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E_∞-ring structures in Motivic Hermitian K-theory.

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Contents

Introduction.

This thesis deals with E_∞ ring structures on the Hermitian K-theory in the motivic setting. Let us start giving an idea of all the words appearing in the title: K-theory, hermitian, motivic and E_∞-ring. The K-theory started as the group completion of an abelian monoid, i.e, we add inverses to the commutative monoid structure getting an abelian group in an universal way. Following up this idea they were defined the higher groups. These groups use to be expressed as the homotopy groups of a space, or as the groups of a cohomology or homology theory. Any generalized cohomology theory is represented by a spectrum, i.e, a sequence of pointed spaces E_n for $n \in \mathbb{Z}$ such that there is a map $\Sigma E_n \to E_{n+1}$ for any $n \in \mathbb{Z}$. Where being represented by the spectrum E means that the cohomology groups are given as the classes of homotopy maps between the suspension spectrum of the space and the spectrum E, i.e, $E^*(X) = [\Sigma^\infty X_+, E]$. The hermitian K-theory is a branch of the K-theory whom study the K-theory of rings with involution, i.e, $\big(R, \bar{(\ )}\big)$ such that $\bar{\bar{a}} = a$, or the categories with duality, which are a categorical generalization of the rings with involution. A category with duality is a category $\mathcal{C}$ with a contravariant endofunctor $*\colon \mathcal{C} \to \mathcal{C}^{op}$ such that applying twice this functor is isomorphic to the identity. Having a duality in a category we can consider symmetric bilinear forms on the category. The category of non-degenerated symmetric bilinear forms, the so-called hermitian category, has a structure of symmetric monoidal category (which means, morally, a category with the structure of a commutative monoid) respect to the direct sum of non-degenerated symmetric bilinear forms. We can group complete such a structure and get the hermitian K-theory of a category with duality. In our case, we are going to compute the spectrum representing the hermitian K-theory in the motivic setting. The motivic homotopy theory is a good context to make algebraic topology over algebraic varieties. We consider the category of smooth varieties with a base Noetherian, separated and with finite Krull dimension scheme S, that we denote by $\mathrm{Sm}\ |_S$, This category is intractable since it is not closed under colimits, so we consider the simplicial presheaves over it, $s\mathrm{PSh}\,(\mathrm{Sm}\ |_S)$. To such a category we apply two Bousfield localizations, which is a way to invert morphisms with some desired properties. The first one is respect to the Nisnevich topology, a topology between the étale and the Zariski topologies, and the other is to become the affine line $\mathbb{A}^1$ weak equivalent to the point. The resulting category is denoted by $\mathsf{Spc}(S)$. Morally, this category has the same behavior that the category of smooth manifolds with interval object the usual interval. As we said the K-theory can be expressed, in some cases, as a cohomology theory. If the category that we want to group complete has a second symmetric monoidal structure (usually called multiplicative structure and denoted by $\otimes$) we would like to know if such a structure is preserved along the group completion giving rise to a ring structure in the spectrum and the associated cohomology theory. In homotopy theory it use to be really uncommon to have real strict algebraic structures, but we can require to the algebraic structures to be verified up to homotopy, i.e, a strict algebraic structure in the homotopy category. However, to require just up to homotopy turns to be too weak, so we also require to the operation to verify the properties up to higher homotopies. The idea of an operad is that it is a sequence of spaces that parametrizes an ary of operations. The E_∞-operads are the operads that parametrize a commutative monoid structure up to homotopy and up to higher homotopy coherences. An algebra respect to an E_∞-operad is an object with such an operation. Our multiplicative spectra will be the commutative ring spectra, i.e, the E_∞-algebras in the category of spectra.

In this thesis we are going to see how to obtain a spectrum representing the Hermitian K-theory in the motivic setting. So, we are going to consider the category of algebraic vector bundles over a scheme X in $\mathrm{Sm}\mid_S$, that we denote by $Vect(X)$. For a scheme X this category has two natural symmetric monoidal structures, the first one the direct sum $\oplus$ and the second one the tensor product $\otimes$. Moreover, there is a natural duality in the algebraic vector bundles given by $\mathrm{Hom}_{\mathcal{O}_X}(-, \mathcal{O})$. This two monoidal structures are interrelated since the tensor product distributes respect to the direct sum, leading to what is known as rig category. This ring monoidal structure can be transformed in a bipermutative structure, i.e, a category where some of the coherence conditions of the two symmetric monoidal structures are verified by identities instead of just isomorphisms. This will be our category to deal with. Since we want to work in the Hermitian context we are going to consider the Hermitian category of this last category, morally, the category of non-degenerated bilinear forms. It still exists a bipermutative structure in the Hermitian category inherited from the original one. As one use to do in K-theory, we will consider the category of isomorphisms in such a category, without considering all the morphisms which are not isomorphisms. This one will be the category with respect to whom we want to get the spectrum representing the Hermitian K-theory. To this end, we are going to use infinite loop space machines. We are going to employ two different machines and see that the corresponding spectra have a multiplicative structure on them.

This bipermutative structure exists just when the scheme that we are considering is affine, otherwise there exists not bipermutative structure since the vector bundles in a general scheme does not split. Thus the first step will be to extend this structure to schemes in $\mathrm{Sm}\mid_S$ via a stable version of the descent theorem, which allow us to extend properties from affine to more general schemes. This will be done in Section 1.3.

In the early seventies they appeared the infinite loop space machines. Morally, they give the group completion of an E_∞-space and we can obtain the associated spectrum to it. This is encoded in a functor

$$E_\infty\left(Spc\right) \to \mathrm{Sp}$$

Through the seventies it was proved that all these infinite loop space machines turn out to be equivalents. At the start of the eighties Peter May proved that such infinite loop space machine respects the multiplicative structures ([May82a]), refining to a functor

$$\mathrm{Rig}_{E_\infty}\left(Spc\right) \xrightarrow{Gp} E_\infty\left(\mathrm{Sp}\right)$$

The infinite loop space machine appearing in Section 2.4 is the May's one, the more classic. In fact, this is a multiplicative infinite loop space machine, and it preserves the second operation. This construction is combinatorial, it use concrete operads and defines the associated spectrum in a explicit way. From the bipermutative structure we will produce an E_∞-ring space, a space with the action of two E_∞-operads at the same time, where these operads are interrelated by an action of one into the other. We will group complete this E_∞-ring space via the May's machine and will get the associated spectrum. Since we are working with motivic spaces this will produce a presheaf of spectra that we will denote by **KH**:

$$\mathbf{KH}\colon \mathrm{Sm}\mid_S \to \mathrm{Spt}^1_{\mathbb{S}}$$

The infinite loop space machine that we have applied preserves the multiplicative structure. Therefore, at the end the presheaf will be not just a presheaf of spectra but a presheaf of E_∞-ring spectra. We realize that this is a presheaf of usual $\mathbb{S}^1$-spectra, but the spectra in the motivic setting is given by smashing with two circles, the simplicial, the one of the usual spectra, and the tate circle. Other way to think about motivic spectra is as $\mathbb{P}^1$-spectra. We will get the definitive presheaf of E_∞-ring $\mathbb{P}^1$-spectra by applying the delooping construction in Section 2.5.

The second infinite loop space machine will be the Hermitian direct sum K-theory functor. This is a new Hermitian infinite loop space machine. They were many generalizations of the infinite loop space machines since the seventies. In [Nik13], Gepner, Groth and Nikolaus defined a direct sum K-theory using infinity categories. This is not a combinatorial construction, instead of explicit operads they define the infinite loop space machine as a sequence of functors and universal properties. The price to pay for this construction is that we need to use infinity categories. The Hermitian direct sum K-theory is an Hermitian version of such construction. First we will define the infinity categories with duality. We will define them as the fixed points respect to a C_2-action in the infinity category of the small infinity categories Cat_∞. Our Hermitian K-theory gives rise to a functor

$$K_h \colon \mathsf{SymMonCat}_\infty^{hC_2} \to \mathrm{Sp}.$$

from symmetric monoidal infinity categories with duality to spectra. This functor is in fact a lax symmetric monoidal functor, so it refines to a functor

$$\mathrm{Rig}_{E_\infty}(\mathsf{Cat}_\infty)^{hC_2} \to E_\infty(\mathrm{Sp}).$$

Then we will see that the usual categories with duality can be seen as infinity categories with duality. It is also proved a recognition principle for preadditive infinity categories with duality to express them in a natural way as a rig infinity category with duality. Once we have applied the descent theorem, this recognition principle covers the case of the algebraic vector bundles in the motivic setting and it gives rise to a second presheaf of spectra

$$\mathbf{KQ} \colon \mathrm{Sm}\mid_S \to \mathrm{Spt}^1_{\mathbb{S}}$$

To get the desired presheaf of E_∞-ring spectra we will apply the delooping construction. Such a construction was already done for the usual motivic K-theory in [Ost10] to get an uniqueness result. We will apply this construction to the motivic setting, but it can be done in a broader context. We will try to keep the most general as possible along the construction. For our case, this construction allow us to create a new spectrum which is nothing else that the old $\mathbb{S}^1$-spectrum in each entry, but this new one is a $\mathbb{P}^1$-spectrum and it has structure of commutative algebra. We can apply this delooping construction to any of the presheaves of spectra giving rise to a E_∞-ring $\mathbb{P}^1$-spectrum $\mathbf{KQ}_{\mathbb{P}^1}$.

At the end of the Thesis there is an uniqueness result. In [PW10a] Panin and Walter construct an spectrum representing the Hermitian K-theory in the motivic setting which has like spaces the fibrant replacement of the Schlichting's Waldhausen-like construction. This spectrum has a structure of commutative monoid in the stable motivic homotopy category. They proved also a uniqueness result for such structure, which states that it is the unique commutative monoid up to homotopy such that it restricts to a concrete pairing when reducing to the usual Grothendieck-Witt groups (the 0-groups of the Hermitian K-theory). We will define the pairings in a preadditive infinity category with duality and then we will see that this agrees with pairings defined by Panin and Walter. The pairing given by the tensor product is preserved along the whole construction and agrees with the Panin-Walter one. Therefore we will get the same uniqueness result for our spectrum.

Contents.

The first chapter consists of three sections. The first two cover the necessary background to understand the motivic Hermitian K-theory. The chapter starts with a section which covers the basic background for motivic homotopy theory. In this section they are motivated and constructed the unstable and stable motivic homotopy categories. The next section introduces the reader to the K-theory with emphasis in the hermitian K-theory. This two first section can be skip by the reader with enough knowledgeable on the subject.

The last section in the first chapter, Section 1.3.1, starts with the study of the structures in the presheaves of algebraic vector bundles in the motivic setting. Then, the subsection 1.3.2 explains the transition from affine schemes to more general schemes for presheaves of spectra. After a short review of the actual context of motivic Hermitian K-theory and the known constructions, the subsection 1.3.4 deals with the construction of a Bott element for the Hermitian K-theory via the inclusion of the quaternionic projective line, similarly to the usual K-theory case, that will be used later for the delooping construction.

The second chapter is divided in five parts. Again the first two sections cover a necessary background. In Section 2.1 they are introduced the operads (subsection 2.1.2) and the infinite loop space machines (subsection 2.1.1). Section 2.2 expounds the concepts in higher category theory needed for the infinite loop space machine in Section 2.3.

In Section 2.3, they are defined the infinity categories with duality (subsection 2.3.2), it is explained the pass from usual categories with duality to infinity categories with duality (subsection 2.3.3), it is constructed the direct sum Hermitian K-theory functor (subsection 2.3.4) and the preservation of the multiplicative structure on it (subsection 2.3.5), and it is proved a recognition principle for this machinery (subsection 2.3.6). At the end we will explain how does it produces our desired presheaf of $\mathbb{S}^1$-spectra $\mathbf{KQ}$ (subsection 2.3.7).

In Section 2.4 it is constructed the presheaf of spectra $\mathbf{KH}$. This will be done applying May's infinite loop space machine to our case. In subsection 2.4.1, it will be explained the passage from permutative categories to E_∞-spaces and how to get its associated spectrum. The next subsection develops the case of the bipermutative categories, the E_∞-ring spaces and the E_∞-ring spectra associated to them, covering our special case for getting the presheaf $\mathbf{KH}$.

The last section, Section 2.5, is the delooping construction and the uniqueness result. The section has a long introduction where it is explained the delooping construction in a more general that our case for the Hermitian presheaf of spectra. The first subsections (subsections 2.5.1, 2.5.2, 2.5.3 and 2.5.4) explain the tools appearing in the construction. In subsection 2.5.1, they are explained the model structures employed for getting a rectification from E_∞-algebras to commutative algebras, since we want to take modules over $\mathbf{KQ}_{\mathbb{S}^1}$ during the delooping construction. Then, they will be explained the categories that we are going to use during the delooping (subsection 2.5.2), the day convolution product (subsection 2.5.3) and the model structures that we will use along the construction (subsection 2.5.4). In the subsection about the model structures for the delooping construction (subsection 2.5.4), since we need to work for connective spectra in our particular delooping construction we will define the model structures involved in the delooping construction and shortly after we will do a correction of them. Subsection 2.5.5 is the delooping construction. Here, it is done for our concrete case, but as it will be remarked at the start of the section it can be done in a more general context. The last subsection proves the uniqueness result by comparison with the Panin-Walter spectrum (subsection 2.5.6).

Chapter 1

Motivic Hermitian K-theory.

This thesis deals with multiplicative structures of the Hermitian K-theory in the motivic setting. In this first chapter the reader will be put into context. In the first section we are going to define the motivic homotopy theory. Our attention will be concentrated in the stable case. The unstable case will be quickly established and the stable will be more deeply developed. We will see the triangulated structure in the stable motivic homotopy category and how the spectra category can be seen as a $\mathbb{P}^1$-spectra category, therefore covering the required background for the motivic algebraic K-theory. In the second, it will be addressed the Hermitian K-theory. Starting from a basic review of its history to the last constructions in the Hermitian K-theory. In the last section it will be study in short the presheaf of K-theory spectra and the structures that we want to group complete. We will see that there is a bimonoidal, in fact bipermutative, structure in such a presheaf for affine schemes and it will be used further on in the second chapter to apply the multiplicative loop space machines. Just after there is a short subsection covering the required background in Hermitian K-theory in the motivic setting. In the next subsection, this bipermutative structure will be extended by Nisnevich descent construction from affine to more general schemes, meaning the schemes of the motivic context. At the end there is Bott element subsection in which it will be define a Bott element for the Hermitian K-theory by the inclusion of Quaternionic projective line in analogy to the Bott element for the usual motivic K-theory.

1.1 Motivic homotopy theory.

In this section, we present the basic background and tools used later in motivic homotopy theory, also called $\mathbb{A}^1$-homotopy theory. The motivic homotopy theory is the homotopy theory for smooth schemes, the role of the $\mathbb{A}^1$-homotopy theory for smooth schemes is the equivalent to the usual homotopy theory for topological spaces for manifolds. If in the case of the manifolds we have the site of smooth manifolds and we take as interval object the real line $\mathbb{R}$, in the motivic case, for a given Noetherian and separated scheme with finite Krull dimension S, we consider the Nisnevich site of smooth schemes of finite type over S, $\mathrm{Sm}\,|_S$, and we take as interval object the affine line $\mathbb{A}^1$. Here the affine line $\mathbb{A}^1$ plays the role of the interval object and we get an homotopy theory induced by a good behavior of an interval object. To add colimits to the category $\mathrm{Sm}\,|_S$ we consider the simplicial presheaves over it and we do two localizations, one respect to the Nisnevich site and an other to make the affine line contractible. As one use to do for Bousfield localizations we fix the fibrant objects such that they verify the desired properties, here they are Nisnevich sheaves up to homotopy and to become $\mathbb{A}^1$ weak equivalent to the point. This will give rise to the so-called *unstable motivic homotopy category.* There are many different categories worthy of this name but all of them result Quillen equivalents and broadly constructed via this two localizations.

