbb{P}^n/\mathbb{P}^{n-1} \simeq S^1 \wedge \mathbb{A}^n \setminus 0$ *(where the left hand side denotes the cofiber of* $k'_{n-1}\colon \mathbb{P}^{n-1} \to \mathbb{P}^n$*).*

Proof. This can be seen by pasting a further pushout diagram to the pushout diagram from Prop.3.1.25 (the upper square in the following diagram):

$$\begin{tikzcd} \mathbb{A}^n \setminus 0 \arrow[r] \arrow[d] & 1 \arrow[d] \\ \mathbb{P}^{n-1} \arrow[r, "k'_{n-1}"] \arrow[d] & \mathbb{P}^n \arrow[d] \\ 1 \arrow[r] & \mathbb{P}^n/\mathbb{P}^{n-1} \end{tikzcd}$$

□

Lemma 3.1.34. $S^1 \wedge \mathbb{A}^n \setminus 0 \simeq (S^1 \wedge \mathbb{G})^{\wedge n} \simeq (\mathbb{P}^1)^{\wedge n}$

Proof. The left hand side is the cofiber of the map $\mathbb{A}^n \setminus 0 \to 1$. That map is the n-fold Day convolution power of $\mathbb{G} \to 1$ in $\mathcal{X}^{\Delta^1}$. By Cor. 2.4.3 we can equivalently take the cofibers of the morphisms $\mathbb{G} \to 1$ and then their smash product. For the second equivalence use Prop. 3.1.26. □

Since we mainly need the combination of the previous two lemmas we record as a statement of its own:

Corollary 3.1.35. $\mathbb{P}^n/\mathbb{P}^{n-1} \simeq (\mathbb{P}^1)^{\wedge n}$

Notation 3.1.36. For the next result (Prop. 3.1.40) we need a notation for the several canonical embeddings $\mathbb{P}^1 \to \mathbb{P}^n$. Consider the diagram $d_n : K_n \to \mathcal{X}$ defining $\mathbb{P}^n$. For every $i, j \in \{1, \ldots n\}$, $i \neq j$ the restriction of d_n to the subdiagram $\{i\} \leftarrow \{i, j\} \to \{j\}$ is isomorpic to the diagram defining $\mathbb{P}^1$. Passage to colimits yields a map for which we introduce the notation $g_{i,j} : \mathbb{P}^1 \to \mathbb{P}^n$. Thus $g_{0,1} = k_{n-1} \circ \ldots \circ k_1$ and $g_{n-1,n} = k'_{n-1} \circ \ldots \circ k'_1$ in our earlier notation of Def. 3.1.24. Put differently the map $g_{i,j}$ is the one appearing in the pushout diagram

$$\begin{tikzcd} \mathbb{G} \arrow[r] \arrow[d] & 1 \arrow[d] \arrow[ddr, "p_j"] & \\ 1 \arrow[r] \arrow[drr, "p_i"'] & \mathbb{P}^1 \arrow[dr, dotted, "g_{i,j}"] & \\ & & \mathbb{P}^n \end{tikzcd}$$

◊

Remark 3.1.37. In algebraic geometry $g_{i,j}$ is the embedding $\mathbb{P}^1 \to \mathbb{P}^n$ which takes the two homogeneous coordinates of $\mathbb{P}^1$, inserts them into the ith and jth homogeneous coordinates of $\mathbb{P}^n$ and fills up the rest with zeros. One can easily give explicit $\mathbb{A}^1$-homotopies showing that all of these embeddings are homotopic. We see no reason for the embeddings to be equivalent in our general setting. ◊

Lemma 3.1.38. *After composition with the canonical embedding $\mathbb{P}^n \to \mathbb{P}^\infty$ all the embeddings $g_{i,j}$ become equivalent.*

Proof. The several maps that are compared in the statement arise from maps $\mathbb{A}^2 \setminus 0 \to \mathbb{A}^\infty \setminus 0$ by quotienting out the $\mathbb{G}$-action. But since $\mathbb{A}^\infty \setminus 0$ is contractible by Thm. 3.1.16, there is only one such map up to equivalence. □

We will need a description of the maps $g_{i,j}$ in terms of the Day convolution formalism of Section 2.4.

Lemma 3.1.39. *Consider the square*

$$\begin{tikzcd} \mathbb{G} \arrow[r] \arrow[d] & 1 \arrow[d, "p_j"] \\ 1 \arrow[r, "p_i"'] & \mathbb{P}^n \end{tikzcd}$$

as a morphism $(\mathbb{G} \to 1) \to (1 \xrightarrow{p_j} \mathbb{P}^n)$ in the category $\mathcal{X}^{\Delta^1}$ of arrows. Applying the cofiber functor $cof\colon \mathcal{X}^{\Delta^1} \to \mathcal{X}_*$ to this morphism yields the morphism $g_{i,j}\colon \mathbb{P}^1 \to \mathbb{P}^n$.

Proof. Taking cofibers of the vertical morphisms results in the following cubical diagram in which the dotted arrow is the one from the claim.

$$\begin{tikzcd} \mathbb{G} \arrow[rr] \arrow[dr] \arrow[dd] & & 1 \arrow[dr, equal] \arrow[dd, "p_j", near start] & \\ & 1 \arrow[rr, equal] \arrow[dd] & & 1 \arrow[dd, "p_j"] \\ 1 \arrow[rr, "p_i", near start] \arrow[dr] & & \mathbb{P}^n \arrow[dr, equal] & \\ & \mathbb{P}^1 \arrow[rr, dotted] & & \mathbb{P}^n \end{tikzcd}$$

Now one sees that the dotted arrow is the map from the pushout of $(1 \leftarrow \mathbb{G} \to 1)$ to $\mathbb{P}^n$ induced by the square of the claim, which is by definition $g_{i,j}$. □

The following result, in the scheme setting due to Morel, is crucial for the computation of the oriented cohomology theory of projective spaces, see Thm. 3.2.25.

Proposition 3.1.40. *The n-fold diagonal map followed by the quotient map to the smash product factors as follows:*

$$\begin{tikzcd} \mathbb{P}^n \arrow[r, "\bar{\Delta}"] \arrow[d, "cof(k'_{n-1})"'] & (\mathbb{P}^n)^{\wedge n} \\ \mathbb{P}^n/\mathbb{P}^{n-1} \arrow[r, "\simeq"] & (\mathbb{P}^1)^{\wedge n} \arrow[u, "(k_{n-1}\circ\ldots\circ k_1)^{\wedge n}"'] \end{tikzcd}$$

Proof. The map $\bar{\Delta}$ from the claim is the n-fold diagonal map followed by the quotient map giving the smash product: $\mathbb{P}^n \to (\mathbb{P}^n)^n \to (\mathbb{P}^n)^{\wedge n}$. We use the several points $p_i\colon 1 \simeq d_n(\{i\}) \to \mathbb{P}^n$, $i = 0, \ldots, n$ of $\mathbb{P}^n$ discussed in Remark 3.1.23 and consider the composite $h\colon K_n \hookrightarrow (\Delta^1)^n \xrightarrow{p_1 \times \ldots \times p_n} \mathcal{X}^n \xrightarrow{\times} \mathcal{X}$. By Corollary 2.4.3 we have $(\mathbb{P}^n)^{\wedge n} \simeq (\mathbb{P}^n/1)^{\wedge n} \simeq (\mathbb{P}^n)^n/\operatorname{colim}(h)$.

Consider the pullback

$$\begin{tikzcd}
F \arrow[r] \arrow[d] & \operatorname{colim}(h) \arrow[d] \\
\mathbb{P}^n \arrow[r, "diag"] & (\mathbb{P}^n)^n
\end{tikzcd}$$

Note that since $\mathcal{X}$ is cartesian closed, the pullback functor $diag^*\colon \mathcal{X}_{/(\mathbb{P}^n)^n} \to \mathcal{X}_{/\mathbb{P}^n}$ preserves both colimits and limits. Since the forgetful functor $\mathcal{X}_{/x} \to \mathcal{X}$ (given by composing with $!\colon x \to 1$) is a left adjoint, it preserves colimits, i.e. colimits in slice categories are the underlying colimits. Thus $F \simeq \operatorname{colim}(diag^*(h))$.

We are in the situation of special case 1 of Section 2.3: $\mathbb{P}^n$ is a colimit of a diagram $d\colon K_{n+1} \to \mathcal{X}$ and the maps $v(\{i\}) \to \operatorname{colim} d$ are those occurring in the definition of the diagram h. Hence we obtain, as in that section, a map of diagrams $w \to diag^*(h)$, where $w\colon K_n \to \mathcal{X}$ is d restricted to the full simplicial subset of K_{n+1} with vertices $\{S \subseteq \{0, \ldots, n\} \mid S \neq \emptyset, 0 \notin S\}$. This induces a morphism of colimits $\mathbb{P}^{n-1} \simeq \operatorname{colim}(w) \to \operatorname{colim}(diag^*(h)) \simeq F$ over $\mathbb{P}^n$. The map $\mathbb{P}^{n-1} \simeq \operatorname{colim}(w) \to \mathbb{P}^n$ is the map k'_{n-1}, by definition of the latter (Def. 3.1.24).

By passing to the cofibers of the vertical maps in the above diagram we obtain maps $\mathbb{P}^n/\mathbb{P}^{n-1} \to \mathbb{P}^n/F \to (\mathbb{P}^n)^n/\operatorname{colim}(h) \simeq (\mathbb{P}^n)^{\wedge n}$ providing the desired factorization:

$$\begin{tikzcd}
\mathbb{P}^{n-1} \arrow[r] \arrow[dr, "k'_{n-1}"'] & F \arrow[r] \arrow[d] & \operatorname{colim}(h) \arrow[d] \\
 & \mathbb{P}^n \arrow[r, "diag"] \arrow[d, "cof"] \arrow[dl, "cof(k'_{n-1})"'] & (\mathbb{P}^n)^n \arrow[d, "cof"] \\
\mathbb{P}^n/\mathbb{P}^{n-1} \arrow[r] & \mathbb{P}^n/F \arrow[r] & (\mathbb{P}^n)^{\wedge n}
\end{tikzcd}$$

Now using Corollary 3.1.35 we know that Δ factors through a map $(\mathbb{P}^1)^{\wedge n} \to (\mathbb{P}^n)^{\wedge n}$. However, to see that this map is indeed $(k_{n-1} \circ \ldots \circ k_1)^{\wedge n}$, one has go back to proposition 3.1.25 to remember the pushout from which the equivalence $(\mathbb{P}^1)^{\wedge n} \simeq \mathbb{P}^n/\mathbb{P}^{n-1}$ was obtained. Pasting this from the left side to the above diagram we obtain

$$\begin{tikzcd}
\mathbb{A}^n \setminus 0 \arrow[r] \arrow[dr, "!"'] & \mathbb{P}^{n-1} \arrow[r] \arrow[dr, "k'_{n-1}"'] & F \arrow[r] \arrow[d] & \operatorname{colim}(h) \arrow[d] \\
 & 1 \arrow[r, "p_0"] \arrow[dl, "cof(!)"'] & \mathbb{P}^n \arrow[r, "diag"] \arrow[d, "cof"] \arrow[dl, "cof(k'_{n-1})"'] & (\mathbb{P}^n)^n \arrow[d, "cof"] \\
(\mathbb{P}^1)^{\wedge n} \arrow[r, "\simeq"] & \mathbb{P}^n/\mathbb{P}^{n-1} \arrow[r] & \mathbb{P}^n/F \arrow[r] & (\mathbb{P}^n)^{\wedge n}
\end{tikzcd} \qquad (*)$$

Remember that for every $i \in \{1, \ldots, n\}$ we have the morphism $f_i\colon (\mathbb{G} \to 1) \to (1 \xrightarrow{p_i} \mathbb{P}^n)$ in $\mathcal{X}^{\Delta^1}$ from Lemma 3.1.39. We will show that taking the Day convolution product in $\mathcal{X}^{\Delta^1}$ of all these morphisms results in the square

$$\begin{tikzcd}
\mathbb{A}^n \setminus 0 \arrow[r] \arrow[d] & \mathrm{colim}(h) \arrow[d] \\
1 \arrow[r, "p_0"'] & (\mathbb{P}^n)^n
\end{tikzcd}$$

occuring as the upper half of the diagram $*$. For this remember the definition of the Day convolution product: given n arrows one forms their product, resulting in a hypercube diagram $(\Delta^1)^n \to \mathcal{X}$. Then on takes the canonical map from the colimit of the restriction of the hypercube to K_n to the terminal object. This recipe is functorial, thus taking the products of the morphisms f_i results in a map of hypercube diagrams. The source diagram is the diagram d_n defining $\mathbb{A}^n \setminus 0$, the target diagram is h. On the object $\{i\}$ of $(\Delta^1)^n$ the morphism of diagrams is given by

$$\mathbb{G} \simeq 1 \times \ldots \times \mathbb{G} \times \ldots \times 1 \xrightarrow{p_i \times \ldots \times ! \times \ldots \times p_i} \mathbb{P}^n \times \ldots \times 1 \times \ldots \times \mathbb{P}^n$$

where the “different” factor appears in th ith place. On the object $\emptyset$ of $(\Delta^1)^n$ the morphism is given by $1 \xrightarrow{p_0^n} (\mathbb{P}^n)^n$.

Since by Lemma 2.3.8 the target hypercube is a right Kan extension of its lowest two levels, the map of diagrams is completely determined by these maps. Thus it suffices to show that the map of diagrams inducing the morphisms of the top row of the diagram $*$ coincides, on the lowest two levels with these maps.

This is indeed the case: the morphism at the object $\emptyset$ factors through the diagonal, as $1 \xrightarrow{p_0} \mathbb{P}^n \xrightarrow{diag} (\mathbb{P}^n)^n$, hence the morphism at the object $\{i\}$ factors through the pullback along the diagonal as follows

$$\begin{tikzcd}
\mathbb{G} \arrow[r] \arrow[d] & 1 \arrow[r, "p_i \times \ldots \times id \times \ldots \times p_i"] \arrow[d, "p_i"'] & \mathbb{P}^n \times \ldots \times 1 \times \ldots \times \mathbb{P}^n \arrow[d, "id \times \ldots \times p_i \times \ldots \times id"] \\
1 \arrow[r, "p_0"'] & \mathbb{P}^n \arrow[r, "diag"'] & (\mathbb{P}^n)^n
\end{tikzcd}$$

That the right hand square is a pullback square follows from Lemma 2.3.7.

These are exactly the maps which induced the morphisms of the top row of diagram $*$.