The motivic theory supposes a blend between the algebraic geometry and the algebraic topology. We are working with $\mathsf{sSet}^{(\mathrm{Sm}|_S)^{op}}$ and since the Nisnevich topology is subcanonical we can introduce both inside: $\mathrm{Sm}\,|_S$ via the Yoneda embedding and the spaces (sSet) via the constant presheaves. The current text is concentrated in the stable case. Since this setting comes from two different branch we get two different circles, the simplicial circle and the Tate circle, whom smashing we can define a suspension. We will stabilize respect to both in the category of motivic spaces yielding the stable the category of motivic spectra. Taking the associated homotopy category we get the *stable motivic homotopy category*, $\mathsf{SH}(S)$, and we will see that there exists a space, the *Tate sphere* $\mathbb{A}^1/(\mathbb{A}^1-0)$, such smashing with it is equivalent to do it with both circles. In fact, we will get the same with the projective line $\mathbb{P}^1$. So in the stable case we will consider the $\mathbb{P}^1$-spectra and the triangulated structure in $\mathsf{SH}(S)$.

We are going to suppose that the reader has a basic background in algebraic geometry. This background is almost entirely covered by Hartshorne's book [Har77]. We assume the reader is familiar with algebraic varieties and schemes ([Har77, Chap. *I* and Chap. *II*]). We also assume to be familiar with model categories, stable homotopy theory and their basic constructions (as Bousfield localizations) as well as basic topos theory, more precisely sites. Some of this tools will be remember in the next subsection or at least the references to texts that assists the general comprehension.

1.1.1 Introduction.

We start defining a Grothendieck topology in a category and the respective site, the pair given by the category and the topology. Instead of closed sets and open sets the Grothendieck topologies are defined by the morphisms (coverings or sieves). The other main point, and more important in practice, is that a site allows us to define sheaves and therefore also topoi. The Grothendieck topologies are defined by sieves, subfunctors of representable functors, and the pretopologies are given by coverings. The second one is more intuitive but it requires fibered coproducts in the category. Any pretopology generates a topology considering the collection of all sieves that contains a covering family. Since the categories that we are going to work with have fibered coproducts we are not going to distinguish pretopologies and topologies, all will be defined using coverings.

Definition 1.1.1. 1. Let $\mathcal{C}$ be a category. A *Grothendieck topology* is a collection of families of morphisms in $\mathcal{C}$ that we call coverings and we denote $Cov(X)$ for any X in $\mathcal{C}$ such that they verify the following axioms:

   (a) Isomorphisms: If $f : Y \to X$ is an isomorphism in $\mathcal{C}$ then $\{f\}$ is a covering family, $\{f\} \in Cov(X)$.

   (b) Existence of fibered products: For any X in $\mathcal{C}$, any morphism $Z \to X$ and $f : Y \to X$ appearing in a covering of X, the fibered product $Y \times_X Z \to X$ exits.

   (c) Base change: For any $\{X_\alpha \to X\}_\alpha \in Cov(X)$ and any morphism $Y \to X$ in $\mathcal{C}$, $\{X_\alpha \times_X Y \to X\}_\alpha$ is a covering family of X.

   (d) Local character: If $\{X_\alpha \to X\}_\alpha \in Cov(X)$ and if for all α $\{X_{\beta,\alpha} \to X_\alpha\}_\beta$ is a covering family of X_α, then the family of composites $\{X_{\beta,\alpha} \to X\}_{\beta,\alpha}$ is covering family of X.

   The function that assigns to each object $X \in \mathcal{C}$ the collection of covering families $Cov(X)$ will be denoted by J. A *site* is the pair $(\mathcal{C}, J)$.

2. Given a category $\mathcal{C}$ a *presheaf* is a functor $\mathcal{F} : \mathcal{C} \to \mathsf{Set}^{op}$. If instead of just a category we have a site $(\mathcal{C}, J)$, a presheaf is a *sheaf* if for each family of coverings $\{X_\alpha \to X\}_\alpha$ the diagram

$$\mathcal{F}(X) \rightrightarrows \Pi_{\alpha\in A}\mathcal{F}(X_\alpha) \longrightarrow \Pi_{\beta,\alpha\in A}\mathcal{F}(X_\alpha \times_X X_\beta)$$

   is a equalizer. We say that $\mathcal{F}$ is a *separated presheaf* if the first arrow in the diagram is just an injective morphism.

3. A topology in a category $\mathcal{C}$ is called *subcanonical* if all the presentable presheaves are sheaves. The finest such a category is called *canonical*.

Let S be a Noetherian, separated and regular scheme of finite Krull dimension. We consider the category of all smooth schemes of finite type over S, $\mathrm{Sm}\mid_S$, with all the morphisms between schemes (not just smooth). This category is intractable since it is not closed under colimits, thus we consider the category of simplicial presheaves over it, $\mathsf{sSet}^{(\mathrm{Sm}|_S)^{op}} = s\mathrm{PSh}\left((\mathrm{Sm}\mid_S)\right)$. We are going to provide a site structure to this category, the Nisnevich topology, which is a topology between the Zariski topology and the étal topology (for more details [Mil98]).

Definition 1.1.2. Given $f : U \to X$ a morphism between schemes and let $x \in X$ we say that f is *completely decomposed at x* if it exists $u \in U$ with $f(u) = x$ such that the induced map in the residue fields, $k(x) \to k(u)$ is an isomorphism. A *Nisnevich morphism* is an étal map $f : Y \to X$ such that is completely decomposed $\forall x \in X$. A *Nisnevich covering* is a family of morphisms $\{f_i : U_i \to X\}_{i\in I}$ such that I is finite and the morphisms $\coprod_{i\in I} U_i \to X$ are Nisnevich morphisms of finite type. These coverings form a basis for a topology in $\mathrm{Sm}\mid_S$, the *Nisnevich topology.* The category $\mathrm{Sm}\mid_S$ with the Nisnevich topology is the *Nisnevich site* $(\mathrm{Sm}\mid_S)_{Nis}$.

Remark.

1. The Nisnevich topology is clearly coarser than the étal topology and finer than the Zariski one.

2. The Nisnevich topology is subcanonical ([sORV02, Chap. II]). Since it is a subcanonical topology if we consider the sheaves in $(\mathrm{Sm}\mid_S)_{Nis}$, $\mathrm{Sh}_{Nis}\left(\mathrm{Sm}\mid_S\right)$, by the Yoneda lemma we have a full and faithful embedding $Y : \mathrm{Sm}\mid_S \to \mathrm{Sh}_{Nis}\left(\mathrm{Sm}\mid_S\right)\ ,\ X \mapsto Hom_{\mathrm{Sm}|_S}\left(-, X\right)$.

3. If we consider the simplicial sheaves in the Nisnevich site, $s\mathrm{Sh}\left((\mathrm{Sm}\mid_S)\right)$, any simplicial set can be seen as a constant Nisnevich sheaf on $\mathrm{Sm}\mid_S$. Therefore $s\mathrm{Sh}_{Nis}\left(\mathrm{Sm}\mid_S\right)$ and $s\mathrm{PSh}_{Nis}\left(\mathrm{Sm}\mid_S\right)$ contain $\mathrm{Sm}\mid_S$ and sSet.

We can develop the motivic homotopy theory in both categories, for simplicial sheaves and for simplicial presheaves, but resulting homotopy categories turn out to be equivalents. We choose the sheaves case in what follows. We denote our category of motivic spaces as $\mathsf{Spc}(S) = s\mathrm{Sh}_{Nis}\left(\mathrm{Sm}\mid_S\right)$. Further on we are going to stabilize respect to two circles. To this end we consider the pointed case of $\mathsf{Spc}(S)$, $\mathsf{Spc}_\bullet(S)$ and the pointed functor $\mathsf{Spc}(S) \xrightarrow{+} \mathsf{Spc}_\bullet(S)\ ,\ X \mapsto X_+ = X \coprod S$. $\mathsf{Spc}_\bullet(S)$ has a symmetric monoidal structure induced by the smash product levewise in sSet, with unit S_+. In fact, this monoidal structure is closed monoidal and there is an internal hom functor, Hom.

Definition 1.1.3. The *Simplicial circle*, $\mathbb{S}^1_s$, is the coequalizer of

$$\Delta[0] \rightrightarrows \Delta[1]$$

Definition 1.1.4. The *Tate circle* $\mathbb{S}^1_t$ or $\mathbb{G}_m$, is $\mathbb{A}^1 \smallsetminus 0$, pointed by the global section given by the identity.

1.1.2 Unstable motivic homotopy category.

In the next subsections we are going to define model categories and localize them fixing the fibrant objects, which may be Nisnevich sheaves up to homotopy and affine line $\mathbb{A}^1$ may be equivalent to the point. We have defined the category of motivic spaces as $\mathsf{Spc}(S) = s\mathrm{Sh}_{Nis}(\mathrm{Sm}\,|_S)$. We start with the (global) projective model structure in motivic spaces and then we modify it to get the desired model structure. This subsection covers the unstable case unlike the followings.

Definition 1.1.5. A motivic pointed space $X \in \mathsf{Spc}_\bullet(S)$ is *locally fibrant* if $X(\emptyset) = *$, for all $U \in \mathrm{Sm}\,|_S$ $X(U)$ is fibrant, in the usual model structure on simplicial sets, and it is a *Nisnevich sheaf up to homotopy*, i.e, for all pullback square

$$\begin{tikzcd} W \arrow[r] \arrow[d] & V \arrow[d, "p"] \\ U \arrow[r, "i"] & Z \end{tikzcd}$$

where i is an open embedding, p is étal and the restriction $p\,|_{p^{-1}(Z\setminus U)}\colon p^{-1}(Z\setminus U) \to (Z\setminus U)$ induces an isomorphism of reduced schemes (this kind of diagrams are called *upper distinguised square*), the induced square

$$\begin{tikzcd} X(W) \arrow[r] \arrow[d] & X(V) \arrow[d, "p"] \\ X(U) \arrow[r, "i"] & X(Z) \end{tikzcd}$$

is a homotopy pullback square.

Definition 1.1.6. For any $X \in \mathsf{Spc}_\bullet(S)$ we denote by $* \rightarrowtail KX \widetilde{\twoheadrightarrow} X$ the cofibrant resolution in the projective model structure over $\mathsf{Spc}_\bullet(S)$. We say that a morphism $f : X \to Y$ is a *local equivalence* if for all locally fibrant $Z \in \mathsf{Spc}_\bullet(S)$ the induced morphism

$$\mathrm{Hom}_{\mathsf{Spc}_\bullet(S)}(KY, Z) \to \mathrm{Hom}_{\mathsf{Spc}_\bullet(S)}(KX, Z)$$

is a pointwise equivalence.

The *local cofibrations* are the cofibrations in the projective model structure and the *local fibrations* are the morphisms with right lifting property respect to acyclic local cofibrations.

These morphisms give rise to a model structure, the *local model structure.* To get the unstable motivic category we need $\mathbb{A}^1$ to be contractible.

Definition 1.1.7. 1. An $\mathbb{A}^1$-*fibrant* object $X \in \mathsf{Spc}_\bullet(S)$ is a locally fibrant object such that the map by the projection $X\left(\mathbb{A}^1 \times_S -\right) \to X$ is an equivalence pointwise.

2. A $\mathbb{A}^1$-*weak equivalence* is a map $f : X \to Y$ in $\mathsf{Spc}_\bullet(S)$ such that for all $\mathbb{A}^1$-fibrant Z the map

$$\mathrm{Hom}_{\mathsf{Spc}_\bullet(S)}\left(KY, Z\right) \to \mathrm{Hom}_{\mathsf{Spc}_\bullet(S)}\left(KX, Z\right)$$

is a pointwise equivalence.

3. The $\mathbb{A}^1$-*cofibrantions* are the local cofibrations and the $\mathbb{A}^1$-*fibrations* are the morphisms with right lifting property respect to acyclic $\mathbb{A}^1$-cofibrations.

4. These three types of maps are the $\mathbb{A}^1$-structure in $\mathsf{Spc}_\bullet(S)$, they form a model structure with homotopy category the *unstable motivic homotopy category* $\mathcal{H}_\bullet(S)$.

1.1.3 $\mathbb{S}^1_1$-stable homotopy category.

Now we are going to stabilize $\mathsf{Spc}_\bullet(S)$ respect to $\mathbb{S}^1_s$. We start by defining the s-spectra, which are like the classic spectra, whose definition will be remembered later in Definition 1.2.2, but with motivic spaces.

Definition 1.1.8. An *s-spectrum* E is a sequence of pointed motivic spaces $\{E_n\}_{n\geq 0}$ with structure maps

$$\mathbb{S}^1_s \wedge E_n \xrightarrow{\sigma^s_n} E_{n+1}$$

And maps between s-spectra the compatible ones. The resulting category is denoted as $\mathrm{Spt}_s(S)$.

Remark. $\Sigma^\infty_s X$ is the s-spectrum given by $\left(\mathbb{S}^1_s\right)^n \wedge X$ as the n-th term and identities as structure maps. This defines the functor $\Sigma^\infty_s : \mathsf{Spc}_\bullet(S) \to \mathrm{Spt}_s(S)$.

Definition 1.1.9. For $(X, x) \in \mathsf{Spc}_\bullet(S)$, the *sheaf of homotopy groups* is the sheaf

$$\left(U \mapsto \Pi_n(X(U), x_{|U})\right)^\sim$$

This is a Nisnevich sheaf of abelian groups for $n \geq 2$.

The suspension morphisms of $\mathsf{sSet}_\bullet$ provide suspension morphisms in the sheaves of homotopy groups. For $E \in \mathrm{Spt}_s(S)$ and $m > n$ we get

$$\Pi_{n+m}(E_m) \xrightarrow{\Sigma_s^1} \Pi_{n+m+1}\left(\mathbb{S}_s^1 \wedge E_m\right) \xrightarrow{\sigma_n^s} \Pi_{n+m+1}(E_{m+1})$$

Definition 1.1.10. The *sheaf of stable homotopy groups* of $E \in \mathrm{Spt}_s(S)$ are:

$$\Pi_n(E_m) := colim_{m>n}\Pi_{n+m}(E_m).$$

We say that a morphism $f : E \to E'$ in $\mathrm{Spt}_s(S)$ is a *s-stable weak equivalence* if it induces an isomorphism in the sheaves of stable homotopy groups $\hat{f}_n : \Pi_n(E) \xrightarrow{\sim} \Pi_n(E')$.

Definition 1.1.11. We define the *s-stable homotopy category*, $\mathsf{SH}_s(S)$, as the category obtained from $\mathrm{Spt}_s(S)$ by inverting the s-stable weak equivalences.

Remark.

1. $\mathrm{Spt}_s(S)$ has a proper simplicial model structure with homotopy category $\mathsf{SH}_s(S)$.
2. The simplicial suspension functor $\Sigma_s^1 : \mathsf{Spc}_\bullet(S) \to \mathsf{Spc}_\bullet(S)$, given by smashing with the simplicial circle, becomes an equivalence in $\mathsf{SH}_s(S)$.