To finish the proof, denote by $\boxtimes$ the Day convolution product in $\mathcal{X}^{\Delta^1}$, and by $cof\colon \mathcal{X}^{\Delta^1} \to \mathcal{X}_*$ the cofiber functor. Then we have that the lowest map of the diagram is given by

$cof((\mathbb{A}^n \setminus 0 \to 1) \to (\mathrm{colim}(h))) \simeq cof((\mathbb{G} \to 1)^{\boxtimes n} \to \boxtimes_{i=1}^n (1 \xrightarrow{p_i} \mathbb{P}^n)) \simeq cof(\boxtimes_{i=1}^n ((\mathbb{G} \to 1) \to (1 \xrightarrow{p_i} \mathbb{P}^n))) \simeq \bigwedge_{i=0}^n cof((\mathbb{G} \to 1) \to (1 \xrightarrow{p_i} \mathbb{P}^n))) \simeq \bigwedge_{i=0}^n g_{0,i}$

where the last equivalence comes from Lemma 3.1.39 and the equivalence before that comes from the monoidality of the cofiber functor established in Section 2.4. □

Remark 3.1.41. In Morel’s original proof, reproduced in [NSØ09, Lemma 2.9], the map $\mathbb{P}^{n-1} \to F$ from the above proof is actually an equivalence. There the pullback arises as a Nisnevich (even Zariski) square with $F = \{[a_0 : \ldots : a_n] \mid a_1 \neq 0 \text{ or } \ldots \text{ or } a_n \neq 0\} \simeq \mathbb{A}^1 \times \{[a_1 : \ldots : a_n] \mid a_1 \neq 0 \text{ or } \ldots \text{ or } a_n \neq 0\} \simeq \mathbb{A}^1 \times \mathbb{P}^{n-1} \simeq \mathbb{P}^{n-1}$.

We see no reason for $\mathbb{P}^{n-1}$ to be equivalent to F in our general setting. ◊

Proposition 3.1.42. *The* $(n+1)$*-fold diagonal map* $\Delta\colon \mathbb{P}^n \to (\mathbb{P}^n)^{n+1}$ *followed by the quotient map to the smash product is constant.*

Proof. As in the proof of Prop. 3.1.40 we express the $n+1$-fold smash power as the cofiber of the colimit of a diagram h over $(\mathbb{P}^n)^{n+1}$

$$\begin{tikzcd}
\mathbb{P}^n \arrow[r] \arrow[dr, "\simeq"'] & F \arrow[r] \arrow[d] & \mathrm{colim}(h) \arrow[d] \\
 & \mathbb{P}^n \arrow[r, "diag"] \arrow[dl, "cof"'] & (\mathbb{P}^n)^{n+1} \arrow[d, "cof"] \\
\mathbb{P}^n/\mathbb{P}^n \simeq * \arrow[r] & \mathbb{P}^n/F \arrow[r] & (\mathbb{P}^n)^{\wedge(n+1)}
\end{tikzcd}$$

By special case 2 of of Section 2.3 we have a factorization of the identity map of $\mathbb{P}^n$ through the pullback of $\mathrm{colim}\, h$. Passage to cofibers of the maps from the first to the second row shows the claim. □

3.2 Stabilization and the Snaith spectrum

3.2.1 Stabilization

An object x of a symmetric monoidal ∞-category $\mathcal{D}$ is called invertible, if the endofunctor $x \otimes -\colon \mathcal{D} \to \mathcal{D}$ is an equivalence. As proved by Robalo [Rob15, Prop. 2.9] there is an initial symmetric monoidal accessible functor $\Sigma_x^\infty\colon \mathcal{D} \to \mathrm{Stab}_x(\mathcal{D})$ with respect to the object x having the universal property that any other symmetric monoidal accessible functor $\mathcal{D} \to \mathcal{E}$ sending x to an invertible object factors through Σ_x^∞. The category $\mathrm{Stab}_x(\mathcal{D})$ is called the symmetric monoidal presentable stabilization with respect to x. By construction Σ_x^∞ has a right adjoint Ω_x^∞.

Definition 3.2.1. *We denote by $Sp^{\mathbb{P}^1} := Stab_{\mathbb{P}^1}(\mathcal{X})$ the symmetric monoidal presentable stabilization of $\mathcal{X}_*$, with monoidal structure given by the smash product, with respect to smashing with $\mathbb{P}^1$. For $x, y \in Sp^{\mathbb{P}^1}$ we write $[x, y] := Hom_{\mathrm{Ho}(Sp^{\mathbb{P}^1})}(x, y)$. By construction $Sp^{\mathbb{P}^1}$ is a stable presentably monoidal ∞-category. We will denote its monoidal structure by $\wedge$, and keep the notation $\Sigma_{\mathbb{P}^1}^\infty \dashv \Omega_{\mathbb{P}^1}^\infty$ for the adjunction to $\mathcal{X}_*$.*

In an ∞-category that is tensored over spaces we can make sense of the object S^1, and hence of the stabilization with respect to S^1. This is the usual stabilization of [Lur11], and we assume that the reader is familiar with the usual features of stable ∞-categories, e.g. that pushout diagrams coincide with pullback diagrams, that they are enriched in spectra [GH15, Ex. 7..4.14] and that their homotopy categories are triangulated and (co)fiber sequences get sent to exact triangles. For objects x to y in a stable ∞-category $\mathcal{Y}$ we will use the notation $\underline{\mathrm{map}}_{\mathcal{Y}}(x, y)$ for their mapping spectrum.

From the universal property of the stabilization, more precisely from its characterization as a local object in $CAlg(Cat_\infty)$ with respect to the endomorphism $x \otimes -$ in [Rob15, Prop. 2.9], one obtains that inverting several objects consecutively is the same as inverting their tensor product. Thus $\mathrm{Stab}_{\mathbb{P}^1}(\mathcal{X}) \simeq \mathrm{Stab}_{S^1 \wedge \mathbb{G}}(\mathcal{X}) \simeq \mathrm{Stab}_{\mathbb{G}}(\mathrm{Stab}_{S^1}(\mathcal{X})) \simeq \mathrm{Stab}_{S^1}(\mathrm{Stab}_{\mathbb{G}}(\mathcal{X}))$, in particular $\mathrm{Sp}^{\mathbb{P}^1}$ is stable.

Convention 3.2.2. *By construction $Sp^{\mathbb{P}^1}$ is a module (even an algebra) over the monoidal category $\mathcal{X}_*$. In the following we abbreviate $\Sigma_{\mathbb{P}^1}^\infty(x)$ simply with x when it is clear that we consider x as an object in the $\mathbb{P}^1$-stable category. For an object $E \in Sp^{\mathbb{P}^1}$ the notation $x \wedge E$ is unambiguous, as the module structure is given by mapping x into $Sp^{\mathbb{P}^1}$ and then applying the algebra structure.*

Since we can construct $Sp^{\mathbb{P}^1}$ by stabilizing both S^1 and $\mathbb{G}$, the groups of homotopy classes of morphisms are naturally bigraded. As usual we write $S^{a,b} := (\mathbb{P}^1)^a \wedge \mathbb{G}^b \simeq (S^1)^a \wedge \mathbb{G}^{a+b}$. In general there may be more invertible objects in $Sp^{\mathbb{P}^1}$ than just S^1 and $\mathbb{G}$, for example Brauer-Severi varieties in the case of schemes.

Furthermore, from now on we will denote the terminal object of $\mathcal{X}$ by pt, and use the symbol 1 *for units of ring structures.*

Remark 3.2.3. For a pointed object $(p\colon pt \to x) \in \mathcal{X}_*$ we have two choices for embedding it into $\mathrm{Sp}^{\mathbb{P}^1}$: Either apply $\Sigma_{\mathbb{P}^1}^\infty$ right away, or ignore the base point and adjoin a new one, i.e.

form x_+, then apply $\Sigma^\infty_{\mathbb{P}^1}$. In the pointed category $\mathcal{X}_*$ we have a cofiber sequence $pt_+ \xrightarrow{p_+} x_+ \xrightarrow{x}$. As $\Sigma^\infty_{\mathbb{P}^1}$ is a left adjoint, this induces a cofiber sequence in $\mathrm{Sp}^{\mathbb{P}^1}$, hence an exact triangle in $\mathcal{H}(\mathrm{Sp}^{\mathbb{P}^1})$. Since the cofiber sequence is split by the map $x_+ \xrightarrow{!_+} pt_+$, so is the triangle, which yields a splitting $\Sigma^\infty_{\mathbb{P}^1}(x_+) \simeq \Sigma^\infty_{\mathbb{P}^1}(pt_+) \vee \Sigma^\infty_{\mathbb{P}^1}(x)$ in $\mathcal{H}(\mathrm{Sp}^{\mathbb{P}^1})$, and since coproducts are detected in the homotopy category, also in $\mathrm{Sp}^{\mathbb{P}^1}$. ♢

Remark 3.2.4. There is also the non-monoidal stabilization of an ∞-category with respect to an object. If x satisfies condition that the cyclic permutations on $x \otimes x \otimes x$ are homotopic to the identity, then it coincides with the symmetric monoidal stabilization by [Rob15, Cor. 2.22]. Our $\mathbb{P}^1$ will in general not satisfy this cyclic permutation condition.

Note that it is an open question whether the two kinds of stabilization also coincide when the cyclic permutation condition is not satisfied. ♢

Remark 3.2.5. If one starts out with an additive geometric context, e.g. schemes, semiring schemes, derived schemes or log schemes, and then constructs $\mathcal{X}$ by taking simplicial presheaves and localizing with respect to some Grothendieck topology and contracting $\mathbb{A}^1$, one can repeat the usual proof for the cyclic invariance condition of $\mathbb{P}^1$, where the key facts are that elementary matrices operate trivially on $\mathbb{A}^3 \setminus 0$ and that even permutations can be modelled by products of elementary matrices. ♢

Remark 3.2.6 (*Slice filtration*)**.** For the usual abstract reasons there is a slice filtration: We have the (colimit preserving, hence left adjoint) functors $\Sigma^q_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}$ $(q \in \mathbb{Z})$. The composite of this functor with its right adjoint gives a coreflection functor f_q to the subcategory of q-effective objects. Clearly we have a map of functors $f_{q+1} \to f_q$ and applying the cofiber functor to this yields the qth slice functor.

We will make no use of it in this work, but expect it to be an important tool in the study of particular cases. ♢

Remark 3.2.7. The functors f_q are triangulated, hence they preserve the exact triangle $\mathbb{P}^{n-1}_+ \to \mathbb{P}^n_+ \to (\mathbb{P}^1)^{\wedge n}$ obtained from Cor. 3.1.35. For $k \leq n$ we have $f_k((\mathbb{P}^1)^{\wedge n}) \simeq (\mathbb{P}^1)^{\wedge n}$. Hence we obtain the following diagram with exact rows and columns

$$\begin{tikzcd}
f_{k+1}(\mathbb{P}^{n-1}_+) \arrow[r] \arrow[d] & f_{k+1}(\mathbb{P}^n_+) \arrow[r] \arrow[d] & f_{k+1}((\mathbb{P}^1)^{\wedge n}) \simeq (\mathbb{P}^1)^{\wedge n} \arrow[d] \\
f_k(\mathbb{P}^{n-1}_+) \arrow[r] \arrow[d] & f_k(\mathbb{P}^n_+) \arrow[r] \arrow[d] & f_k((\mathbb{P}^1)^{\wedge n}) \simeq (\mathbb{P}^1)^{\wedge n} \arrow[d] \\
s_k(\mathbb{P}^{n-1}_+) \arrow[r, "\simeq"] & s_k(\mathbb{P}^n_+) \arrow[r] & s_k((\mathbb{P}^1)^{\wedge n}) \simeq 0
\end{tikzcd}$$

If the involved functors preserve filtered colimits, it follows that we have equivalences $s_n(\mathbb{P}^{n-1}_+) \xrightarrow{\simeq} s_n(\mathbb{P}^n_+) \xrightarrow{\simeq} \dots \xrightarrow{\simeq} s_n(\mathbb{P}^\infty_+)$, (the filtered colimit condition being needed for the last equivalence only). Note that in this case in particular the zero slice of the sphere spectrum coincides with that of $\mathbb{P}^\infty_+$. When applicable to schemes, this gives an alternative, arguably easier, proof of [Pel13, Thm 4.2]. ◊

Remark 3.2.8. There is a theory of cellular objects as in [DI05]: For a collection $\mathcal{A}$ of objects of $\mathrm{Sp}^{\mathbb{P}^1}$ the *$\mathcal{A}$-cellular objects* are defined as the smallest stable full sub-∞-category of $\mathrm{Sp}^{\mathbb{P}^1}$ containing $\mathcal{A}$ that is closed under colimits. The *finite $\mathcal{A}$-cellular objects* are defined as the smallest full sub-∞-category of $\mathrm{Sp}^{\mathbb{P}^1}$ containing $\mathcal{A}$ that is closed under *finite* colimits.

The statements of [DI05, Section 7] are purely formal and are valid in our context by choosing a presentation of $\mathrm{Sp}^{\mathbb{P}^1}$ by a stable model category (they are mostly instances or adaptations of the techniques of [EKMM97]). We will be interested in the case $\mathcal{A} = \{S^{p,q} \mid p, q \in \mathbb{Z}\}$ and then simply speak of *cellular objects*, resp. *finite cell complexes*. ◊

Lemma 3.2.9. *The projective spaces $\Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^n_+$ are finite cell complexes.*

Proof. This can be shown inductively using the cofiber sequences $\mathbb{P}^{n-1} \to \mathbb{P}^n \to (\mathbb{P}^1)^{\wedge n}$ of Cor. 3.1.35 and the facts that $\Sigma^\infty_{\mathbb{P}^1}(-)_+$ preserves colimits and that if two objects in a cofiber sequences are finite complexes then so is the third. □

3.2.2 The Snaith spectrum and other stable objects

Definition 3.2.10. *An object x of a monoidal ∞-category is* dualizable *if it is dualizable in the homotopy category, i.e. if there exists a $x^\vee$ and maps $\mathbb{S} \to x \otimes x^\vee$, $x \otimes x^\vee \to \mathbb{S}$ satisfying the triangle identities in the homotopy category.*

Invertible objects are dualizable: The tensor inverse of an invertible object is automatically its dual. Hence S^1, $\mathbb{G}$ and their smash powers are invertible.

Proposition 3.2.11. *$\Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^n$ is dualizable in $Stab_{\mathbb{P}^1}(\mathcal{X})$*

Proof. $\Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^1$ is invertible, hence dualizable. Dualizable objects are closed under completion of exact triangles and we have $\Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^{n-1} \to \Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^n \to (\Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^1)^{\wedge n}$ by Corollary 3.1.35 and the fact that the functor $\Sigma^\infty_{\mathbb{P}^1}\colon \mathcal{X} \to \mathrm{Stab}_{\mathbb{P}^1}(\mathcal{X})$ preserves colimits. □

Remark 3.2.12. We have shown that the projective spaces are finite cell complexes, Lemma 3.2.9, and that they are dualizable, Prop. 3.2.11. There is a third common notion of smallness, namely finite presentability, or compactness. In general these notions are independent. We did not suppose that the terminal object is compact, hence projective spaces will not be compact in general. ◊

Since the object $\mathbb{P}^1$ occurs in the colimit defining $\mathbb{P}^\infty$ we have a natural map $i\colon \mathbb{P}^1 \to \mathbb{P}^\infty$. This map, after applying $\Sigma^\infty_{\mathbb{P}^1}(-)_+$, yields a "Bott multiplication map"

$$\nu_\beta\colon \Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^1 \wedge \Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^\infty_+ \to \Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^1_+ \wedge \Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^\infty_+ \to \Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^\infty_+ \wedge \Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^\infty_+ \to \Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^\infty_+$$

where the first map is on the first smash factor given by the inclusion $\Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^1 \to \Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^1 \vee \Sigma^\infty_{\mathbb{P}^1}pt_+ \simeq \Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^1_+$ (see Rem. 3.2.3), the second map is $i_+ \wedge id$ and the third map is the multiplication given by the commutative algebra structure on $\Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^\infty_+$ (which exists because $\mathbb{P}^\infty$

is a commutative algebra by Cor. 3.1.30 and $\Sigma^\infty_{\mathbb{P}^1}(-)_+$ is monoidal by construction). After taking away one copy of the invertible object $S^{2,1} = \Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^1$ the map ν_β corresponds to a map $\cdot\beta\colon \Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^\infty_+ \to S^{-2,-1} \wedge \Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^\infty_+$.