1.1.4 $\mathbb{A}^1$ contractible and $\mathsf{SH}_s^{\mathbb{A}^1}(S)$.

Now we consider $\mathbb{A}^1$ as a interval object and we localize to make it contractible.

Definition 1.1.12.

1. A $\mathbb{A}^1$-*local* s-spectrum is $E \in \mathrm{Spt}_s(S)$ such that $\forall U \in \mathrm{Sm}\,|_S$ and $n \in \mathbb{Z}$ the map induced by the projection $\mathbb{A}^1 \times X \to X$

$$\mathrm{Hom}_{\mathsf{SH}_s(S)}(\Sigma_s^\infty U_+, \Sigma_s^n E) \to \mathrm{Hom}_{\mathsf{SH}_s(S)}\left(\Sigma_s^\infty (U \times \mathbb{A}^1)_+, \Sigma_s^n E\right)$$

is an isomorphism.

2. A $\mathbb{A}^1$-*stable weak equivalence* is a morphism $f : E \to E'$ in $\mathrm{Spt}_s(S)$ such that for any $\mathbb{A}^1$-local s-spectrum $F \in \mathrm{Spt}_s(S)$ the induced map

$$\mathrm{Hom}_{\mathsf{SH}_s(S)}\left(E', F\right) \to \mathrm{Hom}_{\mathsf{SH}_s(S)}(E, F)$$

is a bijection.

3. The motivic s-stable homotopy category $\mathsf{SH}_s^{\mathbb{A}^1}(S)$ is obtained by inverting the $\mathbb{A}^1$-stable weak equivalences in $\mathrm{Spt}_s(S)$.

Remark. 1. For $X \in \mathrm{Sm}\,|_S$ the projection $\mathbb{A}^1 \times X \to X$ induces an $\mathbb{A}^1$-stable weak equivalence of s-spectra. In particular, $\Sigma_s^\infty\left(\mathbb{A}^1, 0\right)$ is contractible.

2. Σ_s^1 stills an equivalence in $\mathsf{SH}_s^{\mathbb{A}^1}(S)$.

1.1.5 Motivic stable homotopy category $\mathsf{SH}(k)$.

Definition 1.1.13. An (s,t)-*bispectrum* E is a sequence of pointed motivic spaces $\{E_{m,n}\}_{n,m\geq 0}$ with two structure maps

$$\mathbb{S}_s^1 \wedge E_{m,n} \xrightarrow{\sigma_n^s} E_{m+1,n}$$

$$\mathbb{S}_t^1 \wedge E_{m,n} \xrightarrow{\sigma_n^t} E_{m,n+1}$$

which are compatibles in the sense that the diagram

$$\begin{tikzcd}
\mathbb{S}_s^1 \wedge \mathbb{S}_t^1 \wedge E_{m,n} \arrow[rr, "\tau\wedge E_{m,n}"] \arrow[d, "\mathbb{S}_s^1\wedge\sigma_t"] & & \mathbb{S}_t^1 \wedge \mathbb{S}_s^1 \wedge E_{m,n} \arrow[d, "\mathbb{S}_t^1\wedge\sigma_s"] \\
\mathbb{S}_s^1 \wedge E_{m,n+1} \arrow[r, "\sigma_s"] & \wedge E_{m+1,n+1} & \mathbb{S}_s^1 \wedge \mathbb{S}_t^1 \wedge E_{m,n} \arrow[l, "\sigma_t"']
\end{tikzcd}$$

where τ interchanges the order of the circles, commutes. The maps between (s,t)-bispectra are the obvious ones. The resulting category is denoted as $\mathrm{Spt}_{s,t}(S)$, and we let $\Sigma_{s,t}^\infty$ denote the corresponding suspension functor.

Remark. $E_n := E_{*,n}$ is a t-spectrum object in $\mathrm{Spt}_s(S)$.

Example 1.1.1. Eilenberg-MacLane spectrum: Let $L(X)$ for $X \in \mathrm{Sm}\,|_S$ be the functor that associates to each $U \in \mathrm{Sm}\,|_S$ the free abelian group generated by the closed irreducible subsets of $U \times X$ which are finite over U and surjective over a connected component of U. The elements of this group use to be called cycles. $L(X)$ turns out to be a Nisnevich sheaf, and forgetting the abelian group structure it is a motivic pointed space. We can extend the functor $L(X)$ from motivic pointed spaces to Nisnevich sheaves. For example $L(\mathbb{S}_s^0 \wedge \mathbb{S}_t^1) = L(\mathbb{A}^1 - 0, 1)$ is the quotient sheaf of abelian groups $L(\mathbb{A}^1 - 0)/L(\mathrm{Spec}(k))$, considered as a pointed space. Other example is the motivic pointed space $L(\mathbb{S}_s^1 \wedge \mathbb{S}_t^1)$ which is equivalent to $L(\mathbb{P}^1, \infty)$. The exterior product of cycles induces a pairing

$$L(\mathbb{S}_s^p \wedge \mathbb{S}_t^q) \wedge L(\mathbb{S}_s^m \wedge \mathbb{S}_t^n) \to L(\mathbb{S}_s^{p+m} \wedge \mathbb{S}_t^{n+q})$$

In particular, we get

$$\mathbb{S}_s^1 \wedge L(\mathbb{S}_s^m \wedge \mathbb{S}_t^n) \xrightarrow{\Gamma(\mathbb{S}_s^1)\wedge id} L(\mathbb{S}_s^1) \wedge L(\mathbb{S}_s^m \wedge \mathbb{S}_t^n) \to L(\mathbb{S}_s^{m+1}, \mathbb{S}_t^n)$$

$$\mathbb{S}^1_t \wedge L(\mathbb{S}^m_s \wedge \mathbb{S}^n_t) \xrightarrow{\Gamma(\mathbb{S}^1_t)\wedge id} L(\mathbb{S}^1_t) \wedge L(\mathbb{S}^m_s \wedge \mathbb{S}^n_t) \to L(\mathbb{S}^m_s, \mathbb{S}^{n+1}_t)$$

where $\Gamma(f)$ denotes the graph of a morphism $f\colon U \to X$. We define the *Eilenberg-MacLane spectrum* $\mathbb{HZ}$ as the (s,t)-bispectrum with motivic pointed spaces $\mathbb{HZ}_{m,n} := L(\mathbb{S}^m_s \wedge \mathbb{S}^n_t)$ and structural maps the morphisms defined just above.

Definition 1.1.14. For any $E \in \mathrm{Spt}_{s,t}(S)$ we can define the *sheaf of bigraded homotopy groups* $\Pi_{p,q}(E)$ as the sheaf that associates to each $U \in \mathrm{Sm}\,|_S$ the sheaf

$$\left(U \mapsto colim_m \mathrm{Hom}_{\mathsf{SH}^{\mathbb{A}^1}_s(S)}\left(\mathbb{S}^{p-q}_s \wedge \mathbb{S}^{q+m}_t \wedge \Sigma^\infty_s U_+, E_m\right)\right)^\sim$$

for $p < q$, $q + m \geq 0$ integers.

Definition 1.1.15.

1. A morphism $f\colon E \to E'$ in $\mathrm{Spt}_{s,t}(S)$ is a *stable weak equivalence* when $\forall p, q \in \mathbb{Z}$ the induced morphism in the sheaves of stable homotopy groups $\Pi_{p,q}(E) \to \Pi_{p,q}(E')$ are isomorphisms.
2. The motivic stable category $\mathsf{SH}(S)$ is the category obtained by inverting the stable weak equivalences in $\mathrm{Spt}_{s,t}(S)$.

Remark.

1. There is a stable proper simplicial model structure on $\mathrm{Spt}_{s,t}(S)$ such that its homotopy category is $\mathsf{SH}(S)$.
2. Σ^∞_s and Σ^∞_t induce equivalences on $\mathsf{SH}(S)$.
3. For $X \in \mathrm{Sm}\,|_S$, $X \times \mathbb{A}^1 \to X$ induces a stable weak equivalence on $\mathsf{SH}(S)$.
4. There is a natural equivalence of functors $\Sigma^\infty_{s,t} \simeq \Sigma^\infty_s \circ \Sigma^\infty_t$.
5. The maps of $\mathsf{SH}(S)$ can be expressed in terms of $\mathsf{SH}^{\mathbb{A}^1}_s(S)$ ([sORV02, Prop. 2.13, Chap. *III*]).

1.1.6 $\mathbb{P}^1$-spectra.

It is well know that the category $\mathsf{SH}(S)$ is additive and triangulated (Definition 1.2.9). In this subsection we will see that the triangulated structure in $\mathsf{SH}(S)$ has by shift the functor Σ^1_s.

Any map in $\mathrm{Spt}_{s,t}(S)$ $f\colon E \to E'$ has an associated cofibration sequence

$$E \to E' \to Cone(S) \to \Sigma^1_s E$$

where the cone of f is the pushout

$$\begin{tikzcd} E \arrow[r, "E\wedge 0"] \arrow[d, "f"] & E \wedge \Delta[1] \arrow[d] \\ E' \arrow[r] & Cone(f) \end{tikzcd}$$

with $\Delta[1]$ pointed at 1. The morphism $Cone(f) \to \Sigma_s^1 E'$ collapses E' to a point.

Definition 1.1.16. A *Distinguised triangle* in $\mathsf{SH}(S)$ is a sequence isomorphic to the image of a cofibration sequence in $\mathrm{Spt}_{s,t}(S)$. This defines a triangulated structure in $\mathsf{SH}(S)$ where the shift is given by $E[1] = \Sigma_s^1 E$.

This model structure allows us, among other things, to define long exact sequence for cohomology theories or Mayer-Vietoris for upper distinguised squares. We can use it also to prove that the following three different suspensions are equivalents.

Now we will see that they are three different ways to express the suspension in $\mathsf{SH}(S)$. We define the *tate sphere* as $\mathbb{T} = \mathbb{A}^1/\left(\mathbb{S}_t^1\right) = \mathbb{A}^1/\left(\mathbb{A}^1 \smallsetminus 0\right)$. The spectrum of $\mathbb{P}^1$ pointed at ∞ is like the spectrum of both circles, and the smashing with $\mathbb{P}^1$ is like smashing with $\mathbb{S}_t^1 \wedge \mathbb{S}_s^1$. In fact, to work with cohomology theories becomes easier with $\mathbb{P}^1$-spectra that with $\mathbb{S}_t^1 \wedge \mathbb{S}_s^1$-spectra, as it will be our case in K-theory.

Lemma 1 (Lemma 2.21, Chap. *III*, [sORV02])**.** In $\mathsf{SH}(S)$, there are canonical isomorphisms

1. $\Sigma_{s,t}^\infty \left(\mathbb{A}^1/\left(\mathbb{A}^1 \smallsetminus 0\right)\right) \simeq \Sigma_{s,t}^\infty \left(\mathbb{S}_t^1 \wedge \mathbb{S}_s^1\right)$

2. $\Sigma_{s,t}^\infty \left(\left(\mathbb{P}^1, \infty\right)\right) \simeq \Sigma_{s,t}^\infty \left(\mathbb{S}_t^1 \wedge \mathbb{S}_s^1\right)$

Remark.

1. The equivalence $\mathbb{T} \simeq \mathbb{S}_t^1 \wedge \mathbb{S}_s^1$ was already true in the $\mathbb{A}^1$-homotopy theory, it follows from the contractibility of $\mathbb{A}^1$. And the same for the projective line.

2. $\mathbb{P}^1$ is just one of our spheres in motivic homotopy theory. We have that $\mathbb{S}_s^2 \simeq \mathbb{S}_s^1 \wedge \mathbb{S}_s^1$. We define the general motivic sphere, the algebro-geometric sphere, by $\mathbb{S}^{a,b} := \left(\mathbb{S}_s^1\right)^{a-b} \wedge \left(\mathbb{S}_t^1\right)^b$.

1.2 Hermitian K-theory.

This section has the objective of providing the needed background, main ideas and motivation for K-theory, in particular Hermitian K-theory. The K-theory began its development at the end of the fifties by

Grothendieck for the study of intersection theory on algebraic varieties. In our days, it stills a motivation via the links between the motivic K-theory and the Chow groups. This section starts with a review from the basic concepts until the develop of the motivic K-theory and the Hermitian K-theory.

The first occurrence of K-theory was in 1957 by Grothendieck as an invariant for a smooth variety X. To each vector bundle he associated its class and the set of all classes was denoted $K(X)$. It is a quotient of the free abelian group on isomorphic classes of vector bundles by the splitness relation in the short exact sequence of vector bundles over X, i.e, sequences of vector bundles $0 \to V' \to V \to V'' \to 0$ with the relation $[V] = [V'] + [V'']$ imposed for its classes. This turns out to be the universal way to associate invariants to a vector bundle in a compatible way with the exactness. $K(X)$ became the first K-group and now a days it is denoted by $K_0(X)$. The idea behind is the group completion of an algebraic structure, to add inverses in an universal way. At the start of sixties it was extended to the topological setting by Atiyah and Hirzebruch. The topological K-theory turned out to be easier given rise to a first definition of the higher groups as the groups of an extraordinary cohomology theory, i.e, a cohomology theory which verifies all the Eilenberg–Steenrod axioms less the dimension axiom. The first subsection covers the definitions of the first K-group and the higher topological K-theory.

The next step in the K-theory develop was the definition of the higher algebraic K-groups. The algebraic case resulted to be more rigid and they were many attempts before the appearing of the first correct one by Quillen. $K_1(X)$ was defined in the topological setting by the vector bundles over the suspension of the original space ΣX. These vector bundles can be constructed by the clutching construction, this in algebraic K-theory comes from the general linear group of the ring, $\mathbf{GL}(R)$. Matrices coming from elementary matrices give rise to equivalent glueings so we have to quotient by these matrices to obtain $K_1(R)$. Using universal central extensions Milnor defined the correct definition of $K_2(R)$ and he also defined the Milnor K-theory for fields, which proved a direct summand of the true K-theory of a ring. Following the idea that $\mathbf{GL}(R)$ gives how the bundles are attached, Quillen defined the first correct definition of the higher K-groups by applying the Kervaire's plus construction to the classifying space of $\mathbf{GL}(R)$. But $\mathbf{GL}(R)$ only talks about how the vector bundles are constructed no the vector bundles themselves and this definition fails for K_0. Using the idea of short exact sequences, Quillen defined the exact categories, morally categories where the short exact sequences makes sense without the need of being abelian categories. This construction, the Q-construction, turns out to be really powerful bringing about theorems that allow us to make computations. The higher K-groups will be defined in the second subsection. There was an other point of view of the group completion up homotopy by Segal, May et al, the infinite loop space machines, which will be developed deeply in the second chapter.

The third subsection talks about the Hermitian K-theory. Before the Grothendieck group completion Witt studied the symmetric bilinear forms and quadratic forms over a field k. When the field has characteristic different of 2 the theories of both become equivalents, otherwise quadratic forms and bilinear forms are not equivalents and one should consider both theories separately. Witt classified the quadratic forms by isometries: two quadratic forms are Witt-equivalents if they differ one each other by the orthogonal sum of a hyperbolic form, and they are isometric if they are Witt-equivalents and have the same dimension. The Witt-equivalences in the set of equivalence classes of quadratic forms with the orthogonal sum gave rise to the Witt group $W(k)$. Later, the group completion was applied for symmetric bilinear forms leading the Grothendieck-Witt group, now a days the first group of the Hermitian K-theory. The Hermitian K-theory followed a similar develop that the algebraic K-theory. We will review broadly its develop in the third subsection.

The last part is dedicated to the motivic algebraic K-theory. As in past cases, the Hermitian version was developed later by Hornbostel ([Hor04]). We will see that in the motivic case there is a $(2,1)$-periodic $\mathbb{P}^1$-spectrum representing the algebraic K-theory. The Hermitian case is also represented by a $\mathbb{P}^1$-spectrum but with $(8,4)$-periodicity, but it will be covered in the Section 1.3 where we will see the different existing models now a days and their properties.

1.2.1 Early K-theory.

In this subsection we are going to give an idea about the K-theory, as well as different points of view for the theory, and remember some basic facts about stable theory. The K-theory appears in different branch of mathematics and it can be understood under different perspectives: as a topological invariant for topological spaces, when working with topological K-theory, as the group completion of an algebraic structure, i.e, to add inverses in an universal way, or as a way to slice an object and later glue pieces, among others. Group completing an algebraic structure will be our main pointed of view in this text. More specifically, we are going to group complete a good definition of an abelian monoid up to homotopy, more concretely an E_∞-algebra (see subsection 2.1.2) via infinite loop space machines (see subsection 2.1).

The K-theory is given by the K-groups. The first K-group is the Grothendieck group, the group completion of a structure, either respect to the Whitney sum for vector bundles in the topological case, or the direct sum of finitely generated projective modules over a ring for the algebraic K-theory. Later they were defined the second and the third k-groups, K_1 and K_2. The higher K-groups use to be the homotopy groups of a space or the stable homotopy groups of a spectrum and it generalizes the second and third groups. Now we are going to concentrate our attention now in the lower case.

Definition 1.2.1. Let $(M,+) \in \mathsf{AbMon}$ be an abelian monoid. We want to get a commutative group from M in an universal way, i.e, we want $M^{Gp} \in \mathsf{Ab}$ and a morphism $\phi \in \mathrm{Hom}_{\mathsf{AbMon}}\left(M, M^{Gp}\right)$ such that for any other abelian group $B \in \mathsf{Ab}$ and morphism $f \in \mathrm{Hom}_{\mathsf{AbMon}}(M,B)$, it exists an unique morphism $f' \in \mathrm{Hom}_{\mathsf{Ab}}\left(M^{Gp}, B\right)$ such that the following diagram commutes

$$\begin{tikzcd} M \arrow[r, "\phi"] \arrow[dr, "f"'] & M^{Gp} \arrow[d, "f'"] \\ & B \end{tikzcd}$$

We say that $\left(M^{Gp}, \phi\right)$, or in short M^{Gp}, is the *group completion* of the abelian monoid M.

Every abelian monoid M has a group completion. We can construct it by taking the free group generated by M, $F(M)$, and then take quotient by the subgroup generated by the relations $[n+m]-[m]-[n]$ for $[m],[n] \in F(M)$.

Remark. By universality we have that this construction defines a functor from AbMon to Ab and, in fact, there is an adjoint $\mathrm{Hom}_{\mathsf{AbMon}}(M,B) \simeq \mathrm{Hom}_{\mathsf{Ab}}\left(M^{Gp}, B\right)$, where $(-)^{Gp}$ is the left adjoint to the forgetful, $(-)^{Gp} \dashv U$.

The original definition of Grothendieck used the abelian structure in the category of algebraic vector bundles over a smooth variety X, concretely the short exact sequences. One consider the free abelian group generated by isomorphism classes $[V]$ of algebraic vector bundles on X modulo the equivalence relation generated by $[V] = [V'] + [V'']$ for each short exact sequence $0 \to V' \to V \to V'' \to 0$. This point of view is closer to the Quillen's definition of higher K-groups using exact categories.

Example 1.2.1.

1. The most simple example is $\mathbb{N}^{Gp} = \mathbb{Z}$, we add inverses to the natural numbers getting the negatives.

2. Even if the group completion started as a theory for varieties it was easier to define for topological spaces. Let us consider the category of isomorphic classes of complex vector bundles over X, a compact and Hausdorff topological space. This category has a symmetric monoidal structure respect to the Whitney sum $\oplus_W$, i.e, the direct sum of vector spaces fiberwise. We define the first K-group

of X as the group completion of such structure, $K^0(X) = (Vect_{\mathbb{C}}(X), \oplus_W)^{Gp}$. This corresponds to the first group of the topology K-theory of X. In fact, the compactness requirement can be weaker and we can define the topological K-theory for a general paracompact and Hausdorff spaces. As a more concrete example we have that $K^0(pt) = \{\mathbb{C}^n - \mathbb{C}^m\} \simeq \mathbb{Z}$, that is, the vector bundles are just of the form $\mathbb{C}^n$ and its possible differences give K^0. And as an immediate consequence we get $K^0(\mathbb{S}^0) = \{n - m, p - q\} \simeq \mathbb{Z} \oplus \mathbb{Z}$.

3. If we consider the category of finitely projective R-modules for R a ring we get a symmetric monoidal structure respect to the direct sum of modules. We define the first group of algebraic K-theory of R as $K_0(R) = (\mathbf{P}(R), \oplus)^{Gp}$. The projective requirement is to get the same behavior that the vector bundles: If P is projective then it exists another module such the direct sum with it gives rise to a free module. The finitely generated requirement corresponds to the compactness requirement. In the second chapter of the Weibel's K-book, [Wei13], one can find many other examples for the algebraic case as well as for the topological one.

Remark. 1. The two general constructions from the example above give two functors

$$K_0 \colon \text{Ring} \to \mathsf{Ab} \qquad\qquad K^0_{\mathbb{C}} \colon \mathsf{Top}^{op}_{cpct,T_2} \to \mathsf{Ab}$$

2. If we group complete a semiring, we get a ring. If we restrict our attention in the example above to compact spaces and commutative rings, the tensor product of modules and vector bundles gives a second operation compatible with the one which is group completed, given rise to a ring categories. Therefore we get that $\left(K^0(X), \oplus_W, \otimes\right)$ and $(K_0(R), \oplus, \otimes)$ are rings. This multiplicative structure in the hermitian case and its preservation will become our primary interest.

3. For compact pointed spaces one can define the reduced K-theory, as the topological K-theory modulo the trivial bundles: for $(X, x_0) = X \in \mathsf{Top}_{*,cpct}$ we define $\tilde{K^0}(X) = \ker\left(K^0(X) \to K^0(pt) \simeq \mathbb{Z}\right)$. The short exact sequence of a pair of spaces (X, A), $(X \setminus A) \to X \to A$, produces a short exact sequence $\tilde{K^0}(X \setminus A) \to \tilde{K^0}(X) \to \tilde{K^0}(A)$ and it can be extended to the left by suspension to $\cdots \to \tilde{K^0}(\Sigma X) \to \tilde{K^0}(\Sigma A) \to \tilde{K^0}(X \setminus A)$. Therefor, the negative reduced groups are defined as $\tilde{K^0}^{-n}(X) = \tilde{K^0}(\Sigma^n X)$ for $n \geq 0$, and then, for $X \in \mathsf{Top}_{cpct}$, we define $K^{-n}(X) = \tilde{K}^{-n}(X_+)$.

Even if the K-theory started from the point of view of algebraic geometry, its version in the topological case developed by Atiyah and Hirzebruch produced the first definition of higher groups. And the intuition behind it give rise to different attempts for rings, included the first correct definition.

First, as we are going to define the topological K-theory, which is a extraordinary cohomology theory, we need to remember some facts about spectra. The stable homotopy theory is a really useful field since Freudenthal proved that the iterated suspension of the space of maps up to homotopy between two pointed spaces tends to stabilize. This gives rise to a theory which classifies and studies the stable maps between two spaces.

Definition 1.2.2. A *spectrum* is a sequence of pointed spaces with the homotopy types of a CW-complexes $E = \{E_n\}_{n\in\mathbb{Z}}$ with structural maps, or bonding maps, $\Sigma E_n \xrightarrow{\sigma_n} E_{n+1}$ for any $n \in \mathbb{Z}$. The maps between two spectra are the sequences of maps between the level spaces such that they respect the structure, i.e, they commute respect to the suspension Σ. The morphisms between two spectra E and F at the level n are denoted by $[E, F]_n$, i.e, $f \in [E, F]_n$ is $f \colon E_r \to F_{r-n}$. In particular, we define the *stable homotopy groups* are $\pi_n^S(E) = [\mathbb{S}, E]_n$. A spectrum $E = \{E_n\}_{n\in\mathbb{Z}}$ is said to be an Ω-*spectrum* if the adjoint of each bonding map $E_n \to \Omega E_{n+1}$ is a homotopy equivalence. A *connective spectrum* is a spectrum whose homotopy groups in negative degrees are zero. One use to denote the category of connective spectra by $\mathrm{Sp}_{\geq 0}$.

Remark.

1. All the concepts in Definition 1.2.2 exist for motivic theory by replacing Σ by Σ_s or Σ_t and the CW-complexes by motivic spaces we recover the different definitions of motivic spectra in Section 1.1.

2. In Section 2.1 the connective spectra will become important since they correspond to infinite loop spaces, i.e, topological spaces that one can deloop infinite times, and therefore they are a model for group like E_∞-spaces. This will explained with more detail in Section 2.1.

Example 1.2.2. As two basic examples we have the suspension spectrum, $E_n = \Sigma^\infty E$ for E a CW-complex, e.g, the sphere spectrum $\mathbb{S}$, and the Eilenberg-MacLane spectrum $HG = K(G,n)_{n\in\mathbb{N}}$, where $K(G,n)$ is defined up to homotopy type as the space with all the homotopy groups zero, less the nth group which is G. The structural maps implicit in this two spectra are the homotopy equivalences given by $\mathbb{S}^n \wedge \mathbb{S}^1 \simeq \mathbb{S}^{n+1}$ for $n \geq 0$ and the adjoints of the equivalences $K(G,n) \simeq \Omega K(G,n+1)$ for $n \geq 0$, respectively. This is an example of Omega spectrum,the structural maps induce $K(G,n) \simeq \Omega K(G,n+1)$ for all n.

Example 1.2.3. *Bott Periodicity.* At the end of the fifties Bott proved that the iterated deloop of the stable unitary group $\mathbf{U} = \lim_{n\to\infty}\mathbf{U}_n$ produces a 2-periodic spectrum and the iterated deloop of the stable orthogonal group $\mathbf{O} = \lim_{n\to\infty}\mathbf{O}_n$ produces a 8-periodic spectrum. More concretely, for $\mathsf{B}\mathbf{U} =$

$\lim_{n\to\infty}\mathsf{B}\mathbf{U}_n$, where $\mathsf{B}\mathbf{U}_n$ is the classifying space for $\mathbf{U}_n$ vector bundles, we have that $\Omega\mathbf{U} \simeq \mathbb{Z}\times\mathsf{B}\mathbf{U}$ and $\Omega(\mathbb{Z}\times\mathsf{B}\mathbf{U}) \simeq \mathbf{U}$. One can deloop also $\mathbb{Z}\times\mathsf{B}\mathbf{O}$ obtaining an 8-periodic spectrum.

The $\mathbf{O}_n$ (resp. $\mathbf{U}_n$) vector bundles correspond to the real (resp. complex) vector bundles of dimension n. Thus, we have $[X, \mathsf{B}\mathbf{U}_n] = \{$isomorphic classes of complex rank n vector bundles over X$\}$, and the same for $\mathsf{B}\mathbf{O}_n$ for real bundles, where $[,]$ here means homotopy classes of maps. To differentiate, in what follows, the real K-theory of a space X will be denoted by $\mathsf{KO}(X)$ and the complex by $\mathsf{KU}(X)$. Following this notation $K^0(X)$ in example 1.2.1 point 2 is $\mathsf{KU}^0(X)$.

Any Ω-spectrum E give rise to an extraordinary cohomology theory and an extraordinary homology theory. The E-cohomology of X is $E^*(X) = [\Sigma^\infty X_+, E]$ and the E-homology is $E_*(X) = \pi_*\left(E\wedge X_+\right)$. The result with the point pt gives the coefficient group for the theory: $E_*(pt) = \pi_*\left(E\right) = [\mathbb{S}, E] = [\Sigma^\infty pt_+, E] = E^*(X)$. In fact, by Brown representability, any cohomology theory arises from a spectrum and any spectrum give rise to a cohomology theory. The spectrum use to give more information because the existence of phantom maps in the cohomology theory (see Remark 2.5 in § 2.5).

Definition 1.2.3. The complex and real topological K-theory are the extraordinary cohomology theories represented by the Ω-spectrum given by $\mathbb{Z}\times\mathsf{B}\mathbf{U}$ in the complex case and $\mathbb{Z}\times\mathsf{B}\mathbf{O}$ in the real one. In particular, for the complex case:

$$\mathsf{KU}^{2n}(X) = [X, \mathbb{Z}\times\mathsf{B}\mathbf{U}], \qquad \mathsf{KU}^{2n-1}(X) = [X, \mathbf{U}].$$

And analogously with 8-periodicity with $\mathbb{Z}\times\mathsf{B}\mathbf{O}$.

Remark.

1. The suspension generates the negative groups $\mathsf{KU}^{-1}(X) = [X, \mathbf{U}] = [X, \Omega\left(\mathbb{Z}\times\mathsf{B}\mathbf{U}\right)] = [\Sigma X, \mathbb{Z}\times\mathsf{B}\mathbf{U}] = \mathsf{KU}^0(\Sigma X)$.

2. The Bott periodicity is given by the *Bott element*. A Bott element for the cohomology theory KU is an element $\beta\in\mathsf{KU}^{-2}(pt)$ such that for any space X and any $n\geq 0$ multiplication by β gives an isomorphism
$$\beta\colon \mathsf{KU}^{-n}(X)\to\mathsf{KU}^{-n-2}(X).$$
We will construct a Bott element for the Hermitian motivic K-theory in Subsection 1.3.4 that will be used later in Section 2.5 for the delooping construction.

After the definition of the topological K-theory they appeared K_1 and K_2 for a ring R. In the topological case K_1 is define on the suspension of the space. The vector bundles on the suspension space are generated by clutching constructions, by glueing two trivial bundles along a strip (see [Wei13] for details). If we think analogously with $P(R)$ the glueing is given by the general linear group of the ring $\mathbf{GL}(R)$, but the glueings given by elementary matrices produce the same glueings, so one has to kill them. The first correct definition of $K_1(R)$ was given by H. Bass:

$$K_1(R) = \mathbf{GL}(R)/[\mathbf{GL}(R), \mathbf{GL}(R)]$$

where $[\mathbf{GL}(R), \mathbf{GL}(R)]$ is the group of elementary matrices $E(R)$. Using universal sequences Milnor defined $K_2(R)$, but it was not possible to go further. In the following subsection we will see the higher groups.