Definition 3.2.13. *We define* $\mathbb{P}^\infty[\beta^{-1}] := \operatorname{colim}(\Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^\infty_+ \xrightarrow{\cdot\beta} S^{-2,-1} \wedge \Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^\infty_+ \xrightarrow{\cdot\beta} S^{-4,-2} \wedge \Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^\infty_+ \xrightarrow{\cdot\beta} \dots)$. *We call* $\mathbb{P}^\infty[\beta^{-1}]$ *the* Bott inverted infinite projective space *or* Snaith spectrum. *When it is convenient we sometimes abbreviate* $K := \mathbb{P}^\infty[\beta^{-1}]$.

Proposition 3.2.14. *The Snaith spectrum is a commutative algebra. The natural map* $\Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^\infty \to \mathbb{P}^\infty[\beta^{-1}]$ *is an algebra map, initial among algebra maps* $\Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^\infty \to A$ *under which* ν_β *becomes an equivalence.*

Proof. By definition $\mathbb{P}^\infty[\beta^{-1}]$ is the module localization along the map $\cdot\beta$. By [BNT15, Lemma C.2], saying that the cyclic invariance condition is always satisfied in additive ∞-categories, it coincides with the algebra localization. □

Remark 3.2.15. By definition the Snaith spectrum is cellular in the sense of Rem. 3.2.8. ◇

Remark 3.2.16. By construction we have a multiplication map ν_β on $\mathbb{P}^\infty[\beta^{-1}]$ which is an equivalence and fits into a commutative square

$$\begin{tikzcd}
\Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^1 \wedge \Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^\infty_+ \arrow[r, "\nu_\beta"] \arrow[d] & \Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^\infty_+ \arrow[d] \\
\Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^1 \wedge \mathbb{P}^\infty[\beta^{-1}] \arrow[r, "\nu_\beta"] & \mathbb{P}^\infty[\beta^{-1}]
\end{tikzcd}$$

We denote the $\mathbb{P}^1$-desuspension of this map by $\cdot\beta\colon \mathbb{P}^\infty[\beta^{-1}] \to S^{-2,-1} \wedge \mathbb{P}^\infty[\beta^{-1}]$. It can indeed be understood as multiplication with an element β, see Rem. 3.2.22.

We denote a homotopy inverse for the map ν_β by $\cdot\beta^{-1}\colon \mathbb{P}^\infty[\beta^{-1}] \to \Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^1 \wedge \mathbb{P}^\infty[\beta^{-1}]$. ◇

3.2.3 Cohomology theories

Definition 3.2.17. *A* cohomology theory *on* $\mathcal{X}$ *is a representable functor* $Sp^{\mathbb{P}^1} \to$ *Spectra. If the representing object is* E*, then for an object* $x \in \mathcal{X}_*$ *we abbreviate again* $E(x) := \underline{map}_{Stab_{\mathbb{P}^1}(\mathcal{X})}(\Sigma^\infty_{\mathbb{P}^1}(x), E)$ *and call this spectrum the* E-cohomology *of* x.

Since $Sp^{\mathbb{P}^1}$ *is the stabilization with respect to two objects, the homotopy groups of* $E(x)$ *are bigraded. For* $a, b \in \mathbb{Z}$ *we abbreviate* $E^{a,b}(x) := \pi_0\underline{map}_{Stab_{\mathbb{P}^1}(\mathcal{X})}(\Sigma^\infty_{\mathbb{P}^1}(x), (\mathbb{P}^1)^a \wedge \mathbb{G}^b \wedge E) \simeq [x, (\mathbb{P}^1)^a \wedge \mathbb{G}^b \wedge E]$

Remark 3.2.18. Note that our notion of cohomology theory is by definition one that comes with a $\mathbb{P}^1$*-suspension isomorphism*. While this is all that we will consider in this work, in concrete situations it often makes sense to also consider cohomology theories given by representable functors $\mathrm{Stab}_{S^1}(\mathcal{X}) \to$ Spectra and regard it as a special property that the representing spectrum of such a cohomology theory be given by the infinite $\mathbb{P}^1$-loop spectrum of a $\mathbb{P}^1$-spectrum. ◇

Remark 3.2.19. In the case where $\mathcal{X}$ arises as a localization of simplicial presheaves on some category Sm/S of "schemes", one is usually interested in cohomology theories on Sm/S, not on all objects of $\mathcal{X}$. Given a funcor on Sm/S with good properties one would like it to be representable by a spectrum. If one is able to extend it to the full subcategory of compact objects of $\mathrm{Sp}^{\mathbb{P}^1}$, then a sufficient condition for this can sometimes be obtained along the lines of [NS11]; this is for example the case for the site of smooth monoid schemes. ◊

If the representing object E has a monoid structure (i.e. is a "ring spectrum"), we obtain a multiplicative cohomology theory. In this case the evaluation of a multiplicative cohomology theory at an object $x \in \mathcal{X}$, embedded into $\mathcal{X}_*$ as x_+, results in a graded ring:

$$\begin{aligned} E^{a,b}(x) \times E^{c,d}(x) &= [\Sigma^\infty x_+, S^{a,b} \wedge E] \times [\Sigma^\infty x_+, S^{c,d} \wedge E] \\ &\to [\Sigma^\infty x_+ \wedge \Sigma^\infty x_+, S^{a,b} \wedge S^{c,d} \wedge E \wedge E] \\ &\to [\Sigma^\infty x_+, S^{a+c,b+d} \wedge E] \\ &= E^{a+c,b+d}(x) \end{aligned}$$

where the first arrow given by the smash product, and the second by precomposing with the diagonal $\Sigma^\infty \Delta_+ \colon \Sigma^\infty x_+ \to \Sigma^\infty (x \times x)_+ \simeq \Sigma^\infty x_+ \wedge \Sigma^\infty x_+$ and postcomposing with the multiplication map of the ring spectrum and identifying $S^{a,b} \wedge S^{c,d}$ with $S^{a+c,b+d}$. Note that the latter identification involves twist maps and may introduce signs.

Moreover, the cohomology $E^{*,*}(Y)$ of any object $Y \in \mathrm{Sp}^{\mathbb{P}^1}$ is an $E^{*,*}(pt_+)$-module via

$$\begin{aligned} E^{a,b}(pt_+) \times E^{c,d}(x) &= [\Sigma^\infty pt_+, S^{a,b} \wedge E] \times [Y, S^{c,d} \wedge E] \\ &\to [\Sigma^\infty pt_+ \wedge Y, S^{a,b} \wedge S^{c,d} \wedge E \wedge E] \\ &\to [Y, S^{a+c,b+d} \wedge E] \\ &= E^{a+c,b+d}(Y) \end{aligned}$$

where the first arrow is given by smashing both maps in question and the second by composing with the multiplication map and noting that $\Sigma^\infty_{\mathbb{P}^1}(pt_+)$ is the unit for the smash product of $\mathbb{P}^1$-spectra.

Clearly maps $Y_1 \to Y_2$ in $\mathrm{Sp}^{\mathbb{P}^1}$ induce $E^{*,*}(pt_+)$-module homomorphisms $E^{*,*}(Y_2) \to E^{*,*}(Y_1)$ and maps of the form $f_+ \colon x_+ \to y_+$ for $x, y \in \mathcal{X}$ induce $E^{*,*}(pt_+)$-algebra homomorphisms.

As usual we have a reduced and an unreduced version of a cohomology theory, depending on whether we add an extra base point to a pointed object or not.

Using the notion of cellularity of Rem. 3.2.8 we have a *Künneth spectral sequence*, due to Dugger and Isaksen, available as a computational tool:

Theorem 3.2.20 ([DI05, Thm. 8.6]). *Let $A, B \in Sp^{\mathbb{P}^1}$, A a finite cell complex, and $E \in CAlg(Sp^{\mathbb{P}^1})$. Then there exists a strongly convergent tri-graded Künneth spectral sequence of the form*

$$\left[Tor_a^{E^{*,*}(pt_+)}(E^{*,*}(A), E^{*,*}(B))\right] \Rightarrow E^{b-a,c}(A \wedge B)$$

Proof. The original statement of [DI05, Thm. 8.6] is formulated for motivic spectra, but its proof is purely formal and carries over to our setting. □

Remark 3.2.21. As an alternative to the above result, one can establish a bigraded version of the content of [Lur11, Sections 8.2.1, 8.2.2], following the instructions in the proof of [DI05, Prop. 7.7]. On the other hand, the bigrading is actually not that important for what follows, since we mainly treat the cohomology theory represented by the Snaith spectrum, and this has a built in $(2, 1)$-periodicity. $\diamond$

Remark 3.2.22 (Snaith cohomology and $\mathbb{G}$-bundles)**.** We review the construction of the Snaith spectrum in terms of the corresponding cohomology theory. To begin with, we can regard classes of morphisms which factor through $\mathbb{P}^\infty_+$ as $\mathbb{G}$-bundles. Up to a $\mathbb{P}^1$-shift every class in $K^{*,*}(x_+)$ for a *compact* object $x \in \mathcal{X}$ is of this kind, by the definition of $\mathbb{P}^\infty[\beta^{-1}]$ as a colimit of copies of $\mathbb{P}^\infty_+$.

For each $n \in \mathbb{N} \cup \{\infty\}$ there is a class $\mathbf{t}_n \in K^{0,0}(\mathbb{P}^n_+)$ corresponding to the map $\mathbb{P}^n_+ \to \mathbb{P}^\infty \to \mathbb{P}^\infty[\beta^{-1}]$. The first of these maps, the canonical "embedding" $\mathbb{P}^n_+ \to \mathbb{P}^\infty$, is the map that in geometry classifies the tautological bundle $\mathcal{O}(-1)_{\mathbb{P}^n}$, so one can think of $\mathbf{t}_n$ as the class $[\mathcal{O}(-1)_{\mathbb{P}^n}]$of the tautological bundle. Note that $(\mathbf{t}_n)|_{\mathbb{P}^{n-1}} = \mathbf{t}_{n-1}$.

Also for each $n \in \mathbb{N} \cup \{\infty\}$ there is a class in $K^{0,0}(\mathbb{P}^n_+)$ corresponding to the map $\mathbb{P}^n_+ \to pt_+ \to \mathbb{P}^\infty_+ \to \mathbb{P}^\infty[\beta^{-1}]$. The composition of the first two of these maps, the constant map to $\mathbb{P}^\infty$ factoring through the canonical point $p_0 \colon pt \to \mathbb{P}^\infty$, is the map that in geometry classifies the trivial bundle. The point $p_0 \colon pt \to \mathbb{P}^\infty$ is the unit map for the commutative algebra structure on $\mathbb{P}^\infty \simeq B\mathbb{G}$, hence $pt_+ \to \mathbb{P}^\infty_+ \to \mathbb{P}^\infty[\beta^{-1}]$ is the unit for the commutative algebra $\mathbb{P}^\infty[\beta^{-1}]$. Since precomposition of the latter map with $!_+ \colon \mathbb{P}^n_+ \to pt_+$ is a ring homomorphism, it preserves the unit, so the class of the constant map is the ring unit $1 \in K^{0,0}(\mathbb{P}^n)$.

Note that by definition the two classes we just considered coincide for $n = 0$ i.e. for $\mathbb{P}^0_+ = pt_+$.

By Rem. 3.2.3 we have a split exact sequence $K^{0,0}(\mathbb{P}^n) \to K^{0,0}(\mathbb{P}^n_+) \to K^{0,0}(pt_+)$. Since the second map sends $\mathbf{t}_n$ to $\mathbf{t}_n|_{pt_+} = \mathbf{t}_0 = 1$, we have the decomposition $\mathbf{t}_n = (\mathbf{t}_n - 1) + 1$ into a $K^{0,0}(\mathbb{P}^n)$-part and a $K^{0,0}(pt_+)$-part. The multiplication map $\nu_\beta \colon \mathbb{P}^1 \wedge \mathbb{P}^\infty_+[\beta^{-1}] \to \mathbb{P}^\infty_+[\beta^{-1}]$ occurring in the bottom line of the diagram in Rem. 3.2.16 is, by definition of the upper line (see before Def. 3.2.13) and the commutativity of the diagram, given by multiplication with the class $\mathbf{t}_1 - 1 \in K^{0,0}(\mathbb{P}^1)$.

Under the $\mathbb{P}^1$-desuspension isomorphism, the element $\mathbf{t}_1 - 1 \in K^{0,0}(\mathbb{P}^1)$ corresponds to an element $\beta := S^{-2,-1} \wedge [\mathbf{t}_1 - 1] \colon pt_+ \to S^{-2,-1} \wedge \mathbb{P}^\infty[\beta^{-1}] \in K^{-2,-1}(pt_+)$. This is called the *Bott element*. The map $\cdot\beta \colon \mathbb{P}^\infty[\beta^{-1}] \to S^{-2,-1} \wedge \mathbb{P}^\infty[\beta^{-1}]$ from Rem. 3.2.16 induces maps $K^{*,*}(Y) \to K^{*-2,*-1}(Y)$ on cohomology groups of any $Y \in \mathrm{Sp}^{\mathbb{P}^1}$. It is now clear from the definitions that these latter maps are given by multiplication (using the $K^{*,*}(pt_+)$-module structure) with the element $\beta \in K^{-2,-1}(pt_+)$. Likewise we have an inverse element $\beta^{-1} \in K^{2,1}(pt_+)$ multiplication with which gives the map $K^{*,*}(Y) \to K^{*+2,*+1}(Y)$ induced by the map $\cdot\beta^{-1} \colon \mathbb{P}^\infty[\beta^{-1}] \to \Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^1 \wedge \mathbb{P}^\infty[\beta^{-1}]$ of Rem. 3.2.16. $\diamond$

3.2.4 Oriented ring spectra

The definition of "Oriented Cohomology Theory", as in [NSØ09], Def. 2.7, makes sense in our setting. It(s Chern class version) reads:

Definition 3.2.23. *An orientation on a commutative algebra E in $Sp^{\mathbb{P}^1}$ is a class $ch \in E^{2,1}(\mathbb{P}^\infty) = [\Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^\infty, \mathbb{P}^1 \wedge E]$, such that $ch|_{\mathbb{P}^1} = \Sigma_{\mathbb{P}^1}(1)$. Here "restriction" to $\mathbb{P}^1$ means pullback along the canonical map $\mathbb{P}^1 \hookrightarrow \mathbb{P}^\infty$ and $\Sigma_{\mathbb{P}^1}(1)$ means the transport of the unit $1 \in E^{0,0}(pt_+)$ to $E^{2,1}(\mathbb{P}^1)$ via the $\mathbb{P}^1$-suspension isomorphism $[\mathbb{S}, E] \to [\mathbb{P}^1, \mathbb{P}^1 \wedge E]$.*

Proposition 3.2.24. *The Bott inverted infinite projective space $\mathbb{P}^\infty[\beta^{-1}]$ has an orientation given by the canonical map followed by the inverse Bott multiplication map:*

$$ch\colon \Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^\infty \to \mathbb{P}^\infty[\beta^{-1}] \xrightarrow{(-\cdot\beta^{-1})} \mathbb{P}^1 \wedge \mathbb{P}^\infty[\beta^{-1}].$$

Proof. In the following we denote by $1\colon S^0 \to \mathbb{P}^\infty$ the unit map of the algebra $\mathbb{P}^\infty$ and by $i\colon \mathbb{P}^1 \to \mathbb{P}^\infty$ and $j\colon \mathbb{P}^\infty \to \mathbb{P}^\infty[\beta^{-1}]$ denote the canonical maps coming from the colimit definitions of the codomains.