1.2.2 Higher K-theory.

They were many attempts to find a definition of the higher K-groups generalizing $K_0(R)$, $K_1(R)$ and $K_2(R)$, but it was not until the year 1971 when Quillen gave the first correct definition. Motivated by the topological case he defined $K_n(R)$ using $\mathbf{BGL}(R)$. The idea was to define the K-groups as the homotopy groups of $\mathbf{BGL}(R)$. If G is a discrete group we get that$\mathsf{B}\mathbf{G} = K(G, 1)$, where $\mathsf{B}\mathbf{G}$ is the classifying space and $K(G, 1)$ is an Eilenberg–MacLane space. We can consider the general linear group over R as a discrete group, and then we have $\mathbf{BGL} = K(\mathbf{GL}, 1)$ (see [Fri08, § 2.2]). To do it agree with the definition of $K_1(R)$ he changed the π_1 respect to $E(R)$, but it is still failing for $K_0(R)$. With this purpose he used the Kervaire's plus construction ([Ker69]) to change the fundamental group without changing the homology and the cohomology.

Definition 1.2.4. *Plus construction*([Fri08, Thm. 2.9]): Let G be a discrete group and $H < G$ a perfect normal subgroup. Then, it exists a CW-complex $\mathsf{B}\mathbf{G}^+$ and a continuous map $\gamma\colon \mathsf{B}\mathbf{G} \to \mathsf{B}\mathbf{G}^+$ such that

$$\ker(\pi_1(\mathsf{B}\mathbf{G}) \to \pi_1(\mathsf{B}\mathbf{G}^+)) = H$$

and such that $\tilde{H}_*(hmtpfib(\gamma), \mathbb{Z}) = 0$, where$hmtpfib(\gamma)$ denotes the homotopy fiber of γ. Moreover, γ is unique up to homotopy.

Since the subgroup of elementary matrices of $\mathbf{GL}(R)$ is the commutator of $\mathbf{GL}(R)$, it is a perfect subgroup of $\mathbf{GL}(R)$. Therefor, we can apply the plus construction respect to $E(R)$.

Definition 1.2.5. *Higher K-groups of a ring R.* Let $\mathbf{BGL}(R)^+$ be the plus construction of $\mathbf{BGL}(R)$ respect to $E(R)$. Then

$$K_i(R) := \pi_i(\mathbf{BGL}(R)^+) \qquad i > 0$$

Remark. 1. $\mathbf{BGL}(R)^+ \times \mathbb{Z}$ is a Γ-space, a good version of commutative monoid in homotopy theory. We will see an overview of infinite loop space machines in Section 2.1.1 applied to these kind of spaces.

2. This definition did not give any negative K-groups as we had in the topological case.

3. The classifying space of the general group $\mathbf{BGL}(R)$ is connected, so it fails for $i = 0$. Morally, this is because $\mathbf{GL}$ only gives information about how they are glueing the bundles and not about the bundles themselves. To extend it to 0 one use to correct the space $\mathbf{BGL}(R)^+$ taking the cartesian product with $K_0(R)$:
$$K_i(R) \coloneqq \pi_i(\mathbf{BGL}(R)^+ \times K_0(R))$$

The next definition for higher K-groups that we are going to see is the one for exacts categories. The idea behind the definition of the exact categories ([Qui72]) is that they are morally the categories where the concept of exact sequences makes sense. The first definition for K_0 Grothendieck used implicitly the short exact sequences in an abelian category. Any abelian category is an exact category in a natural way but it is still a hard requirement, and not all interesting categories verify the abelian axioms. Quillen also proved that this definition recover the old ones ([Qui74b]), and the requirements are enough weak to be verified by $(Vect_{\mathbb{C}}(X), \otimes_W)$ and $(\mathbf{P}(R), \otimes)$. Moreover, Quillen proved really useful theorems for this construction, e.g, Localization Theorem or Dévissage ([Qui74a]).

Definition 1.2.6. A *exact category* is a pair $(\mathcal{P}, S)$ where $\mathcal{P}$ is a full additive subcategory of an abelian category $\mathcal{P} \subset \mathcal{A}$ and S is a set of sequences of the form
$$0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0 \tag{1.1}$$
,that we call *short exact sequences* of $\mathcal{P}$, such that

1. S is the class of all sequences in $\mathcal{P}$ which are short exact sequences in $\mathcal{A}$.

2. $\mathcal{P}$ is closed under extensions: if we have a exact sequence in $\mathcal{A}$ of the form of 1.1 with A and C in $\mathcal{P}$, then B is isomorphic to an object in $\mathcal{P}$.

We say that a map is an *admissible monomorphism* and we denote it by $\rightarrowtail$ if it is the morphism i for some short exact sequence in $\mathcal{P}$. A map is an *admissible epimorphism* and we denote it by $\twoheadrightarrow$ if it is the morphism j for some short exact sequence.

Example 1.2.4. Examples of exact categories are: the category of left modules over a ring R, $\mathrm{Mod}(R)$, the category of finitely generated projective left modules over a Noetherian ring R, $P(R)$, the category of quasi-coherent sheaves on a variety X, $QCoh(R)$, and the category of coherent sheaves on a Noetherian variety X, $Coh(R)$, among others.

The idea of the Q-construction comes from the definition of Grothendieck for K_0 but it produces a category instead an abelian group. If $\mathcal{C}$ is an abelian category, $Q\mathcal{C}$ still have the same objects but morphisms defined in terms of short exact sequences in $\mathcal{C}$. The K-groups of the exact category are the homotopy groups of $\Omega\mathbf{B}Q\mathcal{C}$, i.e, the loop space of the classifying space of $Q\mathcal{C}$.

Definition 1.2.7. *Q-construction:* Let $\mathcal{P}$ be an exact category. We construct a new category $Q\mathcal{P}$ with same objects but the morphisms in QP between $A, B \in QP$ are the following

$$\mathrm{Hom}_{QP}\left(A \twoheadleftarrow C \rightarrowtail B/\sim\right)$$

where $\sim$ is an equivalence relation between short exact sequences such that two short exact sequences $A \twoheadleftarrow C \rightarrowtail B$ and $A' \twoheadleftarrow C' \rightarrowtail B'$ are equivalents if it exists a commutative diagram

$$\begin{tikzcd}
A \arrow[d, "\wr"] & C \arrow[l, two heads] \arrow[r, tail] \arrow[d] & B \arrow[d, "\wr"] \\
A' & C' \arrow[l, two heads] \arrow[r, tail] & C'
\end{tikzcd}$$

Definition 1.2.8. *Higher K-groups for an exact category.* Let $\mathcal{P}$ be an exact category, $Q\mathcal{P}$ its Q-construction and $\mathsf{B}Q\mathcal{P}$ geometric realization of the nerve. Then

$$K_i\left(\mathcal{P}\right) = \pi_i\left(\Omega\mathsf{B}Q\mathcal{P}\right) \quad i \geq 0.$$

This construction gives the same result that the Quillen's plus construction but it has the advantage of being more general and automatically functorial by definition. With this construction Quillen proved some of the most important theorems for computations in K-theory: Additivity ([Qui72, Thm. 2]) [[Qui72] Resolution ([Qui72, Cor. 2]) Dévissage ([Qui72, Thm. 4]) and Localization ([Qui72, Thm. 5]).

Remark. *Waldhausen construction ($\mathcal{S}_\bullet\mathcal{C}$):* There is a third construction due to Waldhausen ([Wal85]) that applies to more general categories that the exact categories, the Waldhausen categories. The Wauldhausen categories are categories with cofibrations and weak equivalences. All the exact categories give rise to a Waldhausen category. Moreover, this construction produces a connective Ω-spectrum instead of a space and almost all the theorems stated for the Q-construction can be also proved for it.

Remark. There is an other important construction due to Quillen, the $S^{-1}S$-construction. It group completes a symmetric monoidal category. The geometric realization of the subcategory of the isomorphisms is a $\mathcal{H}$-commutative monoid (Definition 2.1.2), a commutative monoid up to homotopy, and this construction group complete it (see [Sri91] for details).

There is a construction for triangulated categories. It does not give always the same information that the Q-construction or the Waldhausen construction, as one can see in [Sch02], but we will use the Balmer's triangulated Witt-groups later so we present a short overview. A triangulated category is an axiomatization of a derived category of an abelian category A, in brief, the category that one obtains inverting the quasi-isomorphisms in the category $\mathsf{Ch}^b(A)$ of bounded chain complexes. From the point of view of the homotopy theory it is the axiomatization of the stable homotopy category. If we have classic stable homotopy category SH the invertible functor given by the reduced suspension Σ defines the shift of a triangulated structure where the n-shift is Σ^n and the distinguished triangles are the cofibration sequence.

Definition 1.2.9. A *triangulated category* is a pair $(\mathcal{T}, [1])$ for an additive category $\mathcal{T}$ and an auto-equivalence $(-)[1]\colon \mathcal{T} \to \mathcal{T}$, the *shift*, together with a class of diagrams $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$, the *distinguished triangles* that satisfy the following conditions.

1. The distinguished triangles are closed under isomorphisms.
2. Rotation: If (f, g, h) is a distinguished triangle then $(g, h, -f[n])$ is also a distinguished triangle.
3. Given a diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r, "h"] \arrow[d, dashed, "\gamma"] & A[1] \arrow[d, "\delta"] \\ A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r, "h'"] & A'[1] \end{tikzcd}$$

   with distinguished triangles as lines and whose left square is commutative then it exists an arrow $\gamma\colon C \to C'$ such that the whole diagram commutes.

4. Octahedral axiom: Let

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$$

$$B \xrightarrow{f'} D \xrightarrow{g'} E \xrightarrow{h'} B[1]$$

$$A \xrightarrow{f' \circ f} D \xrightarrow{g''} F \xrightarrow{h''} A[1]$$

be distinguished triangles. Then it exists an other distinguished triangle $C \xrightarrow{n} F \xrightarrow{m} E \xrightarrow{i} C[1]$ such that the following diagram commutes

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r, "g"] \arrow[d, "f'"] & C \arrow[r, "h"] \arrow[d, "n"] & A[1] \arrow[d, equal] \\
A \arrow[r, "f'\circ f"] \arrow[d, "f"] & D \arrow[r, "g''"] \arrow[d, equal] & F \arrow[r, "h''"] \arrow[d, "m"] & A[1] \arrow[d, "f[1]"] \\
B \arrow[r, "f'"] \arrow[d, "g"] & D \arrow[r, "g'"] \arrow[d, "g''"] & E \arrow[r, "h'"] \arrow[d, equal] & B[1] \arrow[d, "g[1]"] \\
C \arrow[r, "n"] & F \arrow[r, "m"] & E \arrow[r, "i"] & C[1]
\end{tikzcd}$$

commutes.

Given a morphism $A \xrightarrow{f} B$ the object C in the distinguised triangle is determined up to isomorphism. It is called the *cone of* f and denote by $cone(f)$.

For a triangulated category $\mathcal{T}$ the zero-group $K_0(\mathcal{T})$ is defined as the free abelian group on the isomorphic objects quotient the relation $[B] = [A] + [C]$ for any distinguised triangle $A \to B \to C \to A[1]$. A deeper discussion by Amnon Neeman can be find in the Section V.4 of the handbook of K-theory.

1.2.3 Hermitian K-theory.

Now we are going to provide a basic background in Hermitian K-theory in which they are included some constructions and definitions highly used in other parts of the text. As stated before the Hermitian K-theory did not started with the Grothendieck-Witt but with Witt groups. In the thirties the Witt group $\mathrm{W}(k)$, for a given field k with characteristic different of 2, was defined to classify quadratic forms or non-degenerated symmetric bilinear forms over such a field. A symmetric space over k is a pair (β, V) where $\beta\colon V \times V \to k$ is a non-degenerated symmetric bilinear form, while a quadratic space over k is a pair (ϕ, V) such that $\phi\colon V \to k$ verifies $\phi(a \cdot v) = a^2 \cdot \phi(v)$ for any $a \in k$ and any $v \in V$, and the polarization $\phi(v+w) - \phi(v) - \phi(w)$ is a bilinear form. Any symmetric form β on k give rise to a quadratic form $\phi_\beta(v) = \beta(v,v)$ on k while only a quadratic form produces a symmetric form via $\beta_\phi(v,w) = 1/2\,(\phi(v+w) - \phi(v) - \phi(w))$ provided the characteristic of k is not 2.

Given a quadratic space (ϕ, V) one says that it is isotropic if it exists any $v \in V - 0$ such that $\phi(v) = 0$. We say that a quadratic space (ϕ, V) is anisotropic if it does not exists a non-zero vector in V such that ϕ

vanish. A particular case of isotropic space is the hyperbolic quadratic plane: $\mathsf{H} = (k^2, (\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}))$. Witt proved that any quadratic space (ϕ, V) admits a decomposition respect to the orthogonal sum $\phi = \phi_{an} \perp_i \mathsf{H}$, where ϕ_{an} is an isotropic form and $\perp_i \mathsf{H}$ the orthogonal sum of hyperbolic planes (see §5 [Sch85]). Moreover, such decomposition is unique up to isometry, and the summands ϕ_{iso} and $\perp_i \mathsf{H}$ are unique up to isometry. For two quadratic forms ϕ and ψ we say that they are Witt equivalents if $\phi_{an} = \psi_{an}$. The quotient by the orthogonal sum of hyperbolic forms gives the group $(\mathrm{W}(k), \perp)$, in fact $(\mathrm{W}(k), \perp, \otimes)$ is a ring with the tensor product. Later, it was defined the Grothendieck-Witt group $\mathrm{GW}(k)$ as the group completion of the symmetric monoidal groupoid $(Iso(Sym(k)), \perp)$, where $Iso(Sym(k))$ is the core of the category of symmetric forms over k and the $\perp$ orthogonal sum. The Grothendieck-Witt group is the first group of the Hermitian K-theory, as the Grothendieck group is the first group of the K-theory. If we consider the tensor product we get the semiring category denoted by $(Iso(Sym(k)), \perp, \otimes)$.

Along the rest of the text we will focus on the Grothendieck-Witt groups (GW-groups) without paying much attention to the Witt groups. But this branch also has a develop for triangulated and exact categories as the Hermitian case by Knebusch, Balmer, et al. In the seventies Karoubi developed a construction equivalent to the Quillen's plus construction for the Hermitian K-theory using the orthogonal and symplectic groups for rings with involution (see chapter I.4 in the handbook of K-theory for a short introduction). Karoubi's work also gave the fundamental theorem of Hermitian K-theory that supposes a generalization of the Bott periodicity, proving that there is a 8-periodicity in the hermitian groups. This will become a $(8, 4)$-periodicity in the motivic setting as the two periodicity in the usual K-theory becomes a $(2, 1)$-periodicity for motivic K-theory. We will focus the attention on the constructions for triangulated categories with duality and exact categories with duality, since they are going to be used on sections 2.5.6 and 1.3.4.

The develop of the Hermitian K-theory followed the one of the classic K-theory with the special treatment for rings with involution or categories with duality, for categorical constructions. The involution becomes a duality in the category setting. As for the normal K-theory they were developed Hermitian K-theory constructions for triangulated categories (Walter), exact categories (Giffen and Karoubi) and "Waldhausen" categories (Schlichting, see 1.2.15), where we are always considering categories with duality. Let's remember what does it means.

Definition 1.2.10. A *category with duality* is a 3-tuple $(\mathcal{C}, *, \mu)$ where $\mathcal{C}$ is a category, $*\colon \mathcal{C} \to \mathcal{C}^{op}$ is a functor and $\mu\colon Id_{\mathcal{C}} \dot{\to} **$ is a natural transformation such that for every c in $\mathcal{C}$ the formula

$$(\mu_c)^* \circ \mu_{c^*} = Id_{c^*}$$

is verified. This kind of category with duality use to be called *category with weak duality*. And if μ is a natural isomorphism or the identity, it will be called *category with strong duality* or *category with strict duality*, respectively. When it is not specify the type of duality it is understood that the duality is strong.

The categories with duality suppose a generalization of the rings with involution. The non-degeneration of the symmetric bilinear forms for vector spaces can be expressed as an isomorphism between the vector space and its dual space. The categories with duality allow us to define non-degenerated symmetric bilinear forms in a general context. Therefore, we can do Hermitian K-theory for finitely generated projective modules, vector bundles over topological spaces with a duality, algebraic vector bundles over schemes, or simplicial sheaves of schemes as it is our case.

Example 1.2.5.

1. Let R be a ring with an involution $(\bar{\ })\colon R \to R^{op}$, i.e, a ring isomorphism such that $\bar{\bar{a}} = a$, and $P(R)$ the category of finitely generated projective right modules. For a right R-module $M \in P(R)$ the dual module $M^* = Hom_R(M, R)$ has an structure of right module via the morphism given by the involution. Therefore, there is a functor

$$P(R) \xrightarrow{*} P(R)^{op}$$
$$M \longmapsto Hom_R(M, R)$$

   and a natural isomorphism $\mu_M\colon M \xrightarrow{\sim} M^{**}$ given by $\mu_M(x)(f) = \bar{f(x)}$. Thus, the 3-tuple $(P(R), *, \mu)$ is a strong category with duality. Classical examples of rings with involution are the complex numbers with the usual conjugation or any commutative ring with trivial involution.

2. Let us consider $Vect(X)$ the category of algebraic vector bundles over the scheme X, i.e, the locally free $\mathcal{O}_X$-sheaves of finite rank. The functor $Hom_{\mathcal{O}_X}(\ \ , \mathcal{O}_X)$ defines a dualy for $Vect(X)$, and moreover it generalizes the example above for commutative rings, considering them as affine schemes.

3. The last example still can be generalized for a general line bundle. Let X be again a scheme, $\mathcal{L}$ be a line bundle over X and $Vect(X)$ be the category of locally free coherent of $\mathcal{O}_X$-modules. We define the duality functor $*\colon Vect(X) \to Vect(X)$ by $E^* = Hom_{\mathcal{O}_X}(E, \mathcal{O}_X) \otimes \mathcal{L}$, which is nothing but the usual duality twisted by the line bundle $\mathcal{L}$. One defines the identification $\mu\colon E \simeq E^{**}$ as before, and the particular case $\mathcal{L} = \mathcal{O}_X$ recovers the last example.

Since we are going to work with categories with different dualities it is necessary to define the functors that preserve a duality.

Definition 1.2.11. Let $(\mathcal{C}, *, \mu)$ and $(\mathcal{D}, \vee, \nu)$ be two categories with duality. A *duality preserving functor* is a functor $F\colon \mathcal{C} \to \mathcal{D}$ between the underlying categories together with a natural transformation $\lambda\colon F \circ \mu \to \nu \circ F$ such that for every c in $\mathcal{C}$ the diagram

$$\begin{tikzcd} F(c) \arrow[r, "\nu_{F(c)}"] \arrow[d, "F(\mu_c)"] & \vee \circ \vee^{op} \circ F(c) \arrow[d, "\vee(\lambda_c)"] \\ F \circ * \circ *^{op}(c) \arrow[r, "\lambda_{c^*}"] & \vee \circ F^{op} \circ *^{op}(c) \end{tikzcd}$$

commutes. For other duality preserving functor (G, α) the composition is given by $(G, \alpha) \circ (F, \lambda)$: $(G \circ F, \alpha_F \circ G(\lambda))$. We say that F is an *equivalence of categories with duality* if it exists a duality preserving functor $\left(F^{-1}, \lambda'\right) : (\mathcal{D}, \vee, \nu) \to (\mathcal{C}, *, \mu)$ and natural isomorphisms $\zeta\colon F^{-1} \circ F \xrightarrow{\sim} Id_{\mathcal{C}}$ and $\zeta'\colon F \circ F^{-1} \xrightarrow{\sim} Id_{\mathcal{D}}$ such that $\lambda'_{F(c)} \circ F^{-1}(\lambda_c) = \zeta_c^* \circ \zeta_{c^*}$, and analogously for ζ'.

As mentioned above, the classic definition of the Hermitian K-theory, i.e, the Grothendieck-Witt group for a field, is obtained by Grothendieck group completion of the abelian monoid of isometric classes of non-degenerated symmetric bilinear forms over the field and the orthogonal product. The role of the non-degenerated symmetric bilinear forms is played by the Hermitian objects in the categorical setting.

Definition 1.2.12. Let $(\mathcal{C}, *, \mu)$ be a category with duality. We define the *Hermitian category* $\mathcal{C}_h$ as the category with objects the pairs (M, ϕ) where $M \in \mathcal{C}$ and $\phi\colon M \to M^*$ is an isomorphism such that $\phi = \phi^* \mu$, and morphisms the maps $\alpha\colon (M, \phi) \to (N, \theta)$ where $\alpha\colon M \to N$ is a morphism in $\mathcal{C}$ such that $\alpha^* \theta \alpha = \phi$. The objects in $\mathcal{C}_h$ are called hermitian objects or *symmetric forms.*

Remark. One can also consider the skew-symmetric forms. Changing the sing of a given category with duality $(\mathcal{C}, *, \mu)$ we obtain $(\mathcal{C}, *, -\mu)$, and taking the hermitian category of it we get the skew-symmetric forms.

Taking the first point of the Example 1.2.5 the K-theory obtained from this category with duality is the classic one for rings. More concretely, if we use the Quillen plus construction, $K^h(\mathcal{C}) := \mathcal{B}(i\mathcal{C}_h)^+$ and then we take the homotopy groups for the category of projective finitely generated projective we recover the Hermitian groups for the a ring with involution.

Since in the later section we will use the triangulated Witt-groups dues to Balmer for extending the Schlichting groups to negative groups we are going to make a quick review of it. In the seventies Knebusch defined the Witt group for a scheme X, recovering the original definition for $\mathrm{W}(\mathrm{Spec}(k)) = \mathrm{W}(k)$. But it was not until the nineties that it became a cohomology theory thanks to Balmer's work for triangulated categories with duality.

Definition 1.2.13. An *additive category with duality* is a category with duality $(\mathcal{C}, *, \nu)$ such that $\mathcal{C}$ is an additive category and the functor $*$ is an additive functor, i.e, $(A \otimes B)^* = (A)^* \otimes (B)^*$. A *triangulated category with duality* $(\mathcal{C}, [1], *, \nu)$ is an additive category with duality whose is moreover a triangulated category (Definition 1.2.9) such the following compatibility conditions are verify:

1. The duality is a exact functor, i.e, it verifies $[1] \circ * \simeq * \circ [1]^{-1}$ and for any distinguished triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$ in $\mathcal{C}$ the following triangle is also distinguised

$$C^* \xrightarrow{g^*} B^* \xrightarrow{f^*} A^* \xrightarrow{h^*[1]} C^*[1]$$

2. The natural transformation of the duality structure is compatible with the triangulated structure: $\nu_{M[1]} = \nu_M[1]$.

There is a definition of Witt-groups for exact categories with duality dues to Knebusch where the role of the hyperbolic spaces is played by the metabolic spaces. The analogous definition also exists for triangulated categories.

Definition 1.2.14. A hermitian object (M, ϕ) in a triangulated category with duality $(\mathcal{C}, [1], *, \nu)$ is called *metabolic* if it exists a 3 tuple (L, α, β) for $\alpha \colon L \to M$ such that

$$L \xrightarrow{\alpha} M \xrightarrow{\alpha^* \circ \phi} L^* \xrightarrow{\beta} L[1]$$

is a distinguised triangle and such that $\beta \colon L^* \to L[1]$ is symmetric, i.e, $\beta^*[1] = \nu_{L[1]} \circ \beta$. We will denote the classes of metabolic objects by $M(\mathcal{C}, *, \nu)$, which form a subcommutative monoid in the isometric classes of hermitian objects $Iso(\mathcal{C}_h)$. We define the triangulated Witt-group of $(\mathcal{C}, *, \nu)$ as the quotient

$$\mathrm{W}(\mathcal{C}) = Iso(\mathcal{C}_h) / M(\mathcal{C}, *, \nu)$$

.

Remark. This definition recovers the classic Witt-groups but it also extend them and produces higher Witt-groups by shifting the duality. This produces a cohomology theory. Given a triangulated category with duality $(\mathcal{C}, *, \nu)$ the shifted duality is again a triangulated category with duality $(\mathcal{C}, *, \nu)[n]$. The n-th shifted Witt group is defined as $\mathrm{W}^{[n]}(\mathcal{C}) = \mathrm{W}(\mathcal{C}, *, \nu)[n]$.

This construction enabled Balmer to prove important theorems (see [Bal00] and [Bal01]). We will use later this construction to prolong the Schlichting hermitian groups to negative values in subsection 2.5.6.

The definition of the Hermitian K-theory for exact categories with duality is due to Gille and Karoubi. Using this version of the Q-construction they also proved many of the theorems proved by Quillen in [Qui72]. The proof of the localization theorem is due to Schlichting who used a version of the Waldhausen construction for exact categories with duality and weak equivalences. Since in the motivic Hermitian K-theory one of the models used along the text is a fibrant replacement of such construction we will jump directly to that one and related concepts.

Definition 1.2.15. *Schlichting's waldhausen categories: exact categories with weak equivalences and duality.*

1. An *exact category with weak equivalences and duality* is a quadruple $(\mathcal{C}, *, \omega, \mu)$, where $\mathcal{C}$ is a exact category (Def. 1.2.6), $(\mathcal{C}, *, \mu)$ is a category with duality (Def. 1.2.10), and $\omega \in Mor(\mathcal{C})$ is a set of morphisms such that: the identities are contained, it is closed under isomorphisms , retracts, pushouts along inflations, pullback along coflations and composition, and the 2 out of three property for composition. Moreover, we require a compatibility property: the functor $*\colon \mathcal{C} \to \mathcal{C}^{op}$ is a exact functor, i.e, it sends coflations to coflations, weak equivalences to equivalences (in particular $*(\omega) \subset \omega$) and $\mu\colon Id \to **$ is a natural equivalence, i.e, $\mu_X \subset \omega$ for all $X \in \mathcal{C}$.

2. A symmetric form (X, ϕ) in a exact category with weak equivalences and duality is called *non-singular* if ϕ is a weak equivalence. In such a case we say that (X, ϕ) is a *symmetric space.* A *symmetric form* of degree n in $(\mathcal{C}, *, \omega, \mu)$ is a pair (X, ϕ) with $\phi\colon X \to X^*[n]$ a weak equivalence which is symmetric for the shifted duality, $\phi = \phi^*$.

3. Given a exact category with weak equivalences and duality $(\mathcal{C}, *, \omega, \mu)$ we denote the space obtained by the Schlichting construction by $\mathsf{KO}(\mathcal{C}, *, \omega, \mu)$ and when we consider $(\mathcal{C}, *, \omega, \mu)$ shifted by n as

$$\mathsf{KO}^{[n]}(\mathcal{C}, *, \omega, \mu) = \mathsf{KO}\left((\mathcal{C}, *, \omega, \mu)[n]\right)$$

   These are the Hermitian K-theory spaces for the n-th shifted duality and the shifted Hermitian groups are

$$\mathsf{KO}^{[n]}_i(\mathcal{C}, *, \omega, \mu) = \pi_i\left(\mathsf{KO}^{[n]}(\mathcal{C}, *, \omega, \mu)\right).$$

Remark. 1. There is a natural 4-periodicity: $\mathsf{KO}^{[n]}(\mathcal{C}, *, \omega, \mu) \simeq \mathsf{KO}^{[n+4k]}(\mathcal{C}, *, \omega, \mu)$.

2. The Grothendieck-Witt groups are the zero homotopy groups of the Hermitian spaces

$$\mathsf{KO}_0^{[n]}(\mathcal{C}, *, \omega, \mu) = \mathrm{GW}^{[n]}(\mathcal{C}, *, \omega, \mu).$$

3. We recover the Witt group of the homotopy category $\mathsf{Ho}(\mathcal{C})$ for Balmer's triangulated categories with duality when $\mathcal{C}$ is $\mathbb{Z}[1/2]$-linear.

Theorem 1.2.1 (Localization, [Sch12])**.** *Let $(\mathcal{C}, *, \omega, \mu)$ be a exact category with weak equivalences and duality, and $\mathcal{C}'$ be the thick triangulated subcategory which is stable under the duality. Let $\mathcal{C}_1$ be the full exact subcategory with the same objects as $\mathcal{C}'$ and ω_1 be the set of all morphisms in $\mathcal{C}$ whose mapping cones are in $\mathcal{C}_1$. Then, we have the following fibration sequences:*

$$\mathsf{KO}(\mathcal{C}_1, *, \omega, \mu) \to \mathsf{KO}(\mathcal{C}, *, \omega, \mu) \to \mathsf{KO}(\mathcal{C}, *, \omega_1, \mu).$$

1.2.4 Motivic K-theory.

The motivic K-theory is an special case of motivic cohomology theory represented by an spectrum KGL. Motivic cohomology theory plays the role of the singular cohomology theory in algebraic geometry and it is one of the main tools in motivic homotopy theory. In fact, the original motivation to develop the motivic theory for Voevodsky was to prove Milnor's conjecture in K-theory. A motivic cohomology theory will form a bigraded theory, reminding the simplicial and tate circles. The more natural way to think about motivic cohomology in a first glance is as a bigraded cohomology theory, but for facility we will work with a $\mathbb{P}^1$-spectrum representing the theory.

Definition 1.2.16. A $\mathbb{P}^1$-*suspension motivic cohomology theory* is a set of additive presheaves

$$h^n \colon \mathcal{H}_\bullet(k)^{op} \to \mathsf{Ab}$$

for $n \in \mathbb{Z}$, with a $\mathbb{P}^1$-suspension isomorphism

$$\sigma_{\mathbb{P}^1} \colon h^n \to h^{n+1}\Sigma_s \circ \Sigma_t$$

These two kinds of motivic cohomology theories result to be equivalents at the end. A nice introduction can be find in [Mor02]. In motivic homotopy theory the most usual cohomology theories are cohomology theories represented by a $\mathbb{T}$-spectrum or $\mathbb{P}^1$-spectrum, and all the cohomologies in what follows will be of such a type.

Definition 1.2.17. Given a $\mathbb{P}^1$-spectrum E and a pointed motivic space X_+ the *associated bigraded motivic cohomology theory of* E applied to X is

$$\begin{aligned} E^{p,q}(X_+) &\coloneqq \mathrm{Hom}_{\mathsf{SH}(k)}\left(\Sigma^\infty_{s,t}X, \mathbb{S}_s^{p-q} \wedge \mathbb{S}_t^q \wedge E\right) \\ &= \mathrm{Hom}_{\mathsf{SH}(k)}\left(\Sigma^\infty_{\mathbb{P}^1}X, \mathbb{S}_s^{p-2q} \wedge (\mathbb{P}^1)^{\wedge q} \wedge E\right) \end{aligned}$$

Example 1.2.6. The archetypical example is the one given by the Eilenberg-Mclane spectrum $\mathsf{H}\mathbb{Z}$ ([sORV02, Def. 3.5, Chap. 3]). The integral motivic cohomology groups for a $\mathbb{P}^1$-spectrum E are given by $\mathsf{H}\mathbb{Z}^{p,q}(E) \coloneqq \mathrm{Hom}_{\mathsf{SH}(k)}(E, \mathbb{S}_s^{p-q} \wedge \mathbb{S}_t^q \wedge \mathsf{H}\mathbb{Z})$. Other important example is the cobordism spectrum MGL also explained in the same chapter that $\mathsf{H}\mathbb{Z}$ in [sORV02].