To see that ch does indeed give an orientation, we need to identify $ch|_{\mathbb{P}^1} = ch \circ i$ with $id_{\mathbb{P}^1} \wedge (j \circ 1)$. For this consider the following diagram which can be easily seen to be commutative in the homotopy category:

$$\begin{tikzcd}
\mathbb{P}^1 \arrow[r, "\simeq"] \arrow[ddrr, bend right, "i"'] & \mathbb{P}^1 \wedge S^0 \arrow[r, "\mathbb{P}^1\wedge 1"] \arrow[d, "i\wedge S^0"'] & \mathbb{P}^1 \wedge \mathbb{P}^\infty \arrow[r, "\mathbb{P}^1\wedge j"] \arrow[d, "i\wedge\mathbb{P}^\infty"'] & \mathbb{P}^1 \wedge \mathbb{P}^\infty[\beta^{-1}] \arrow[d, "(j\circ i)\wedge\mathbb{P}^\infty[\beta^{-1}]"'] \arrow[dd, bend left, "\cdot\beta"] \arrow[ddd, bend left=60, "id"] \\
& \mathbb{P}^\infty \wedge S^0 \arrow[r, "\mathbb{P}^\infty\wedge 1"] \arrow[dr, "\simeq"'] & \mathbb{P}^\infty \wedge \mathbb{P}^\infty \arrow[r, "j\wedge j"] \arrow[d, "\mu"'] & \mathbb{P}^\infty[\beta^{-1}] \wedge \mathbb{P}^\infty[\beta^{-1}] \arrow[d, "\mu"'] \\
& & \mathbb{P}^\infty \arrow[r, "j"] \arrow[dr, "ch"'] & \mathbb{P}^\infty[\beta^{-1}] \arrow[d, "\cdot\beta^{-1}"] \\
& & & \mathbb{P}^1 \wedge \mathbb{P}^\infty[\beta^{-1}]
\end{tikzcd}$$

The claim follows by comparing the two outer paths from the top left to the bottom right. □

Theorem 3.2.25. *Let E be an oriented ring spectrum with Chern class $ch \in E^{2,1}(\mathbb{P}^\infty)$. Let $c_n \in E^{2,1}(\mathbb{P}^n) = [\Sigma^\infty_{\mathbb{P}^1}(\mathbb{P}^n), \Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^1 \wedge E]$ be the class given by $\mathbb{P}^n \to \mathbb{P}^\infty \xrightarrow{ch} \Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^1 \wedge E$. Then there is an isomorphism of $E^{*,*}(pt)$-algebras $E^{*,*}(\mathbb{P}^n) \cong E^{*,*}(pt)[c_n]/((c_n)^{n+1})$*

Proof. The proof of [NSØ09, Thm. 2.10] can be transferred to our setting. It proceeds by induction on n. For $n = 0$ the statement is trivial. Suppose that we know the statement for $\mathbb{P}^{n-1}$. The cofiber sequence $\mathbb{P}^{n-1} \to \mathbb{P}^n \to (\mathbb{P}^1)^{\wedge n}$ obtained by Corollary 3.1.35 gives the long exact sequence in the lower row of the following diagram of $E^{*,*}(pt_+)$-module maps:

$$\begin{tikzcd}
\dots \arrow[r] & E^{*-2n,*-n}(pt_+) \arrow[r] \arrow[d] & \bigoplus_{i=0}^{n} E^{*-2i,*-i}(pt_+) \arrow[r] \arrow[d] & \bigoplus_{i=0}^{n-1} E^{*-2i,*-i}(pt_+) \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & E^{*,*}((\mathbb{P}^1)^{\wedge n}) \arrow[r] & E^{*,*}(\mathbb{P}^n) \arrow[r] & E^{*,*}(\mathbb{P}^{n-1}) \arrow[r] & \dots
\end{tikzcd}$$

The right hand vertical map is given by $(a_0, \dots, a_{n-1}) \mapsto a_0 + a_1 c_{n-1} + \dots + a_{n-1}(c_{n-1})^{n-1}$, and is an isomorphism by induction hypothesis. The middle vertical map is given by $(a_0, \dots, a_n) \mapsto$

$a_0 + a_1 c_n + \ldots + a_{n-1}(c_n)^{n-1} + a_n(c_n)^n$. The left hand vertical map is the n-fold $\mathbb{P}^1$-suspension isomorphism. The upper row is given by the inclusion of the nth summand and the projection to the first n summands, making it exact.

Commutativity of the right hand square: By definition of the classes c_i the class $c_n : \mathbb{P}^n \to \mathbb{P}^\infty \to E$ gets mapped to $c_{n-1} : \mathbb{P}^{n-1} \to \mathbb{P}^n \to \mathbb{P}^\infty \to E$. By hypothesis $E^{*,*}(\mathbb{P}^{n-1}) \cong E^{*,*}(pt_+)[c_{n-1}]/((c_{n-1})^n)$, and the map $E^{*,*}(\mathbb{P}^n) \to E^{*,*}(\mathbb{P}^{n-1})$ is an $E^{*,*}(pt_+)$-algebra map, so $(c_n)^k$ gets mapped to $(c_{n-1})^k$ and in particular $(c_n)^n$ gets mapped to $(c_{n-1})^n = 0$.

Commutativity of the left hand square: The upper map followed by the right hand map sends $a \in E^{*-2n,*-n}(pt_+)$ to $a(c_n)^n$. Since all maps are $E^{*,*}(pt_+)$-module maps, it is enough to consider the case $a = 1$ and confirm that going first down, then right does also send 1 to $(c_n)^n$.

The map down is the n-fold $\mathbb{P}^1$-suspension isomorphism. Applying $\mathbb{P}^1$-suspension once to the unit $1 : \mathbb{S} \to E$ yields $c_1 = ch|_{\mathbb{P}^1} : \mathbb{P}^1 \to \mathbb{P}^1 \wedge E$ by the definition of orientation. Before applying $\mathbb{P}^1$-suspension n times, note that the unit $1 : \mathbb{S} \to E$ is equivalent to

$$\mathbb{S} \simeq \mathbb{S}^{\wedge n} \xrightarrow{1^{\wedge n}} E^{\wedge n} \xrightarrow{\mu_n} E.$$

Thus the n-fold suspension of the unit is

$$(\mathbb{P}^1)^{\wedge n} \xrightarrow{(c_1)^{\wedge n}} (\mathbb{P}^1)^{\wedge n} \wedge E^{\wedge n} \xrightarrow{id \wedge \mu_n} (\mathbb{P}^1)^{\wedge n} \wedge E.$$

which by definition of the classes c_n can be factorized as

$$(\mathbb{P}^1)^{\wedge n} \xrightarrow{(g_{0,1})^{\wedge n}} (\mathbb{P}^n)^{\wedge n} \xrightarrow{(c_n)^{\wedge n}} (\mathbb{P}^1)^{\wedge n} \wedge E^{\wedge n} \xrightarrow{\mu_n} (\mathbb{P}^1)^{\wedge n} \wedge E.$$

By definition of the multiplication in the cohomology ring, $(c_n)^n$ is the class of the composition of the upper horizontal maps in the following diagram

$$\begin{tikzcd}
\mathbb{P}^n \arrow[r, "\Delta"] \arrow[d, "cof"'] & (\mathbb{P}^n)^{\wedge n} \arrow[r, "(c_n)^{\wedge n}"] & (\mathbb{P}^1 \wedge E)^{\wedge n} \simeq (\mathbb{P}^1)^{\wedge n} \wedge E^{\wedge n} \arrow[r, "id \wedge \mu_n"] & (\mathbb{P}^1)^{\wedge n} \wedge E \\
(\mathbb{P}^1)^{\wedge n} \arrow[ur, "\bigwedge_{i=0}^n g_{0,i}"'] & & &
\end{tikzcd}$$

The factorization indicated by the triangle is the one given by Prop. 3.1.40. Since the maps c_n factor through the embedding to $\mathbb{P}^\infty$, by Lemma 3.1.38 all the different $g_{0,i}$ become equivalent after composing with c_n, hence the diagonal map can be replaced with $g_{0,1}$.

This shows that the n-fold $\mathbb{P}^1$-suspension of the unit, after precomposition with the cofiber map (which is the lower right horizontal map in the first diagram), becomes $(c_n)^n$.

To the left and to the right the diagram continues with horizontal maps given by the respective long exact sequences vertical maps given by and shifted versions of the left and right hand vertical ismorphisms. By the 5-Lemma the middle map is an isomorphism of $E^{*,*}(pt_+)$-modules. Hence we know that it is a ring with underlying set $\{a_0 + a_1 c_n + \ldots + a_{n-1}c_n^{n-1} + a_n c_n^n \mid a_i \in E^{*,*}(pt_+)\}$. By Prop. 3.1.42 it satisfies $c_n^{n+1} = 0$, therefore it is isomorphic, as a ring, to $E^{*,*}(pt_+)[c_n]/((c_n)^{n+1})$. □

The pattern of the proof of Thm. 3.2.25 is classical in topology. It has been applied to motivic spectra in [NSØ09] and [GS09]. Some input along the lines of Prop. 3.1.40 is needed in all these proofs but gets increasingly subtle passing from topology to motivic homotopy theory to our general setting.

Proposition 3.2.26. *Let E be an oriented spectrum. For any $X \in Sp^{\mathbb{P}^1}$ we have an isomorphism of $E^{*,*}(pt_+)$-modules $E^{*,*}(X \wedge \mathbb{P}^n_+) \cong E^{*,*}(X)[c_n]/((c_n)^{n+1})$.*

Proof. By Lemma 3.2.9 $\Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^n_+$ is a finite cell complex. Therefore we have the Künneth spectral sequence of Thm. 3.2.20. But the higher Tor-terms vanish since by Thm. 3.2.25 $E^{*,*}(\mathbb{P}^n_+)$ is a free $E^{*,*}$-module (this is an instance of [DI05, Rem. 8.7]). □

Theorem 3.2.27. *Let E be an oriented spectrum. There is an isomorphism of graded rings $E^{*,*}(\mathbb{P}^\infty_+) \cong E^{*,*}(pt_+)[[c]]$.*

Proof. From the colimit expression $\mathbb{P}^\infty \simeq \mathrm{colim}(\mathbb{P}^1 \to \mathbb{P}^2 \to \mathbb{P}^3 \to \ldots)$ we have the Milnor exact sequence

$$0 \to \lim^1_{n\in\mathbb{N}} E^{*+1,*}(\mathbb{P}^n) \to E^{*,*}(\mathbb{P}^\infty_+) \to \lim_{n\in\mathbb{N}} E^{*,*}(\mathbb{P}^n) \to 0.$$

Since the transition maps in the limit diagram are surjective, the $\lim^1$-term vanishes, hence $E^{*,*}(\mathbb{P}^\infty_+) \cong \lim_{n\in\mathbb{N}} E^{*,*}(pt_+)[c]/(c^n) \cong E^{*,*}(pt_+)[[c]]$. □

Proposition 3.2.28. *For an oriented spectrum E we have $E^{*,*}((\mathbb{P}^\infty \times \mathbb{P}^\infty)_+)) \cong E^{*,*}(pt_+)[[x,y]]$ where x, resp. y are the pullbacks along the first, resp. second, projection of the orientation classes of $\mathbb{P}^\infty$.*

Proof.
$$\begin{aligned}
E^{*,*}(pt_+)[[x,y]] &\cong E^{*,*}(pt_+)[[x]] \otimes_{E^{*,*}(pt_+)} E^{*,*}(pt_+)[[y]] \\
&\cong (\lim_{n\in\mathbb{N}} E^{*,*}(pt_+)[x]/(x^n)) \otimes_{E^{*,*}(pt_+)} E^{*,*}(pt_+)[[y]] \\
&\cong (\lim_{n\in\mathbb{N}} E^{*,*}(pt_+)[x]/(x^n)) \otimes_{E^{*,*}(pt_+)} E^{*,*}(\mathbb{P}^\infty_+) \\
&\cong \lim_{n\in\mathbb{N}} (E^{*,*}(\mathbb{P}^\infty_+)[x]/(x^n)) \\
&\cong \lim_{n\in\mathbb{N}} E^{*,*}(\mathbb{P}^n_+ \wedge \mathbb{P}^\infty_+) && \text{(Prop. 3.2.26)} \\
&\cong E^{*,*}(\mathrm{colim}_{n\in\mathbb{N}}(\mathbb{P}^n_+) \wedge \mathbb{P}^\infty_+) && \text{($\lim^1$-term vanishes)} \\
&\cong E^{*,*}(\mathrm{colim}_{n\in\mathbb{N}}(\mathbb{P}^n_+ \wedge \mathbb{P}^\infty_+)) && \text{($\wedge$ preserves colimits)} \\
&\cong E^{*,*}((\mathbb{P}^\infty \times \mathbb{P}^\infty)_+)
\end{aligned}$$
□

Prop. 3.2.28 implies that we get a formal group law on $E^{*,*}((\mathbb{P}^\infty \times \mathbb{P}^\infty)_+)$ for any oriented cohomology theory.

For what follows, note the expression of the classes c_n through virtual line bundles: with the notation $\mathbf{t}_n \colon \mathbb{P}^n \to \mathbb{P}^\infty$ for the canonical map of Rem. 3.2.22 we have $c_n = \beta^{-1} \cdot (\mathbf{t}_n - 1)$. Note that this makes sense for $n = \infty$.