Remark. Note that the definition of the motivic cohomology theory depends only of the scheme X and not on any base scheme, since the morphisms in the category $Sm\mid_S$ or $Sm\mid_{\mathrm{Spec}(k)}$ are all the morphisms between, without any restriction to smoothness.

Now, we are going to see the represented cohomology theory of the motivic K-theory, but before we remember the concept of Bott element that will be widely used along this text. A Bott element for a p-periodic cohomology theory E is an element $\beta \in E^p(pt)$ such that it is invertible respect to the multiplicative structure in the cohomology ring of the point pt, i.e, the product with β produces an isomorphism: $(-)\cdot\beta\colon E^*(pt) \simeq E^{*+p}(pt)$.

Let $\mathrm{Gr}\,(r,n)$ be the Grassmannian scheme, scheme parametrizing the r-dimensional subspaces of all the n-dimensional vector space. One can consider it as a motivic spaces, taking direct limit respect to n with the natural inclusions and then one gets the motivic space Gr_r. There is again a natural sequences of monomorphisms in respect of which we can take the sequential colimit and obtain Gr, the infinite dimensional Grassmannian. One can also think about it as the union of all the Grassmannians. Let us consider the motivic space $\mathbb{Z} \times \mathrm{Gr}$, when the base scheme is regular there exists an isomorphism in the pointed motivic unstable homotopy category $\mathcal{H}_\bullet(k)$

$$\mathbb{Z} \times \mathrm{Gr} \to \Omega_{\mathbb{P}^1}(\mathbb{Z} \times \mathrm{Gr})$$

taking the adjoint to a lift of such a map we get the Bott element

$$\mathbb{P}^1 \wedge (\mathbb{Z} \times \mathrm{Gr}) \xrightarrow{\beta} (\mathbb{Z} \times \mathrm{Gr}) \tag{1.2}$$

The Bott element and its existence are explained with detail in [Ron09, § 1.3]. In this subsection we just use it for the bonding maps.

Definition 1.2.18. The algebraic K-theory spectrum is the $(2,1)$-periodic $\mathbb{P}^1$-spectrum

$$\mathsf{KGL} = (\mathbb{Z} \times \mathrm{Gr}, \mathbb{Z} \times \mathrm{Gr}, \dots)$$

with the maps defined in equation 1.2 (just above) as bonding maps. The associated motivic cohomology theory admits many notations, and along of the text we will use any of them. The representation of the algebraic K-theory in the unstable case is given by

$$\mathbf{BGL}^{2i,i}(X) = \mathsf{KGL}^{2i,i}(X) = \mathrm{Hom}_{\mathcal{H}_{\bullet}(k)}(X_+, \mathbb{Z} \times \mathrm{Gr})$$

The algebraic K-theory spectrum represents the K-theory in the stable category:

$$\mathsf{KGL}^{p,q}(X) = \mathrm{Hom}_{\mathsf{SH}(k)}\left(\Sigma^{\infty}_{\mathbb{P}^1} X_+, \mathbb{S}^{p-2q}_s \wedge (\mathbb{P}^1)^{\wedge q} \wedge \mathsf{KGL}\right)$$

Remark.

1. The Bott element mentioned before is a lift from an isomorphism in the unstable motivic homotopy so there is an isomorphism $\mathbb{P}^1 \wedge \mathsf{KGL} \simeq \mathsf{KGL}$ giving rise to a $(2,1)$-periodicity on the theory KGL.

2. If $K_n(X)$ are the algebraic K-groups defined by Quillen for a scheme X then we have $\mathsf{KGL}^{p,q}(X) = K_{2q-p}(X)$.

3. As it was proved in [Ost10] the multiplicative structure in this theory is an E_∞-ring structure. In such a paper there is also an uniqueness result. We will see that it also exists such structure for the Hermitian case in Sections 2.3 and 2.4 , and in 2.5 we will see the definitive spectrum, its E_∞-ring structure and, at some level, its uniqueness.

1.3 Motivic K-theory presheaf of spectra.

In this Section we will start by presenting our Hermitian K-theory presheaf, some of their properties and expose a general view of what we are going to do with this presheaf in the other sections (subsection 1.3.1). In the previous sections we have already seen that it exists a Hermitian K-theory in the motivic case, now we are going to define our Hermitian K-theory presheaf and the structure on it in the affine case and then the pass to general schemes.

We will consider the category of algebraic vector bundles over an affine scheme $X \in Aff\,|_S$. This category has two natural monoidal structures coming from the direct sum and the tensor product over vector bundles.

These monoidal structures are interrelated and we get what is normally called bipermutative structure. We will explain how is it obtained the bipermutative structure in the first subsection 1.3.1. This presheaf of bipermutative categories will give rise to our presheaf of $\mathbb{S}^1$-spectra after applying group completion and taking the associated spectrum in chapter 2.

We will move from affine schemes to general schemes by comparing the homotopy categories of presheaves of spectra with Nisnevich descent property. We will see that there is no loss of information going to general schemes. We remark that in what follows outside of subsection 1.3.2 we will use always $Sm\mid_S$ as long as confusion is not caused. We will see using Nisnevich descent process, in subsection 1.3.2, that such structure can be relocated in $Sm\mid_S$.

Then, we will do a review of the models and some background for the motivic hermitian K-theory in subsection 1.3.3. This will cover the necessary to address the following subsection, 1.3.4. In this last subsection we are going to define a Bott element for the motivic hermitian K-theory. This will be not done in the spectrum constructed in this text but in the Panin-Walter one. It such a model it is easily to define and moreover, given the explicit form of the spaces in the spectrum, we can define the Bott element as in the usual motivic K-theory by replacing the inclusion of the projective line by the inclusion of the quaternionic projective line.

1.3.1 Structures in the hermitian presheaf.

We start by defining the category $Sm\mid_S$. Let S be a Noetherian, separated and regular scheme of finite Krull dimension. Let us denote by $Sm\mid_S$ the smooth separated schemes of finite type over S, with the Nisnevich topology. Here we have required the typical requirements on the category where we are going to take simplicial sheaves with the intention of doing motivic homotopy theory. We will restrict our attention to the subcategory $Aff\mid_S$ of $Sm\mid_S$. For example, to apply the infinite loop space machines from 2.4 to get an E_∞-*ring space* we may assume that we are working with affine schemes because, in other case, our Hermitian K-theory construction and the usual one for exact categories would not agree, since for general schemes the exact sequences do not split. By the subsection 1.3.2 below we will see that this is not a hard requirement since at the homotopy level the presheaves of spectra turn to be equivalents in both cases. So, outside of that section we will always talk about the structures in $Aff\mid_S$ like they would be in $Sm\mid_S$.

What we want to show in this text is that there is the Hermitian K-theory presheaf of spectra taking values in E_∞-rings. Let us consider $Vb(X)$, the category of algebraic vector bundles over X, for $X \in Sm\mid_S$.

Note that when we talk about algebraic vector bundles over a given scheme X we mean the locally free $\mathcal{O}_X$-modules of finite rank over X. This category has two natural monoidal structures: first the direct sum (or Whitney sum, to be precise),which is nothing but the direct sum at the fiber level, and the second the tensor product, also defined fiberwise. With the direct sum and the tensor product of vector bundles over an affine scheme X we have a bisymmetric monoidal category, i.e, this is a bimonoidal category such that the compatible monoidal structures are symmetric. In fact, we get a bisymmetric structure with some of the operation properties defined strictly, i.e, the natural isomorphisms are identities, giving rise to a bipermutative structure (as it is explained later below).

Moreover, this category has a duality given by $\mathrm{Hom}_{\mathcal{O}_X}(-, \mathcal{O}_X)$. So we can consider its Hermitian K-theory, i.e, we will take the duality into consideration. Let us write this structures like a definition since we are going to refer to it along all the text.

Definition 1.3.1. The category of algebraic vector bundles in a scheme $X \in Sm\mid_S$ is a bipermutative category with duality that we denote by

$$(Vb(X), \mathrm{Hom}_{\mathcal{O}_X}(-, \mathcal{O}_X), \nu_X, \oplus, \otimes)$$

where $\oplus$ is the Whitney sum, $\otimes$ the tensor product fiberwise and the duality is given by the duality in (2) in example 1.2.5.

Remark.

1. When we talk about the bipermutative structure we write $(Vb(X), \oplus, \otimes)$, and when we treat this category as a category with duality we use $(Vb(X), \mathrm{Hom}_{\mathcal{O}_X}, (-, \mathcal{O}_X), \nu_X)$ or just $(Vb(X))$. There are other structures in this category, e.g, like the structure of exact category with duality and weak equivalences (Definition 1.2.15). We will use such a structure when talking about the Schlichting construction in Section 2.5. In such a case it will be denoted by $(Vb(X), \omega_X \mathrm{Hom}_{\mathcal{O}_X}(-, \mathcal{O}_X), \nu_X)$, where ω_X denotes the weak equivalences.

2. In the last section of this text we are going to use the category of bounded chain complexes of big vector bundles over X, which will be denoted by $(Ch(Vb(X)), \omega_X, \mathrm{Hom}_{\mathcal{O}_X}(-, \mathcal{O}_X), \nu_X)$. The resulting hermitian K-theories turn out to be equivalents. The use of this one instead of the one defined just above has some advantages like the uniform behavior of the weights for chain complexes, as it is explained in [PW10b]. The resulting hermitian K-theories turn out to be equivalents, as can find again in [PW10b].

In order to give a flavor of the monoidal structures that we are going to use: Morally a monoidal category can be thought of as a "category with a monoid structure" and a symmetric monoidal category can be thought of as a "category with a commutative monoid structure", since they are a monoid and a commutative monoid in the category of categories, respectively. If we have two monoidal structures on the same category such that one is symmetric monoidal and the other is monoidal (not necessarily symmetric) and distributes over the first we get what is normally called bimonoidal or "ring category". Confusion could arise because the term "ring category" since the monoidal structures don't have the divisibility property, but this term refers to the facts that there is an operation such that distributes over the commutative one. In the context of monoidal categories, one must worry about how strong the coherent conditions are. Mainly there are two possibilities: the normal one, where the operation properties are defined by natural isomorphisms subject to certain coherence conditions, and the strict one, where the natural isomorphisms that define the operation properties are identities. When in a symmetric monoidal category we require strictness for the associativity and the unit we get the so-called permutative category. Adding to a permutative category a second symmetric category that distributes over it, again with the coherent laws for the associativity and the unit required to be strict, we get a bipermutative category.

Once we have specify the bipermutative structure over the category of algebraic vector bundles over X, we will define the Hermitian K-theory for this bipermutative category, but taking in consideration the duality. This will be developed in sections 2.4 and 2.3 in two different ways. To define the motivic Hermitian K-theory we will use two different process that will be develop in the second chapter, but the main structure is the following. We consider the category in Definition 1.3.1, get the hermitian category (definition 1.2.12), take the category of isomorphism and group complete it. The resulting category is still a bipermutative category, and therefore we can compute the usual K-theory for a bipermutative category.

In both construction we are going to group complete and obtain the associated spectrum. To differentiate the two constructions we are going to use two notations. Both of the resulting presheaf can be used for the delooping construction in Section 2.5, as soon as we mean either of the two constructions we will employ $\mathbf{KQ}_{\mathbb{S}^1}$, to accentuate that it is a $\mathbb{S}^1$-spectrum. We will use $\mathbf{KH}$ for the one obtained by the classic group completion for bipermutative categories (section 2.4), getting the presheaf

$$\mathbf{KH}\colon\ (Sm\,|_S)^\circ \longrightarrow E_\infty\text{-rings}$$

and $\mathbf{KQ}$ for the one obtained using the hermitian direct sum K-theory for infinity categories with duality (section 2.3)

$$\mathbf{KQ}\colon\ (Sm\,|_S)^\circ \longrightarrow E_\infty\text{-rings}$$

Remark. We may remark that such presheaves are presheaves of E_∞-ring $\mathbb{S}^1$-spectra. To represent the Hermitian K-theory in the motivic setting we need a $\mathbb{P}^1$-spectrum. This definitive E_∞-ring $\mathbb{P}^1$-spectrum , as well as some uniqueness result, will be done in Section 2.5.

We are going to establish now what we mean exactly by compatibility. Let $(\mathcal{C}, \oplus, 0)$ and $(\mathcal{C}, \otimes, 1)$ be two symmetric monoidal structures on a category. Let us also consider that $\oplus$ distributes respect to $\otimes$, i.e, it is a bimonoidal category. Therefore, there are a left and a right distributivity laws, denote by δ and δ' respectively. We will always require δ' to be strict, i.e, an identity, so it remains to specify the left law. We use the definition in [May09a, § 12], where the left distributivity morphism $\delta\colon A \otimes (B \oplus C) \to (A \otimes B) \oplus (A \otimes C)$ is a natural isomorphism. They exit other weaker definitions for the compatibility, e.g, Laplaza's definition in [Lap72] in which δ is just a monomorphism, but to require δ be an isomorphism is enough hard for our purposes and it is fulfilled for our case with the direct sum and the tensor product. Although there are other weaker definitions for the distributivity our requirement over δ is usually the weaker one that may be found in the major part of literature. At the start of the section we have explained the main definitions in monoidal categories, but let us remember what permutative and bipermutative categories are exactly. For consistency, we will use the definition given by May in [May77].

Definition 1.3.2. A *permutative category* is a symmetric monoidal category $(\mathcal{C}, \oplus, 0)$ in which the natural isomorphisms for associativity and unicity are identities , i.e, $A \oplus (B \oplus C) = (A \oplus B) \oplus C$ and $A \oplus 0 = 0 \oplus A = A$. We call *bipermutative category* to a permutative category with a second permutative structure $(\mathcal{C}, \otimes, 1)$ such that the following conditions are satisfied:

1. The identity of $\oplus$, 0, is a strict two-sided zero object for $\otimes$.
2. The right distributive laws, δ', fulfill in a strict way (it is an identity) and the following diagram, where $c_\otimes$ is the commutative isomorphism, commute:

$$\begin{tikzcd} A \otimes (B \oplus C) \arrow[r, "\delta"] \arrow[d, "c_\otimes"] & (A \otimes B) \oplus (A \otimes C) \arrow[d, "c_\otimes \oplus c_\otimes"] \\ (B \oplus C) \otimes A \arrow[r, "\delta'"] & (B \otimes A) \oplus (C \otimes A) \end{tikzcd}$$

3. For the left distributivity law we require to δ just to be an isomorphism and the additional coherence diagram

$$\begin{tikzcd} (A \oplus B) \otimes (C \oplus D) \arrow[r, "\delta"] \arrow[dd, "Id"] & ((A \oplus B) \otimes C) \oplus ((A \oplus B) \otimes D) \arrow[d, "c_\otimes \oplus c_\otimes"] \\ & (A \otimes C) \oplus (B \otimes C) \oplus (A \otimes D) \oplus (B \otimes D) \arrow[d, "Id \oplus c_\oplus \oplus Id"] \\ (A \otimes (C \oplus D)) \oplus (B \otimes (C \oplus D)) \arrow[r, "\delta \oplus \delta"] & (A \otimes C) \oplus (A \otimes D) \oplus (A \otimes D) \oplus (B \otimes D) \end{tikzcd}$$

is required to commute for A, B, C and D in $\mathcal{C}$.