Proposition 3.2.29. *The formal group law of the Snaith spectrum is given by*

$$f(x,y) = x + y + \beta xy \in K^{*,*}(pt_+)[[x,y]] \cong K^{*,*}((\mathbb{P}^\infty \times \mathbb{P}^\infty)_+)$$

Proof. Denoting the multiplication map by $\mu\colon \mathbb{P}^\infty \times \mathbb{P}^\infty \to \mathbb{P}^\infty$, we need to calculate the pullback of the orientation class x along μ. For this remember the definition of x as $(\mathbf{t}_\infty - 1)\cdot\beta^{-1}$ and the splitting $\mathbf{t}_\infty = (\mathbf{t}_\infty - 1) + 1$ into the restrictions to $\mathbb{P}^\infty$, resp pt_+ (see Rem. 3.2.22). By the construction of $\mathbb{P}^\infty[\beta^{-1}]$ the class $\mathbf{t}_\infty$ is the class of a ring map, therefore the right rectangle in the following diagram commutes:

$$
\begin{tikzcd}
\mathbb{P}^\infty \times \mathbb{P}^\infty \arrow[r] \arrow[d, "\mu"'] & \mathbb{P}^\infty_+ \wedge \mathbb{P}^\infty_+ \arrow[r, "\mathbf{t}_\infty \wedge \mathbf{t}_\infty"] \arrow[d, "\mu_+"'] & \mathbb{P}^\infty[\beta^{-1}] \wedge \mathbb{P}^\infty[\beta^{-1}] \arrow[d, "\cdot"] \\
\mathbb{P}^\infty \arrow[r, hook] \arrow[rr, bend right, "\mathbf{t}_\infty - 1"'] \arrow[rrd, bend right=40, "x"'] & \mathbb{P}^\infty_+ \arrow[r, "\mathbf{t}_\infty"] & \mathbb{P}^\infty[\beta^{-1}] \arrow[d, "\cdot\beta^{-1}"] \\
 & & S^{2,1} \wedge \mathbb{P}^\infty[\beta^{-1}]
\end{tikzcd}
$$

We first consider the pullback along the right hand square. This gives us

$$(\mu_+)^*(\mathbf{t}_\infty) = [\mathbf{t}_\infty \circ m_+] = [\mu \circ (\mathbf{t}_\infty \wedge \mathbf{t}_\infty)] = \mathrm{pr}_1^*(\mathbf{t}_\infty) \cdot \mathrm{pr}_2^*(\mathbf{t}_\infty).$$

Next we see what happens to the reduced class under pullback:

$$\mu^*(\mathbf{t}_\infty - 1) = \mu_+^*(\mathbf{t}_\infty) - 1 = \mathrm{pr}_1^*(\mathbf{t}_\infty) \cdot \mathrm{pr}_2^*(\mathbf{t}_\infty) - 1 = \mathrm{pr}_1^*(\mathbf{t}_\infty - 1) \cdot \mathrm{pr}_2^*(\mathbf{t}_\infty - 1) + \mathrm{pr}_1^*(\mathbf{t}_\infty - 1) + \mathrm{pr}_2^*(\mathbf{t}_\infty - 1)$$

Here the first equality is given by switching from the path along the middle curved arrow and μ to the path along $\mathbf{t}_\infty$, μ_+ and the inclusion $\mathbb{P}^\infty \times \mathbb{P}^\infty \to (\mathbb{P}^\infty \times \mathbb{P}^\infty)_+ \simeq \mathbb{P}^\infty_+ \wedge \mathbb{P}^\infty_+$ — pullback along the latter has the effect of subtracting 1 from a cohomology class.
For the second equality we switched further to the path along the multiplication map, the map $\mathbf{t}_\infty \wedge \mathbf{t}_\infty$ and the inclusion. For the third equality we used that pr_1^* and pr_2^* are ring maps.

Finally, multiplying both sides of the equation with β^{-1}, using that all pullback maps are $K^{*,*}(pt_+)$-module maps, and using $x = (\mathbf{t}_\infty - 1) \cdot \beta^{-1}$, we obtain $\mu^*(x) = \mu^*((\mathbf{t}_\infty - 1) \cdot \beta^{-1}) = \mathrm{pr}_1^*(\mathbf{t}_\infty - 1)\beta^{-1} \cdot \mathrm{pr}_2^*(\mathbf{t}_\infty - 1)\beta^{-1}\beta + \mathrm{pr}_1^*(\mathbf{t}_\infty - 1)\beta^{-1} + \mathrm{pr}_2^*(\mathbf{t}_\infty - 1)\beta^{-1} = \beta xy + x + y$ as claimed. □

3.3 Cohomology operations

3.3.1 Adams operations

In usual ring theory it is a fact that monoid rings are Ψ-rings: If M is a commutative monoid and R a commutative ring, then the free R-module on M has a ring structure and power operations $\Psi^k\colon R[M] \to R[M]$, $\Sigma_{i=0}^n a_i m_i \mapsto \Sigma_{i=0}^n a_i (m_i)^k$, which are ring homomorphisms. These obviously satisfy $\Psi^k \circ \Psi^l = \Psi^{kl}$ and hence commute. Following the usual intuition which compares the formation of a suspension spectrum to the formation of a free module, we can establish the same in our setting.

Proposition 3.3.1. *$\Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^\infty_+$ is a $(Comm \otimes \mathbf{\Psi})$-algebra (in the terminology of the end of Remark 2.6.4.*

Proof. The $(\mathrm{Comm} \otimes \mathbf{\Psi})$-algebra structure on $\mathbb{P}^\infty$ that we established in Remark 2.6.4 is preserved under the monoidal functor $\Sigma^\infty_x(-)_+$. □

Corollary 3.3.2. *We have a sequence of commuting algebra endomorphisms ψ^k $(k \in \mathbb{N})$ on $\Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^\infty$, satisfying $\psi^k \circ \psi^l \simeq \psi^{kl}$.*

It is natural to ask whether the operations ψ^k are passed on to $\mathbb{P}^\infty[\beta^{-1}]$. This is in general not the case. For example the Adams operations on complex K-theory are unstable operations, and the operation ψ^k only becomes stable after inverting k. The same is true in our setting. For this remember that our stable ∞-category is tensored over topological spectra, hence multiplication with endomorphisms of the sphere spectrum makes sense, and in particular we can invert integers by tensoring with the suitably localized sphere. We will construct an operation Ψ^k on the k-localization of $\mathbb{P}^\infty[\beta^{-1}]$ which is a continuation of ψ^k. For this we make a sequence of observations. For ease of notation we abbreviate $x := \mathbb{P}^1$ and $M_+ := \Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^\infty_+$.

1. The following diagram commutes in the homotopy category (where μ_n denotes the product of n factors coming from the monoid structure of M, and $\mu := \mu_2$):

$$\begin{tikzcd}
x_+ \wedge x_+ \wedge M_+ \arrow[r, "id\wedge\beta_+\wedge id"] \arrow[dr, "\beta_+\wedge\beta_+\wedge id"'] & x_+ \wedge M_+ \wedge M_+ \arrow[r, "id\wedge\mu_+"] \arrow[d, "\beta_+\wedge id\wedge id"] & x_+ \wedge M_+ \arrow[r, "\beta_+\wedge id"] \arrow[d, "\beta_+\wedge id"] & \wedge M_+ \wedge M_+ \arrow[d, "\mu_+"] \\
 & M_+ \wedge M_+ \wedge M_+ \arrow[r, "id\wedge\mu_+"'] \arrow[rr, bend right, "(\mu_3)_+"'] & M_+ \wedge M_+ \arrow[r, "\mu_+"'] & M_+
\end{tikzcd}$$

Indeed, this involves just the functoriality of the monoidal structure $\wedge$ and the compatibility of the several multiplication maps on M. Similar diagrams can be drawn for a larger number of $\mathbb{P}^1$-factors. This tells us that, unsurprisingly, carrying out a number of Bott multiplications step by step or all at once amounts to the same. It means that the lowest area in the next diagram commutes.

2. The following diagram commutes in the homotopy category:

$$\begin{tikzcd}
x \wedge M_+ \arrow[r] \arrow[rrr, bend left, "\cdot\beta"] & x_+ \wedge M_+ \arrow[r, "i\wedge id"] \arrow[d, "\Delta_+\wedge\Delta_+"] \arrow[dd, bend right=60, "\Delta_+\wedge\Psi^k_+"'] & M_+ \wedge M_+ \arrow[r, "\mu"] \arrow[d, "\Delta_+\wedge\Delta_+"] & M_+ \arrow[d, "\Delta_+"] \arrow[dd, bend left=60, "\Psi^k_+"] \\
& x_+^k \wedge M_+^k \arrow[r, "i^{\times k}\wedge id^{\times k}"] \arrow[d, "id\wedge\mu_+"] & M_+^k \wedge M_+^k \arrow[d, "\mu_+"] & M_+^k \arrow[d, "\mu_+"] \\
& x_+^k \wedge M_+ \arrow[rr, bend right, "(\cdot\beta)\circ\ldots\circ(\cdot\beta)\circ(\cdot\beta)\quad (k\text{ times})"'] & M_+ \wedge M_+ \arrow[r, "\mu_+"] & M_+
\end{tikzcd}$$

This says, again in accordance with intuition, that first applying Bott multiplication, then Ψ^k is the same as first applying Ψ^k, then k times Bott multiplication (up to homotopy). This in turn implies the commutativity of the next diagram.

3. Tensoring the previous diagram with x^{-i}, ($i \in \mathbb{N}$) we obtain the ingredients of the following diagram:

$$\begin{tikzcd}
M_+ \arrow[r, "\simeq"'] \arrow[rrrr, bend left, "\cdot\beta"] \arrow[d, "\psi^k"'] & x^{-1} \wedge x \wedge M_+ \arrow[r] \arrow[d, "x^{-1}\wedge x\wedge\psi^k"'] & x^{-1} \wedge x_+ \wedge M_+ \arrow[r] \arrow[d, "x^{-1}\wedge\Delta_+\wedge\psi^k"'] & x^{-1} \wedge M_+ \wedge M_+ \arrow[r, "x^{-1}\wedge\mu_+"'] & x^{-1} \wedge M_+ \arrow[r] \arrow[d, "x^{-1}\wedge\psi^k"] & \ldots \\
M_+ \arrow[r, "\simeq"] & x^{-1} \wedge x \wedge M_+ \arrow[r] & x^{-1} \wedge x_+^k \wedge M_+ \arrow[rr, "id\wedge((\cdot\beta)\circ\ldots\circ(\cdot\beta))"'] & & x^{-1} \wedge M_+ \arrow[r] & \ldots
\end{tikzcd}$$

To the right the diagram continues with copies of the same and an increasing amount of x^{-1}-factors. Again this diagram commutes in the homotopy category. A map between two *linear* diagrams in the homotopy category, as we have it here, can always be lifted to a genuine map of diagrams in the ∞-category.

Clearly the colimit of the upper linear diagram is $M[\beta^{-1}]$. Up to now everything worked for an arbitrary monoid with a pointed map $\beta\colon x \to M$. It is in the calculation of the colimit of the lower diagram where we will now use the fact that our x is $\mathbb{P}^1$, more precisely in the next Lemma.

Lemma 3.3.3. *The composition* $x \hookrightarrow x_+ \xrightarrow{\Delta_+} (x^k)_+$ *is equivalent to the k-fold diagonal in spectra* $x \to \bigvee_{j=1}^k x$.

Proof. From the distributivity of $\wedge$ over $\vee$ we obtain

$$(x^k)_+ \simeq (x_+)^{\wedge k} \simeq (x \vee pt_+)^{\wedge k} \simeq (\bigvee_{i=0}^{k} \bigvee_{j=1}^{\binom{k}{i}} x^{\wedge i}).$$

We precompose this with the map $x \hookrightarrow x_+ \xrightarrow{\Delta_+} (x^k)_+$ and determine what the map to each summand is. By Prop. 3.1.42 the diagonal maps from $\mathbb{P}^1$ to $(\mathbb{P}^1)^{\wedge i}$ are zero for $i \geq 2$. The map to the zeroth summand $x^{\wedge 0} \simeq pt_+$ is zero as well because x maps as the complement of pt_+ into x_+. Finally the maps to the k factors of the form $x^{\wedge 1} \simeq x$ are equivalent to the identity, because we have splits $x \to x_+ \to (x^k)_+ \to x_+ \to x$ where the first map is the inclusion $x \to x \vee pt_+$, the second map is the diagonal in $\mathcal{X}$, the third map is a projection in $\mathcal{X}$ and fourth map is the projecting away the basepoint.

Altogether this shows that the map in question in the k-fold diagonal *for the product in spectra* (not in $\mathcal{X}$) $\square$

Lemma 3.3.4. *The colimit of the lower line in the above diagram is the k-local Snaith spectrum* $\mathbb{P}^\infty[\beta^{-1}]_{(k)}$.

Proof. The claim follows from the fact that the composite map $M_+ \to x^{-1} \wedge M_+$ in the lower line is multiplication with k followed by Bott multiplication.

To see this, note that by Lemma 3.3.3 the composition of the first two maps of the lower line is equivalent to

$$M_+ \simeq x^{-1} \wedge x \wedge M_+ \to x^{-1} \wedge (\bigvee_{j=1}^{k} x) \wedge M_+ \simeq \bigvee_{j=1}^{k} x^{-1} \wedge x \wedge M_+ \simeq \bigvee_{j=1}^{k} M_+$$

i.e. to the k-fold diagonal of M_+ in spectra.

On the other hand, the inclusion of a summand of $\bigvee_{j=1}^{k} M_+$ into $x^{-1} \wedge x_+^k \wedge M_+$ can be split as

$$M_+ \simeq x^{-1} \wedge x \wedge M_+ \hookrightarrow x^{-1} \wedge x_+ \wedge M_+ \hookrightarrow x^{-1} \wedge ((x_+ \wedge M_+) \vee (x^{k-1} \wedge M_+))$$

Now it suffices to notice that the k-fold Bott multiplication map factors as

$$\begin{aligned} x^{-1} \wedge (x^k)_+ \wedge M_+ \quad & \simeq \quad x^{-1} \wedge x_+ \wedge (x^{k-1})_+ \wedge M_+ \\ & \simeq \quad x^{-1} \wedge x_+ \wedge (pt_+ \vee x^{k-1}) \wedge M_+ \\ & \simeq \quad x^{-1} \wedge ((x_+ \wedge M_+) \vee (x^{k-1} \wedge M_+)) \\ & \xrightarrow{x^{-1} \wedge ((\cdot\beta) \vee id)} \quad x^{-1} \wedge (M_+ \vee (x^{k-1} \wedge M_+)) \\ & \simeq \quad x^{-1} \wedge (x^{k-1})_+ \wedge M_+ \\ & \xrightarrow{x^{-1} \wedge ((\cdot\beta)^{\circ(k-1)}} \quad x^{-1} \wedge M_+ \end{aligned}$$

Thus the k-fold Bott multiplication can be split up as taking place on the first summand once and on the second summand the other times. Since the inclusion only meets the first summand, it is connected to one Bott multiplication. Thus the third map in the lower line, restricted to each summand of $\bigvee_{j=1}^{k} M_+$ is 1-fold multiplication by the Bott element.

Therefore the composition of the lower three lines is: k-fold diagonal, followed by k-fold codiagonal, followed by 1-fold Bott multiplication, i.e. multiplication by k followed by Bott multiplication. $\square$

Corollary 3.3.5. *There exist maps of spectra* $\Psi^k : \mathbb{P}^\infty[\beta^{-1}] \to \mathbb{P}^\infty[\beta^{-1}]_{(k)}$ *and* $\Psi^k : \mathbb{P}^\infty[\beta^{-1}]_{(k)} \to \mathbb{P}^\infty[\beta^{-1}]_{(k)}$ *fitting into a commutative diagram*

$$\begin{tikzcd} \Sigma^\infty_{\mathbb{P}^1} \mathbb{P}^\infty_+ \arrow[r] \arrow[d, "\Psi^k"'] & \mathbb{P}^\infty[\beta^{-1}] \arrow[r] \arrow[d, "\Psi^k"] & \mathbb{P}^\infty[\beta^{-1}]_{(k)} \arrow[d, "\Psi^k"] \\ \Sigma^\infty_{\mathbb{P}^1} \mathbb{P}^\infty_+ \arrow[r] & \mathbb{P}^\infty[\beta^{-1}]_{(k)} \arrow[r, equal] & \mathbb{P}^\infty[\beta^{-1}]_{(k)} \end{tikzcd}$$

where the horizontal maps are the canonical maps stemming from the colimit descriptions of their targets.

Proof. The morphism of diagrams constructed in point **3.** induces the left hand square. The upper colimit is $\mathbb{P}^\infty[\beta^{-1}]$ by definition, the lower colimit has been identified in Lemma 3.3.4. By the universal property of k-localization the middle vertical map factors through the canonical map $\mathbb{P}^\infty[\beta^{-1}] \to \mathbb{P}^\infty[\beta^{-1}]_{(k)}$. □

3.3.2 Cohomology operations

We now aim to describe the homotopy classes of endomorphisms of the Snaith spectrum $K := \mathbb{P}^\infty[\beta^{-1}]$. From the colimit definition of $\mathbb{P}^\infty[\beta^{-1}]$ we obtain a limit description of mapping spectra:

$$\begin{aligned}\underline{\mathrm{map}}_{\mathrm{Sp}^{\mathbb{P}^1}}(\mathbb{P}^\infty[\beta^{-1}], S^{s,t} \wedge \mathbb{P}^\infty[\beta^{-1}]) &\simeq \underline{\mathrm{map}}_{\mathrm{Sp}^{\mathbb{P}^1}}(\mathrm{colim}(\mathbb{P}^\infty_+ \xrightarrow{\cdot\beta} \mathbb{P}^\infty_+ \xrightarrow{\cdot\beta} \ldots), S^{s,t} \wedge \mathbb{P}^\infty[\beta^{-1}]) \\ &\simeq \lim\Big(\ldots \xrightarrow{(\cdot\beta)^*} \underline{\mathrm{map}}_{\mathrm{Sp}^{\mathbb{P}^1}}(\mathbb{P}^\infty_+, S^{s,t} \wedge \mathbb{P}^\infty[\beta^{-1}]) \xrightarrow{(\cdot\beta)^*} \underline{\mathrm{map}}_{\mathrm{Sp}^{\mathbb{P}^1}}(\mathbb{P}^\infty_+, S^{s,t} \wedge \mathbb{P}^\infty[\beta^{-1}])\Big)\end{aligned}$$

After applying π_0, this earns us a short exact sequence featuring $[K, S^{s,t} \wedge K] = [\mathbb{P}^\infty[\beta^{-1}], S^{s,t} \wedge \mathbb{P}^\infty[\beta^{-1}]] = \pi_0\underline{\mathrm{map}}_{\mathrm{Sp}^{\mathbb{P}^1}}(\mathbb{P}^\infty[\beta^{-1}], S^{s,t} \wedge \mathbb{P}^\infty[\beta^{-1}])$.

Proposition 3.3.6. *There is an exact sequence of $K^{*,*}(pt_+)$-modules*

$$0 \to lim^1_{n\in\mathbb{N}} K^{*+s,*+t-1}(pt_+)[[x]] \to [K, S^{s,t} \wedge K] \to lim_{n\in\mathbb{N}} K^{*+s,*+t}(pt_+)[[x]] \to 0$$

Proof. This is the Milnor exact sequence associated to the colimit map $[K, S^{s,t} \wedge K] \cong [\mathrm{colim}(\mathbb{P}^\infty_+ \xrightarrow{\cdot\beta^{-1}} \mathbb{P}^\infty_+ \xrightarrow{\cdot\beta^{-1}} \ldots), S^{s,t} \wedge \mathbb{P}^\infty[\beta^{-1}]] \to \lim(\ldots \to K^{*+s,*+t}(\mathbb{P}^\infty) \to K^{*+s,*+t}(\mathbb{P}^\infty))$ □

The right hand term is a limit of $K^{*,*}(pt_+)$-modules whose transition maps are given by pullback along the Bott multiplication map. First we need to understand this pullback map.

Lemma 3.3.7. *The pullback along the Bott multiplication is the map*

$$K^{*,*}(pt_+)[[x]] \to K^{*,*}(pt_+)[[x]], \quad f \mapsto (x\beta + 1) \cdot \frac{\partial f}{\partial x}$$

Proof. We decompose the Bott multiplication map into the following pieces:

$$\mathbb{P}^1 \wedge \mathbb{P}^\infty_+ \to \mathbb{P}^1_+ \wedge \mathbb{P}^\infty_+ \to \mathbb{P}^\infty_+ \wedge \mathbb{P}^\infty_+ \xrightarrow{\mu_+} \mathbb{P}^\infty_+$$

Pullback along these pieces corresponds, after applying Snaith K-theory, to the following sequence of $K^{*,*}(pt_+)$-module maps:

$$K^{*,*}(pt_+)[[x]] \to K^{*,*}(pt_+)[[x,y]] \to K^{*,*}(pt_+)[[x,y]]/(y^2) \to y{\cdot}K^{*,*}(pt_+)[[x]] \cong K^{*,*}(pt_+)[[x]]$$

Here the first map sends x to the formal group law $xy\beta + x + y$. Since it is a ring map, it sends x^n to $(xy\beta + x + y)^n$.

The second map deletes all summands in which a factor of y^2 occurs. Thus $(xy\beta+x+y)^n = (y(x\beta + 1) + x)^n = \Sigma_{i=0}^n \binom{n}{i} y^i(x\beta + 1)^i x^{n-i}$ gets sent to $x^n + ny(x\beta + 1)x^{n-1}$.

The third map projects to the summand where the y occurs and the fourth map, the isomorphism, divides by y. Thus altogether x^n gets sent to $n(x\beta + 1)x^{n-1} = (x\beta + 1) \cdot \frac{\partial x^n}{\partial x}$. Since multiplying with a fixed polynomial and deriving are $K^{*,*}(pt_+)$-linear operations, the map is the one of the claim when restricted to polynomials $K^{*,*}(pt_+)[x] \subseteq K^{*,*}(pt_+)[[x]]$. But polynomials are dense in power series for the limit topology, and Bott multiplication can be assembled as a a colimit of its restrictions to the finite stages $\mathbb{P}^n$ and therefore is continuous. Hence the description of the claim is valid for all power series. $\square$

Up to the unit β these transition maps are the same as those of Riou [Rio10, Prop. 5.2.3], who arrived at them for the same reasons but in a different way (namely by considering known properties of K-theory, not by the Snaith construction). Riou has given sufficient criteria for the $\lim^1$-term to vanish:

Proposition 3.3.8 ([Rio10, Prop. 5.2.5])**.** *Let A be an abelian group and let A^Ω be the linear diagram $\ldots \to A[[x]] \to A[[x]] \to A[[x]]$ with transition maps given by $f \mapsto (1 + x)\frac{\partial f}{\partial x}$. If A is divisible or finite, then $\lim^1 A^\Omega = 0$.*

Later we will be interested in the case where the group is divisible, namely when we study the rationalized Snaith spectrum $K_{\mathbb{Q}}$. Indeed, in this case the transition maps are easily seen to be surjective, since $(1 + x)$ is a unit in the ring of power series and the operator $\frac{\partial}{\partial x}$ is surjective. Hence in this case we have an isomorphism $[K_{\mathbb{Q}}, K_{\mathbb{Q}}] \to \lim_{n\in\mathbb{N}} K_{\mathbb{Q}}^{*,*}(pt_+)[[x]]$, and thus we can define endomorphisms of $K_{\mathbb{Q}}$ spectrum by giving sequences $(f_n)_{n\in\mathbb{N}}$ such that $f_n = (x\beta + 1) \cdot \frac{\partial f_{n+1}}{\partial x}$.

Remark 3.3.9. Riou in [Rio10, Prop. 5.2.8] also shows that when A is a torsion free abelian group with $\mathrm{Hom}(\mathbb{Q}, A) = 0$ (e.g. $\mathbb{Z}_{(k)}$ for any $k \in \mathbb{N}$) the map $\lim(\ldots \to A[[x]] \to A[[x]] \to A[[x]]) \to A[[x]]$ projecting to the last factor is injective. In this case one can identify elements of the limit with particular power series (namely those that are liftable along the whole tower). This is of limited use, because it is hard to give an applicable criterion for the liftability of a power series: As Riou remarks, in the case $A = \mathbb{Z}$ the set of liftable power series is uncountable by a Theorem of Adams and Clarke [AC77, Cor. 3.2]. However, as the authors of [CCW05] state in the introduction of their article, the only known examples are linear combinations of $(1 + x)$ and $\frac{1}{1+x}$. $\diamond$

The nth member of the sequence $(f_n)_{n\in\mathbb{N}}$ corresponding to an endomorphism $F\colon \mathbb{P}^\infty[\beta^{-1}] \to \mathbb{P}^\infty[\beta^{-1}]$ is given by precomposing F with the nth map from the colimit tower:

$$\begin{tikzcd}
\mathbb{P}^\infty_+ \arrow[r, "i"] \arrow[d, "\cdot\beta"'] & \mathbb{P}^\infty[\beta^{-1}] \arrow[r, "F"] & \mathbb{P}^\infty[\beta^{-1}] \\
(\mathbb{P}^1)^{-1} \wedge \mathbb{P}^\infty_+ \arrow[ur, "i_1"] \arrow[d, "\cdot\beta"'] & & \\
(\mathbb{P}^1)^{-2} \wedge \mathbb{P}^\infty_+ \arrow[uur, "i_2"'] \arrow[d, "\cdot\beta"'] & & \\
\vdots & &
\end{tikzcd}$$

By the remark right before Prop. 3.2.29, the class of the upper canonical map $i\colon \mathbb{P}^\infty_+ \to \mathbb{P}^\infty[\beta^{-1}]$ is $(1+\beta x)$. By commutativity of the triangles on the left, the class of the map i_n from the nth level of the colimit tower is that which becomes $(1+\beta x)$ after applying Bott multiplication n times, i.e. it is $\frac{1}{\beta^n}(1+\beta x)$. Hence the sequence corresponding to F is given by $f_n := F(\frac{1}{\beta^n}(1+\beta x))$. We now apply this recipe to the Adams operations.

Proposition 3.3.10. *The image of the class of* $\Psi^k\colon \mathbb{P}^\infty[\beta^{-1}]_{(k)} \to \mathbb{P}^\infty[\beta^{-1}]_{(k)}$ *in* $\lim_{n\in\mathbb{N}} K^{*,*}(pt_+)_{(k)}[[x]]$ *is the sequence* $(\frac{1}{\beta^n k^n}(1+\beta x)^k)_{n\in\mathbb{N}}$.

Proof. The sequence $(f_n)_{n\in\mathbb{N}}$ we are looking for, is $(\Psi^k(\frac{1}{\beta^n}(1+\beta x)))_{n\in\mathbb{N}}$. That is, we want to find an expression for the cohomology class of $\mathbb{P}^\infty_+ \xrightarrow{i_n} \mathbb{P}^\infty[\beta^{-1}] \xrightarrow{\Psi^k} \mathbb{P}^\infty[\beta^{-1}]$. For $n=0$ we know from the compatibility of the ψ^k-operation on $\mathbb{P}^\infty_+$ with the Ψ^k-operation on $\mathbb{P}^\infty[\beta^{-1}]$, i.e. the commutativity of the outer rectangle of the diagram of Corollary 3.3.5, that $\Psi^k(1+\beta x) = \psi^k(1+\beta x) = (1+\beta x)^k$.

For higher n consider the following diagram:

$$\begin{tikzcd}
\mathbb{P}^\infty_+ \arrow[r, "\cdot\beta^n"] \arrow[d, "\psi^k"'] \arrow[rr, bend left, "i"] & (\mathbb{P}^1)^{-n}\wedge\mathbb{P}^\infty_+ \arrow[r, "i_n"] \arrow[d, "(\mathbb{P}^1)^{-n}\wedge\psi^k"] & \mathbb{P}^\infty[\beta^{-1}]_{(k)} \arrow[d, "\Psi^k"] \\
\mathbb{P}^\infty_+ \arrow[r, "\cdot(k^n\beta^n)"'] & (\mathbb{P}^1)^{-n}\wedge\mathbb{P}^\infty_+ \arrow[r, "i_n"'] & \mathbb{P}^\infty[\beta^{-1}]_{(k)}
\end{tikzcd}$$

The commutativity of the diagram was established in the proof of Lemma 3.3.4, where it was shown that $(\mathbb{P}^1)^{-1}\wedge\psi^k\circ(\cdot\beta) = (\cdot(k\beta))\circ\psi^k$, hence $(\mathbb{P}^1)^{-n}\wedge\psi^k\circ(\cdot(\beta^k)^n) = (\cdot(k\beta)^n)\circ\psi^k$. This we can use for the third equality in the following calculation:

$$(1+\beta x)^k = \Psi(1+\beta x) = \Psi(\beta^n\frac{1}{\beta^n}(1+\beta x)) = k^n\beta^n\Psi^k(\frac{1}{\beta^n}(1+\beta x)).$$

It follows that $f_n = \Psi^k(\frac{1}{\beta^n}(1+\beta x)) = \frac{1}{k^n\beta^n}(1+\beta x)^k$ □

The doubtful reader can check that the sequence $(\frac{1}{\beta^n k^n}(1+\beta x)^k)_{n\in\mathbb{N}}$ does indeed satisfy the compatibility condition.

Remark 3.3.11. Riou in [Rio10, Def. 5.3.2] *defines* the Adams operations on rational algebraic K-theory as the endomorphism corresponding to the sequence that we calculated in Prop 3.3.10. For rational K-theory this is not a problem since the $\lim^1$-term of Prop. 3.3.6 vanishes and there is no ambiguity. If one considers k-local K-theory, then there might in general be a non-trivial $\lim^1$-term. In algebraic geometry one can still define Adams operations by observing that over $\mathrm{Spec}(\mathbb{Z})$ the $\lim^1$-term vanishes, as $K_1(\mathbb{Z})$ is finite and hence satisfies the criterion of Prop. 3.3.8. Therefore one obtains a well-defined Ψ^k over $\mathrm{Spec}(\mathbb{Z})$ and then by base change over every scheme. Riou uses this argumentation in several places. This type of argument is possible because in algebraic geometry one has a preferred choice, since there is a deepest base $\mathrm{Spec}(\mathbb{Z})$ over which one has uniqueness up to homotopy. One can then transport these unique maps via base change.

One could try to establish the same for our general situation: here we also have a deepest base, see Section 4.1. Should $K^{1,0}(pt_+)$ over this base turn out to be finite or divisible or be such that the $\lim^1$-term vanishes for some other reason, then the same strategy for a general choice would be applicable. We have, however, no idea whether this is the case. $\diamond$

Suppose that the $\lim^1$-term of Prop. 3.3.6 vanishes. Then the composition of endomorphisms induces a ring structure on $\lim_{n\in\mathbb{N}} K^{*,*}(pt_+)[[x]]$. This ring structure is hard to determine explicitly. In good cases one can go about as follows: Consider two families $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ in $\lim_{n\in\mathbb{N}} K^{*,*}(pt_+)[[x]]$:

$$\begin{tikzcd}
\mathbb{P}^\infty_+ \arrow[d, "\cdot\beta"'] \arrow[drr, bend left, "g_0"] & & \\
(\mathbb{P}^1)^{-1}\mathbb{P}^\infty_+ \arrow[d, "\cdot\beta"'] \arrow[rr, bend left, "g_1"] \arrow[ddr, dotted, "\widetilde{g_1}"'] & \mathbb{P}^\infty_+ \arrow[d, "\cdot\beta"'] \arrow[r, "f_0"] & \mathbb{P}^\infty[\beta^{-1}] \\
(\mathbb{P}^1)^{-2}\mathbb{P}^\infty_+ \arrow[d, "\cdot\beta"'] \arrow[urr, bend left, "g_2"] & (\mathbb{P}^1)^{-1}\mathbb{P}^\infty_+ \arrow[d, "\cdot\beta"'] \arrow[ur, "f_1"] & \\
\vdots & (\mathbb{P}^1)^{-2}\mathbb{P}^\infty_+ \arrow[d, "\cdot\beta"'] \arrow[uur, "f_2"'] & \\
& \vdots &
\end{tikzcd}$$

We want to determine a new family (h_n) describing the composition $(f_n) \circ (g_n)$ of the corresponding endomorphisms. If g_n happens to factor through the m-th stage of the colimit diagram defining $\mathbb{P}^\infty[\beta^{-1}]$, i.e. $g_n \simeq i_m \circ \widetilde{g_n}$ for some $\widetilde{g_n}$, then $h_n = f_m \circ \widetilde{g_n}$. Put differently, h_n is the pullback of f_m along $\widetilde{g_n}$.

This is just to give a general idea. In concrete situations one has to see whether such a computation is possible and feasible. If in the power series $g_n = \Sigma_{i=0}^\infty a_i x^i$ there occur arbitrarily high powers of β^{-1} among the coefficients it means for example that the corresponding map does not factor through a finite stage of the colimit diagram.

3.3.3 Rational splitting d'après Riou

In this section we summarize Riou's proof of the rational splitting of the K-theory spectrum. His construction of projectors in the endomorphism ring carries over to our setting without changes.

Consider the endomorphism ring of the rationalized Snaith spectrum $K_{\mathbb{Q}}$. Here the $\lim^1$-term of Prop. 3.3.6 vanishes and we can describe endomorphisms by compatible sequences of power series from $K^{*,*}_{\mathbb{Q}}(pt_+)[[x]]$. The power series ring has the subring $\mathbb{Q}[[\beta x]]$. This subring is closed under pullback along Bott multiplication, i.e. under the map from Lemma 3.3.7, and so we can consider the inverse limit of the tower of these pullback maps, i.e. of the diagram $\mathbb{Q}^\Omega$ in the notation of Prop. 3.3.8.

We will abbreviate $u := \beta x$. Riou [Rio10, Def. 5.3.3] defines power series $p_n := \frac{1}{n!}\log^n(1+u) \in \mathbb{Q}[[u]]$, and for any $\mathbb{Q}$-vector space A the map $\sigma\colon A^{\mathbb{N}} \to A^{\Omega}$, $(a_n)_{n\in\mathbb{N}} \mapsto \Sigma_{n=0}^{\infty} a_n p_n$, and shows [Rio10, Lem. 5.3.4] that this is an isomorphism of topological groups transforming the shift operator on sequences into the Bott multiplication pullback operator $(\cdot\beta)^*$. Speaking in terms of cohomology theories this map transforms the multiplicative group law into the additive one, which explains the occurrence of the logarithms. It follows that there is an isomorphism of the inverse limits of the two systems $\hat{\sigma}\colon A^{\mathbb{Z}} \cong \lim A^{\Omega}$.

Next, Riou shows [Rio10, Prop. 5.3.7] that in the case $A = \mathbb{Q}$ this last isomorphism is an isomorphism of topogical rings, where $\mathbb{Q}^{\mathbb{Z}}$ is endowed with the obvious product structure and $\lim \mathbb{Q}^{\Omega}$ with the structure as a subring of $[K_{\mathbb{Q}}, K_{\mathbb{Q}}] \cong \lim_{n\in\mathbb{N}} K_{\mathbb{Q}}^{*,*}(pt_+)[[x]]$. In $\mathbb{Q}^{\mathbb{Z}}$ there are the obvious orthogonal projectors and they translate to orthogonal projectors $\pi_n \in [K_{\mathbb{Q}}, K_{\mathbb{Q}}]$ ($n \in \mathbb{Z}$) (as power series these projectors are the p_n from above). We denote the subspace to which π_n projects by $K_{\mathbb{Q}}^{(n)}$.

A calculation [Rio10, Lem. 5.3.8] shows that for every $k \in \mathbb{Z}\backslash\{0\}$ one has $\hat{\sigma}((k^n)_{n\in\mathbb{Z}}) = \Psi^k$. Thus [Rio10, Prop. 5.3.14] one has $\Psi^k \circ \pi_n = \pi_n \circ \Psi^k = k^n \cdot \pi_n$. This means that $K^{(n)}$ is a subspace of the eigenspace of Ψ^k for the eigenvalue k^n, and this simultaneously for all $k \in \mathbb{Z}$.

All of the above considerations rely on formal calculations with power series and are equally valid in our setting. They allow the following definition.

Definition 3.3.12. *The spectrum $H_B := K_{\mathbb{Q}}^{(0)}$ is called the Beilinson spectrum. It has the property that all Adams operations act as identity on it.*

There is one main result for which Riou's arguments do not go through in our setting: The statement [Rio10, Thm. 5.3.10] that the inclusion map $\bigoplus_{n\in\mathbb{Z}} K_{\mathbb{Q}}^{(n)} \to K_{\mathbb{Q}}$ is an isomorphism. Riou shows that both objects represent the same functor, and for this uses that it is enough to check it on finitely presentable objects where it can be seen to be the case. In our setting we do not know that our category is finitely presentable and can not follow this route, but even if we assume finite presentability it is not clear that Riou's arguments carry over. We will arrive at the same result in a different way in the next sections.

3.4 The positive rational stable category

3.4.1 The splitting of the sphere and the Morel spectrum

In this section we work in the rational $\mathbb{P}^1$-stable category $\mathrm{Sp}^{\mathbb{P}^1}_{\mathbb{Q}}$. As before, for an $x \in \mathcal{X}$ we will denote its rationalized suspension spectrum $\Sigma^\infty x_{\mathbb{Q}}$ simply by $x_{\mathbb{Q}}$ again, where no confusion is likely. While we used the notation pt_+ for the sphere spectrum before, when it made sense to emphasize its geometric origin, we will switch to the notation $\mathbb{S}$ now, to emphasize its role as tensor unit in $\mathrm{Sp}^{\mathbb{P}^1}$.

The rational sphere spectrum $\mathbb{S}_{\mathbb{Q}}$ (actually already $\mathbb{S}_{\frac{1}{2}}$, the sphere spectrum with 2 inverted) has an involution which under the isomorphism $- \wedge (\mathbb{P}^1_{\mathbb{Q}})^{\wedge 2}$ corresponds to the twist of two smash copies of $\mathbb{P}^1_{\mathbb{Q}}$:

$$\tau \in [\mathbb{P}^1_{\mathbb{Q}} \wedge \mathbb{P}^1_{\mathbb{Q}}, \mathbb{P}^1_{\mathbb{Q}} \wedge \mathbb{P}^1_{\mathbb{Q}}] \cong [\mathbb{S}_{\mathbb{Q}}, \mathbb{S}_{\mathbb{Q}}]$$

From this we get two idempotent endomorphisms $\frac{1}{2}(id - \tau)$ and $\frac{1}{2}(id + \tau)$ of $\mathbb{S}_{\mathbb{Q}}$. There is a corresponding decomposition $\mathbb{S}_{\mathbb{Q}} \cong \mathbb{S}_{\mathbb{Q}+} \vee \mathbb{S}_{\mathbb{Q}-}$. The summand $\mathbb{S}_{\mathbb{Q}+}$ corresponding to the projector $\frac{1}{2}(id + \tau)$ is called the positive rational sphere or the *Morel spectrum*. On $\mathbb{S}_{\mathbb{Q}+}$ the twist τ acts as identity, while on $\mathbb{S}_{\mathbb{Q}-}$ it acts as $-id$.

Lemma 3.4.1. *The positive rational sphere $\mathbb{S}_{\mathbb{Q}+}$ is a commutative algebra. The multiplication map $\mathbb{S}_{\mathbb{Q}+} \wedge \mathbb{S}_{\mathbb{Q}+} \to \mathbb{S}_{\mathbb{Q}+}$ is an equivalence and the unit map $\mathbb{S}_{\mathbb{Q}} \simeq \mathbb{S}_{\mathbb{Q}+} \vee \mathbb{S}_{\mathbb{Q}-} \to \mathbb{S}_{\mathbb{Q}+}$ is the projection.*

Proof. First note that $\mathbb{S}_{\mathbb{Q}+} \wedge \mathbb{S}_{\mathbb{Q}-} \simeq 0$. Endomorphisms of the tensor unit commute with anything, so the twist τ acts on an object x by $(id \wedge \tau \colon x \wedge \mathbb{S} \to x \wedge \mathbb{S}) \simeq (\tau \wedge id \colon \mathbb{S} \wedge x \to \mathbb{S} \wedge x)$. Thus on $\mathbb{S}_{\mathbb{Q}+} \wedge \mathbb{S}_{\mathbb{Q}-}$ it acts by $id \simeq id \wedge id \simeq \tau|_{\mathbb{S}_{\mathbb{Q}+}} \wedge id \simeq id \wedge \tau|_{\mathbb{S}_{\mathbb{Q}-}} \simeq id \wedge -id \simeq -id$, hence $-id \simeq id$, so the object is zero.

Hence we have an algebra structure given by $\mathbb{S}_{\mathbb{Q}+} \wedge \mathbb{S}_{\mathbb{Q}+} \simeq (\mathbb{S}_{\mathbb{Q}+} \wedge \mathbb{S}_{\mathbb{Q}+}) \vee (\mathbb{S}_{\mathbb{Q}+} \wedge \mathbb{S}_{\mathbb{Q}-}) \simeq \mathbb{S}_{\mathbb{Q}+} \wedge (\mathbb{S}_{\mathbb{Q}+} \vee \mathbb{S}_{\mathbb{Q}-}) \simeq \mathbb{S}_{\mathbb{Q}+} \wedge \mathbb{S}_{\mathbb{Q}} \simeq \mathbb{S}_{\mathbb{Q}+}$ □

Remark 3.4.2. Also note that there are no non-zero morphisms $\mathbb{S}_{\mathbb{Q}+} \to \mathbb{S}_{\mathbb{Q}-}$, again because endomorphisms of the tensor unit commute with everything: Precomposing with τ is the same as postcomposing with tau and gives id in one case and $-id$ in the other. ◊

We can thus form the category $\mathrm{Mod}\,\mathbb{S}_{\mathbb{Q}+}$ of $\mathbb{S}_{\mathbb{Q}+}$-modules. We also call this category the category of *Morel motives*. We have the functor
$(-)_{\mathbb{S}_{\mathbb{Q}+}} \colon \mathrm{Sp}^{\mathbb{P}^1}_{\mathbb{Q}} \to \mathrm{Mod}\,\mathbb{S}_{\mathbb{Q}+}$, $X \mapsto X \wedge \mathbb{S}_{\mathbb{Q}+}$. Since by Lemma 3.4.1 the multiplication map is an equivalence, $\mathrm{Mod}\,\mathbb{S}_{\mathbb{Q}+}$ is a reflective sub-∞-category of $\mathrm{Sp}^{\mathbb{P}^1}_{\mathbb{Q}}$ and the reflection functor is a smashing localization, hence monoidal. Thus being an $\mathbb{S}_{\mathbb{Q}+}$-module is a property, not a structure.

For any spectrum E the unit map $E \simeq E \wedge \mathbb{S}_{\mathbb{Q}} \simeq E \wedge (\mathbb{S}_{\mathbb{Q}+} \vee \mathbb{S}_{\mathbb{Q}-}) \simeq E \wedge \mathbb{S}_{\mathbb{Q}+} \vee E \wedge \mathbb{S}_{\mathbb{Q}-} \to E \wedge \mathbb{S}_{\mathbb{Q}+}$ is again a projection to a wedge-summand of E. The element $\tau \colon \mathbb{S}_{\mathbb{Q}} \to \mathbb{S}_{\mathbb{Q}}$ from above acts on any spectrum E as $id_E \wedge \tau$, and on the summand $E \wedge \mathbb{S}_{\mathbb{Q}+}$ it acts as identity.

Remark 3.4.3. Some of the basic calculations of Morel in [Mor04, Sections 6.1, 6.2] carry over; they govern the interaction of the Hopf map with the permutation of homogeneous coordinates on $\mathbb{P}^1$ and the twist map. We have no use for these facts in this chapter, but they will be interesting in some concrete examples. $\diamond$

3.4.2 $\mathbb{P}^\infty_{\mathbb{Q}+}$ is the free commutative algebra over $\mathbb{P}^1_{\mathbb{Q}+}$

We have previously defined the motivic space $B\mathbb{G}$ as the quotient of the action of $\mathbb{G}$ on the point. We will need the following equivalent description.

Lemma 3.4.4. *The square*

$$\begin{tikzcd} \mathbb{G} \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & B\mathbb{G} \end{tikzcd}$$

is a homotopy pushout in the ∞-category of E_∞-algebras in motivic spaces.

Proof. The action of $\mathbb{G}$ on the point is actually an action coming from an algebra map, namely the unique algebra map to the point. The quotient of an action of $\mathbb{G}$ was by Prop. 2.5.3 given by the relative tensor product over a point. This shows the claim. $\square$

Proposition 3.4.5. *In* $\mathrm{Mod}\,\mathbb{S}_{\mathbb{Q}+}$ *(and hence in $Sp^{\mathbb{P}^1}$) there is an equivalence $Sym(\mathbb{P}^1)_{\mathbb{Q}+} \simeq \mathbb{P}^\infty_{\mathbb{Q}+}$.*

Proof. Applying first Σ^∞_+, then $- \wedge \mathbb{S}_{\mathbb{Q}+}$ (both of which are left adjoints) to the homotopy pushout of Lemma 3.4.4 we get a homotopy pushout in commutative algebras in $\mathrm{Mod}\,\mathbb{S}_{\mathbb{Q}+}$:

$$\begin{tikzcd} \Sigma^\infty_+\mathbb{G}_{\mathbb{Q}+} \arrow[r] \arrow[d] & \mathbb{S}_{\mathbb{Q}+} \arrow[d] \\ \mathbb{S}_{\mathbb{Q}+} \arrow[r] & \Sigma^\infty_+ B\mathbb{G}_{\mathbb{Q}+} \end{tikzcd}$$

We now claim that this homotopy pushout is obtained by applying the free commutative algebra functor $F_{E_\infty}\colon\ \mathrm{Mod}\,\mathbb{S}_{\mathbb{Q}+} \to E_\infty\text{-Alg}$ to the homotopy pushout square

$$\begin{tikzcd} \Sigma^\infty\mathbb{G}_{\mathbb{Q}+} \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & \Sigma^\infty\mathbb{P}^1_{\mathbb{Q}+} \end{tikzcd}$$

To see this, we use that by [Lur11, 3.1.3.14] $F_{E_\infty}(\Sigma^\infty\mathbb{G}_{\mathbb{Q}+}) \simeq \bigvee_{n\in\mathbb{N}}(\Sigma^\infty\mathbb{G}_{\mathbb{Q}+})^{\wedge n}/\Sigma_n$ where Σ_n denotes the permutation group. The wedge-summands with $n \geq 2$ on the right hand side are zero in $\mathrm{Mod}\,\mathbb{S}_{\mathbb{Q}+}$: We have

$$(\Sigma^\infty\mathbb{G}_{\mathbb{Q}+})^{\wedge n} \wedge (S^1)^{\wedge 2} \simeq \mathbb{P}^1_{\mathbb{Q}+} \wedge \mathbb{P}^1_{\mathbb{Q}+} \wedge (\Sigma^\infty\mathbb{G}_{\mathbb{Q}+})^{\wedge n-2}$$

There the twist $\tau \wedge id_{(\Sigma^\infty\mathbb{G}_{\mathbb{Q}+})^{\wedge n-2}}$ acts as identity, since we are in $\mathrm{Mod}\,\mathbb{S}_{\mathbb{Q}+}$. But on the other hand the permutation of the two S^1-factors is $-id_{S^1}$. Hence, in order to get the identity for the $\mathbb{P}^1_{\mathbb{Q}+}$-twist, the twist of two $\mathbb{G}_{\mathbb{Q}+}$s must also be $-id$. But in forming $Sym^n(\mathbb{G}_{\mathbb{Q}+})$ we force the $\mathbb{G}_{\mathbb{Q}+}$-twist to be id, so on the wedge summands $Sym^n(\mathbb{G}_{\mathbb{Q}+})$ with $n \geq 2$ of $Sym^\infty(\mathbb{G}_{\mathbb{Q}+})$ we have $-id = id$ whence these summands are zero.

Therefore we have $F_{E_\infty}(\Sigma^\infty \mathbb{G}_{\mathbb{Q}+}) \simeq \bigvee_{n\in\mathbb{N}} (\Sigma^\infty \mathbb{G}_{\mathbb{Q}+})^{\wedge n}/\Sigma_n \simeq \mathbb{S} \vee \Sigma^\infty \mathbb{G}_{\mathbb{Q}+} \simeq \Sigma^\infty_+ \mathbb{G}_{\mathbb{Q}+}$ and so

$$F_{E_\infty}(0 \leftarrow \Sigma^\infty \mathbb{G}_{\mathbb{Q}+} \to 0) \simeq \mathbb{S}_{\mathbb{Q}+} \leftarrow \Sigma^\infty_+ \mathbb{G}_{\mathbb{Q}+} \to \mathbb{S}_{\mathbb{Q}+}$$

As F_{E_∞} is a left adjoint, it preserves pushouts which gives us

$$\Sigma^\infty_+ \mathbb{P}^\infty_{\mathbb{Q}+} \simeq F_{E_\infty}(\mathrm{colim}(0 \leftarrow \Sigma^\infty \mathbb{G}_{\mathbb{Q}+} \to 0)) \simeq F_{E_\infty}(\Sigma^\infty \mathbb{P}^1_{\mathbb{Q}+}) \simeq Sym(\mathbb{P}^1_{\mathbb{Q}+})$$

□

3.4.3 Splitting of the rational Snaith spectrum

Proposition 3.4.6. $\mathbb{P}^\infty[\beta^{-1}]_{\mathbb{Q}+} \simeq \mathbb{P}^\infty[\beta^{-1}]_{\mathbb{Q}}$

Proof. By construction $\mathbb{P}^\infty[\beta^{-1}]$ is a commutative ring spectrum on which two-fold Bott multiplication $\cdot\beta^2 : \mathbb{P}^1 \wedge \mathbb{P}^\infty[\beta^{-1}] \to \mathbb{P}^\infty[\beta^{-1}]$ is an equivalence. The commutativity implies that the following diagram commutes:

$$\begin{tikzcd}
(\mathbb{P}^1)^{\wedge 2} \wedge \mathbb{P}^\infty[\beta^{-1}] \arrow[r, "\cdot\beta^2", "\simeq"'] \arrow[d, "\tau\wedge id"'] & \mathbb{P}^\infty[\beta^{-1}] \\
(\mathbb{P}^1)^{\wedge 2} \wedge \mathbb{P}^\infty[\beta^{-1}] \arrow[ur, "\simeq", "\cdot\beta^2"'] &
\end{tikzcd}$$

This shows that τ acts as identity even before rationalizing. □

Remark 3.4.7. In general it should be true that oriented rational ring spectra have only the positive part, i.e. the $(\mathbb{P}^1)^{\wedge 2}$-twist acts as identity. The proof of Morel, Some basic properties of the stable motivic homotopy category, Lemma 4.1.9(1) should go through. ◊

Theorem 3.4.8. *The rational Snaith spectrum splits into a sum of shifted copies of the positive rational sphere. The summand corresponding to $n \in \mathbb{Z}$ is the eigenspace for the value k^n of the Adams operation Ψ^k.*

Proof. By Proposition 3.4.5 we have an equivalence

$$\mathbb{P}^\infty_{\mathbb{Q}+} \simeq Sym(\mathbb{P}^1)_{\mathbb{Q}+} \simeq \bigvee_{n\in\mathbb{N}} (\mathbb{P}^1_{\mathbb{Q}+})^{\wedge n} \simeq \bigvee_{n\in\mathbb{N}} \Sigma^{2n,n}\mathbb{S}_{\mathbb{Q}+}$$

Under this equivalence the Bott multiplication map

$$\Sigma^{2,1}\mathbb{P}^\infty_{\mathbb{Q}+} \simeq \Sigma^\infty_+(\mathbb{P}^1) \wedge \Sigma^\infty_+(\mathbb{P}^\infty) \to \Sigma^\infty_+(\mathbb{P}^1) \wedge \Sigma^\infty_+(\mathbb{P}^\infty) \to \Sigma^\infty_+(\mathbb{P}^\infty)$$

corresponds to the wedge sum of the maps $\mathbb{P}^1_{\mathbb{Q}+} \wedge (\mathbb{P}^1)^{\wedge n}_{\mathbb{Q}+} \to (\mathbb{P}^1)^{\wedge n+1}_{\mathbb{Q}+}$ ($n \in \mathbb{N}$) — i.e. it shifts the nth wedge summand up to the $n+1$st summand.

Thus by the colimit construction of $\mathbb{P}^\infty[\beta^{-1}]$ we get an equivalence

$$\mathbb{P}^\infty[\beta^{-1}]_{\mathbb{Q}+} \simeq \bigvee_{n\in\mathbb{Z}} \Sigma^{2n,n}\mathbb{S}_{\mathbb{Q}+}$$

Combining this with the equivalence of Prop. 3.4.6 proves the first claim.

For the claim about the eigenspaces we analyze the action of $\Psi^k = (\frac{1}{k^n\beta^n}(1+\beta x)^k)_{n\in\mathbb{N}}$. Remember that $\mathbf{t} = (1+\beta x)$ is the class of the the inclusion $\mathbb{P}^\infty_+ \to \mathbb{P}^\infty[\beta^{-1}]$. Under our equivalence this corresponds to the inclusion of the summands of positive index into all summands $\bigvee_{n\in\mathbb{N}}(\mathbb{P}^1)^{\wedge n} \to \bigvee_{n\in\mathbb{Z}}(\mathbb{P}^1)^{\wedge n}$. The Bott multiplication map $\cdot\beta\colon (\mathbb{P}^1)\wedge(\bigvee_{n\in\mathbb{N}}(\mathbb{P}^1)^{\wedge n})$ is simply adjoining the extra $\mathbb{P}^1$-smash factor distributively to each summand - i.e. it is shifting up the summand, sending $(\mathbb{P}^1)^{\wedge n}$ to $(\mathbb{P}^1)^{\wedge n+1}$.

Since the domain of our action is now a coproduct, we can analyze the action of Ψ^k summand by summand. On the summand $\mathbb{P}^1_{\mathbb{Q}+}$ it is given by taking k-th power, i.e. by

$$\mathbb{P}^1_{\mathbb{Q}+} \hookrightarrow (\mathbb{P}^1_{\mathbb{Q}+})_+ \xrightarrow{\Delta_+} ((\mathbb{P}^1_{\mathbb{Q}+})^k)_+ \simeq \bigvee_{i=0}^{k}\bigvee_{j=0}^{\binom{k}{i}}(\mathbb{P}^1_{\mathbb{Q}+})^{\wedge i}$$

We can analyze this map summand by summand in the codomain. The map to the 0-th summand is zero, because the base point is not present in the domain, and the maps to the higher summands $(\mathbb{P}^1_{\mathbb{Q}+})^{\wedge i}$, $(i \geq 2)$ are zero by Prop. 3.1.42. The maps to the $\mathbb{P}^1_{\mathbb{Q}+}$-summands are the identity and therefore the whole map together is the k-fold diagonal, i.e. multiplication by k.

By multiplicativity, the map restricted to the summand $(\mathbb{P}^1)^{\wedge n}$ is then multiplication by k^n. ◻

Corollary 3.4.9. *In $Sp^{\mathbb{P}^1}$ there is an equivalence $\mathbb{S}_{\mathbb{Q}+} \simeq H_B$.*

3.5 Functoriality

All constructions of this chapter are functorial in product preserving left adjoints:

Let $(\mathcal{X}, \mathbb{G})$ be a pair, as throughout this chapter, consisting of a cartesian closed presentable ∞-category and a group object therein. Let $F\colon \mathcal{X} \to \mathcal{Y}$ be a product preserving left adjoint to a cartesian closed presentable ∞-category $\mathcal{Y}$. Then $F(\mathbb{G})$ is a group object in $\mathcal{Y}$ and the pair $(\mathcal{Y}, F(\mathbb{G}))$ is a datum as needed for the constructions of this chapter.

First of all, F induces a functor $\mathbb{G}$-Mod$(\mathcal{X}) \to F(\mathbb{G})$-Mod$(\mathcal{Y})$, as can e.g. be seen from a description of $\mathbb{G}$-modules as monoid maps $\mathbb{G} \to map_{\mathcal{X}}(x, x)$ to the internal hom of an object in $\mathcal{X}$. Since our constructions in the category of $\mathbb{G}$-modules used nothing but products and colimits, they are preserved under F, e.g. the $\mathbb{A}^n \setminus 0$ constructed in $\mathcal{X}$ from $\mathbb{G}$ are mapped by F to the $\mathbb{A}^n \setminus 0$ constructed in $\mathcal{Y}$ from $F(\mathbb{G})$.

Also quotienting out a $\mathbb{G}$-action then applying F is the same as first applying F, then quotienting out the $F(\mathbb{G})$-action (this follows from the fact that quotienting out is relative tensor product along the map to the terminal group, and terminal objects are preserved). Thus the $\mathbb{P}^n$ constructed in $\mathcal{X}$ from $\mathbb{G}$ are mapped by F to the $\mathbb{P}^n$ constructed in $\mathcal{Y}$ from $F(\mathbb{G})$.

Passage to pointed objects and stabilization have universal properties which guarantee that F induces left adjoint monoidal functors. Thus these induced functors preserve the Bott multiplication map and the Snaith spectrum, as well as the rational splitting into summands. We obtain in particular an induced functor between the categories of Morel motives based on $(\mathcal{X}, \mathbb{G})$, resp. $(\mathcal{Y}, F(\mathbb{G}))$.

4 Examples

4.1 The initial example

There is a non-full subcategory *Cart* of the ∞-category Pr^L of presentable ∞-categories and left adjoints, whose objects are cartesian closed presentable ∞-categories and whose morphisms are product preserving left adjoints.

One can prove the following:

Proposition 4.1.1. *Left Kan extensions of product preserving functors into a cartesian closed cocomplete category along product preserving functors are product preserving.*

This can be used to show that there is a classifying object for group objects in *Cart*, i.e. an object of *Cart* containing a generic group object, such that every group object in a cartesian closed presentable ∞-category is an image of that one. This classifying object can be very concretely described: It is the presheaf category on the opposite category of finitely generated free commutative monoids in spaces (i.e. the presheaf category on a Lawvere theory).

By the considerations of Section 3.5 all ∞-categories of motives or motivic spectra resulting from our construction receive a functor from this initial one.

4.2 A somewhat general example

As in Isbell duality we can take any algebraic structure (say, presented by a Lawvere theory $\mathbb{T}$) and build a kind of schemes out of it. If there is a morphism from the Lawvere theory of E_∞-algebras to $\mathbb{T}$, then restriction along this morphism gives an E_∞-algebra object in the coresponding category of motivic spaces. By Section 3.5 this should become a functor

$$Mot\colon AlgTheories \text{ under } E_\infty \to (\infty,1)-cat,\ (E_\infty \to \mathbb{T}) \mapsto Motives$$

An instance of this would be the theory of motivic spaces based on C^∞-schemes, since the theory of C^∞-rings receives a morphism from the theory of E_∞-algebras: the group of units.

4.3 Differential K-theory

Take as starting category the category of sheaves of spaces on smooth manifolds and as $\mathbb{G}$ some sheaf deserving the name "$U(1)$-with-connection", e.g. take the functor $BU(1)$ associating to a smooth manifold the category of line bundles with connection on it, then set $\mathbb{G} := \Omega BU(1)$). There is a differential Snaith theorem, see [BNV16, Section 6.3] and it would be intersting to investigate the relation to our general constructions.

4.4 $\mathbb{F}_1$-geometry

This is what motivated this work. There are numerous settings for $\mathbb{F}_1$-geometry, e.g. monoid schemes, Λ-schemes, Lorscheid's blueprints, Durov's generalized schemes, Haran's $\mathbb{F}$-rings. The list is long and contains many non-additive settings which we are able to treat because we made no use of addition anywhere, e.g. in the form of GL_n-actions.

4.5 Further examples

- One may take $\mathbb{Z}/2$-equivariant spaces and set $\mathbb{G}$ to be the sign representation.
- One could take as $\mathcal{X}$ the usual ∞-category of motivic spaces, but with a different group object, e.g. set $\mathbb{G} := \mathbb{P}^\infty$. The resulting theory should have something to do with Brauer groups and forms of projective bundles.
- Derived algebraic geometry and the several versions of $\mathbb{G}_m$ occurring there
- Log geometry

5 Conclusion

We list some natural continuations of this work:

General linear groups: Introduce some linearity by axiomatically postulating group objects GL_n, with embeddings $\Sigma_n \ltimes \mathbb{G}^n \hookrightarrow GL_n$ and $GL_n \to GL_{n+1}$ and actions of GL_n on $\mathbb{A}^n \setminus 0$. One could then build BGL via Grassmannians, Thom spectra, cobordism, think about Landweber exactness.

This would include also nonlinear settings like Haran's $\mathbb{F}$-rings.

Cohomology operations: In [Vis16] Vishik studies the one-graded cohomology theories A^* obtained from bigraded ones by taking the steep diagonal $A^{2*,*}$, in particular theories "of rational type" (roughly: obtained from the diagonal of algebraic cobordism by tensoring with a Landweber exact module over the Lazard ring). He shows in [Vis16, Thm. 3.6] that for such a theory the cohomology operations from A^n to any cohomology theory B^* are in bijection with families of maps $A^n((\mathbb{P}^\infty)^{\times l}) \to B^*((\mathbb{P}^\infty)^{\times l})$ ($l \in \mathbb{N}$) which are compatible with:

- pullbacks for the action of the symmetric group Σ_l
- the diagonals
- the projections
- the maps $\mathrm{Spec}(k) \to \mathbb{P}^\infty$
- the Segre embeddings

All of these maps are available in our setting and one might attempt to prove a version of this result, either in an abstract setting admitting a cobordism spectrum or for modules over the Snaith spectrum in the setting of this work. In any case, Vishik's result is a hint that the objects we constructed are important ones in motivic homotopy theory and its generalizations.

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