The appropriate monoidal functors between permutative categories are the strict symmetric monoidal functors, i.e, the (lax) symmetric functors such that the natural transformations for the unit and the associativity are require to be identities.

Definition 1.3.3. Let $(\mathcal{C}, \otimes, I_{\otimes})$ and $(\mathcal{C}', \oplus, I_{\oplus})$ be symmetric monoidal categories. A *lax symmetric monoidal functor* $F\colon (\mathcal{C}, \otimes) \to (\mathcal{C}', \oplus)$ is a functor together with a natural transformation $\Psi_{A,B}\colon FA \oplus FB \to F(A \otimes B)$ and a morphism $\Psi\colon I_{\oplus} \to FI_{\otimes}$, such that for every three objects A, B and C of $\mathcal{C}$ the appropriate coherent conditions for the commutativity, associativity and left and right unit are satisfied. Such a functor is called a *strict symmetric monoidal functor* if Ψ and $\Psi_{A,B}$ are identities.

A bisymmetric structure (as our structure coming from the direct sum and the tensor product in which we are interested in) is equivalent to a bipermutative category. Let us remember explicitly this result.

Lemma 2. (Prop. 3.5, § VI [May77]) There is a functor Φ from the category of bisymmetric monoidal structures to the category of bipermutative categories and a natural transformation $\pi\colon \Phi\mathcal{P} \to \mathcal{P}$ of bisymmetric monoidal categories such that if $\mathcal{P}$ is bipermutative, then π is a morphism of bipermutative categories.

In the proof Φ is built explicitly, so when we have a bisymmetric category we can automatically suppose that we have a bipermutative category. Once we have a bipermutative category we can apply the construction of 2.4 and obtain an E_∞-ring structure. This would give rise to the presheaf **KH**. We can also apply the construction in 2.3 and obtained the presheaf **KQ**.

1.3.2 From affine to general schemes.

For the construction of the Hermitian K-theory for the category of the algebraic vector bundles we require to the category to have splitness. In general, in the category of algebraic vector bundles for simplicial sheaves on the Nisnevich site of smooth schemes the short exact sequences do not split. But if we restrict to the case of smooth affine schemes the category of algebraic vector bundles has splitness. We will develop all the constructions in the second chapter assuming splitness. This is possible since we will use Theorem 1.3.1 at first step from affine to general schemes all the time. It will assumed some background in Grothendieck topologies, basic topos theory and local model structures.

To extend properties form affine schemes to general scheme the most usual is to use descent property. We start by remember the *cd*-structures and the Grothendieck topology associated to it, since for our purpose it is easier to work with *cd*-structures. We will present two *cd*-structures such that in the cases of $\mathrm{Sm}\,|_S$ and $\mathrm{Aff}\,|_S$ correspond to the Nisnevich topology. We will define the descent property for our case of presheaves of spectra. The idea is that the Nisnevich presheaves correspond to the local objects (i.e, t_{Nis}-local objects) respect to a localization. In some cases, as it was proved by Voevodsky, the descent property is equivalent to the excision property. This is the case for the Nisnevich topology in our categories. At the end, since we want to work with presheaves of spectra, we are going to prove that the categories of Nisnevich presheaves of spectra in $\mathrm{Sm}\,|_S$ and $\mathrm{Aff}\,|_S$ are equivalents. The last part is a variation of [HW15, Lemma 3.3.2], in which paper this subsection is highly inspired. The main result of the section is Theorem 1.3.1.

To develop our criterion for descent we start by remembering the *cd*-structures defined by Voevodsky. This is a way to represent sites for categories with initial objects. This structure is more simple, and moreover the descent property for sheaves can be proved using Mayer-Vietoris property. In fact, in subsection 1.1.2 we have use them implicitly, since the Nisnevich topology has associated a *cd*-structure given by the upper distinguised squares .

Definition 1.3.4. Let $\mathcal{C}$ be a category with an initial object $\emptyset$. A *cd-structure* χ in $\mathcal{C}$ is a class of commutative squares such that it is closed under isomorphisms. A square in the class χ is called χ-*square*, or *distinguised square* when the *cd*-structure is understood. The Grothendieck topology generated by the *cd*-structure χ on $\mathcal{C}$, that we denote by t_χ, is the coarset topology such that

1. The empty sieve covers the zero object $\emptyset$.

2. For any distinguised square

$$\begin{tikzcd} W \arrow[r] \arrow[d] & V \arrow[d] \\ U \arrow[r] & X \end{tikzcd} \tag{1.3}$$

   the sieve on X generated by $\mathcal{U} \to X$ and $V \to X$ is a t_χ-covering sieve.

Example 1.3.1. 1. The *Nisnevich cd-structure* Nis on $\mathrm{Sm}\,|_S$ consists of the cartesian squares

$$\begin{tikzcd} U \times_X V \arrow[r] \arrow[d] & V \arrow[d, "\pi"] \\ U \arrow[r, "j"] & X \end{tikzcd}$$

   where j is an open immersion, π is an ètal map, and for Z the reduced closed complement of $j(U)$, the restriction of π to Z induces an isomorphism $V \times_X Z \simeq Z$. This is just other way to express the distinguised squares in 1.1.5.

2. The *Affine Nisnevich cd-structure* $AffNis$ on Aff $|_S$ consists of the cartesian squares

$$\begin{tikzcd} \mathrm{Spec}(B_f) \arrow[r] \arrow[d] & \mathrm{Spec}(B) \arrow[d, "\pi"] \\ \mathrm{Spec}(A_f) \arrow[r] & \mathrm{Spec}(A) \end{tikzcd}$$

where $f \in A$ and π is an ètal map and induces an isomorphism $A/f \simeq B/f$.

Lemma 3. The Grothendieck topology t_{Nis} on Sm $|_S$ generated by the Nisnevich *cd*-structure Nis and the Grothendieck topology induced by t_{AffNis} on Aff $|_S$ generated by the Affine Nisnevich *cd*-structure are both the Nisnevich topology.

Proof. The affine case corresponds to [HW15, Prop. 2.3.2], and the general smooth case for Sm $|_S$ is [MV99, § 3, Prop. 1.4]. □

Since we want to study the sheaves of spectra on categories with a topology generated by *cd*-structures first we are going to see the case for simplicial presheaves. Later we will pass to the spectra case by taking spectra directly in the target category of simplicial sets. In fact, by the stable Giraud's theorem we can invert the order and consider directly sheaves of spectra. We will interchange the order sometimes along the section without more remarks.

We have to consider model structures in Aff $|_S$ and Sm $|_S$ such that they are related with the site. To do so, we will consider model structures where the weak equivalences are defined locally by the site.

A simplicial presheaf (resp. presheaf of spectra) $\mathcal{F}$ with a Grothendieck topology t is said to satisfy the descent property if the injective fibrant replacement $\mathcal{F} \to \mathcal{F}'$ turns to be a weak equivalence levelwise, i.e, the induced morphism $\mathcal{F}(U) \to \mathcal{F}'(U)$ are weak equivalences in simplicial sets (resp. is a stable weak equivalence). It has not be said explicitly, but we are using the injective model structure for presheaves in both case. In the target category, for the simplicial sets we are considering the standard model structure and for spectra the Bousfield-Friedlander model structure for simplicial spectra. We are going to stay this in other form more convenient for the proof of the Theorem 1.3.1.

Definition 1.3.5. Let $\mathcal{C}$ be a category with a site t. A simplicial presheaf (resp. presheaf of spectra) $\mathcal{F}$ on $\mathcal{C}$ satisfies *t-descent*, or it is *t-local* if, for any $X \in \mathcal{C}$ and every t-covering sieve U of X the restriction morphism

$$\mathcal{F}(X) \to holim_{Y \in U} \mathcal{F}(Y)$$

is a weak equivalence (resp. is a stable weak equivalence).

Remark. There are many other ways to formulate the descent condition different of the t-covering sieves. One can find some of them in [HI04].

Remark. The idea behind the descent condition is the same that the sheafification for usual sheaves. The sheaves are the local objects respect to the localization given by left adjoint to the geometric morphism in the topos of presheaves. The sheaves correspond to the local objects, the fibrant objects. This corresponds to the t-local objects for our case.

For our cases the collections of covering sieves result to be small, so we can define a Bousfield localization for objectwise model structures on the presheaves of spectra of the category. Let us consider the injective global model structure for presheaves of spectra. To get a t-local object we will apply the fibrant replacement functor, which at the homotopy level is nothing but the left adjoint to the inclusion of the category of presheaves which verify the descent property in the category of all the presheaves of spectra. For a topology t we will denote the fibrant replacement functor of this localized model structure by R_t. A presheaf of sets $\mathcal{F}$ can view as a discrete simplicial presheaf and $R_t\mathcal{F}$ corresponds to the sheafification. For the spectra case we just need to compose with the stabilization.

For the Nisnevich topology, the homotopy category of the t-local simplicial presheaves, the homotopy category of model structure defined by Jardine in [Jar87] and homotopy category of the Joyal's model structure used in [MV99] turn to be equivalents in Sm $|_S$, provided S is a Noetherian scheme with finite Krull dimension ([MV99, Lemma 1.18]).

Definition 1.3.6. Let $\mathcal{C}$ be a category with an initial object $\emptyset$ with a *cd*-structure χ. A simplicial presheaf $\mathcal{F}$ is said to satisfy χ*-excision* if $\mathcal{F}(\emptyset)$ is weakly contractible and for every square Q in χ, $\mathcal{F}(Q)$ is homotopy cartesian. A presheaf of spectra $\mathcal{G}$ is said to satisfy the *stable* χ*-excision property* if $\mathcal{G}(\emptyset)$ is weakly contractible and for every square Q in χ, $\mathcal{G}(Q)$ is homotopy cartesian of spectra with respect to stable equivalence.

The following theorem basically states a criterion such that χ-excision and t_χ-descent are equivalents. This kind of theorems are sometimes called descent theorems. We say that an initial object $\emptyset$ is strict if all the morphisms $X \to \emptyset$ are isomorphisms.

Lemma 4. Let $\mathcal{C}$ be a category with a strictly initial object $\emptyset$ and let χ be a *cd*-structure on $\mathcal{C}$ such that

1. every square in χ is cartesian.
2. They exist the pullbacks along χ, and χ is closed under pullbacks.
3. For any distinguised square in χ of the form (1.3) $U \to X$ is an monomorphism.
4. For any distinguised square in χ of the form (1.3), the square

$$\begin{tikzcd} W \arrow[r] \arrow[d] & V \arrow[d] \\ W \times_U W \arrow[r] & V \times_X V \end{tikzcd}$$

is also in χ.

Then a simplicial presheaf $\mathcal{F}$ satisfies the (stable) χ-excision if and only if it satisfies (stable) t_χ-descent.

Proof. The χ-excision for the stable case follows easily from the unstable case and this is [HW15, Thm. 3.2.5]. It only remains to see the stable case. Ex^∞ denotes the fibrant Kan replacement, Q the Ω-spectrification, the fibrant replacement in the category of spectra and G will means the levelwise replacement in each case. A presheaf of spectra $\mathcal{F}$ verifies the stable χ-excision property if and only if $QEx^\infty\mathcal{F}^n$ verifies the χ-excision. $QEx^\infty\mathcal{F} \to G\mathcal{F}$ turns to be a weak equivalence for Ω-spectra, and $G\mathcal{F}^n$ is globally fibrant. Then we can work level wise and the stable χ-excision is equivalent to see that $QEx^\infty\mathcal{F}^n(c) \to G\mathcal{F}^n(c)$, is a weak equivalence for pointed spaces. This is equivalent to see that $\mathcal{F}(c) \to G\mathcal{F}(c)$ is an stable weak equivalence, but this is nothing else that the stable t_χ-descent. $\square$

Remark. The Nisnevich topology in Sm $|_S$ and Aff $|_S$, i.e, the *cd*-structures from the example 1.3.1, verify the requirement of the Theorem 4.

Remark. This lemma have many versions. There are other descents theorems for Nisnevich topology in the stable and unstable cases in [Jar00]. One can find other descent theorems in [Voe10] for *cd*-structures which are bound, regular and complete. The case of theorem is for complete and regular *cd*-structures (see [MV99] for the definitions).

Now, employing equivalence between descent and excision described just above we are going to see that the homotopy categories for spectra presheaves with descent property turn to be equivalents for Sm $|_S$ and

Aff $|_S$. Let $i\colon \text{Aff} |_S \hookrightarrow \text{Sm} |_S$ be the inclusion of the affine schemes in Sm $|_S$. This induce a restriction functor between the categories of simplicial presheaves, that we denote by i^*. This functor preserves cofibrations and weak equivalences objectwise so it exits an adjunction $(i^* \dashv i_*)$ of the form

$$\begin{tikzcd} s\mathrm{PSh}\left(\mathrm{Aff}\, |_S\right) \arrow[r, shift left, "i_*"] & s\mathrm{PSh}\left(\mathrm{Sm}\, |_S\right) \arrow[l, shift left, "i^*"] \end{tikzcd}$$

which is a Quillen adjunction. We can consider the right derived functor of the right adjoint Ri_*, which is a homotopy right Kan extension functor, and then we get an adjunction $(i^* \dashv Ri_*)$ between the respective homotopy categories of simplicial presheaves. The homotopy right Kan extension admits an explicit form

$$Ri_*\mathcal{F}(X) = \mathrm{holim}_{Y \in \mathrm{Aff}|_S}(Y)$$

If we consider presheaves of spectra, the argument is the same but we need to see that Ri_* preserves ω-spectra. If it preserves Ω-spectra we get the same adjunction by working levelwise. To see that it preserves Ω-spectra we just need to see that it preserves the loop spaces. The loop space $\Omega_{\mathbb{S}^1} X$ for a pointed simplicial set is given by the pullback

$$\begin{tikzcd} \Omega_{\mathbb{S}^1} X \arrow[r] \arrow[d] & 1 \arrow[d, "*"] \\ X^{\mathbb{S}^1} \arrow[r, "X^{pt}"] & X^1 \end{tikzcd}$$

The adjunction $(i^* \dashv Ri_*)$ is in fact a Quillen equivalence by [HW15, Lemma 3.3.2]. So this pullback is preserved and therefore the Ω-spectra.

Theorem 1.3.1. *The homotopy categories of t_{Nis}-local presheaves of spectra in* Sm $|_S$ *and t_{AffNis}-local presheaves of spectra in* Aff $|_S$ *are equivalents.*

Proof. The proof is the same that in [HW15, Lemma 3.3.2] but using the stable case of the descent Lemma 4 and the discussion just above. □

1.3.3 Motivic hermitian K-theory.

This section covers the different models that exists, or at least that the author knows, for hermitian K-theory in the motivic setting. We are going to define them and highlight the desired proprieties of each one without going into specifics about the constructions. As we have seen in the subsection 1.2.4 the algebraic K-theory is represented in the unstable and the stable motivic homotopy categories. In the first one by

$\mathbb{Z}\times \mathrm{Gr}$ and by a periodicity theorem ([Voe98, Thm. 6.8]) extends to the stable case and it is represented by KGL. Such a spectrum is the algebraic counterpart of the complex topology K-theory, so it is represented by a $(2,1)$-periodic $\Omega_{\mathbb{P}^1}$-spectrum.

The motivic Hermitian K-theory is the algebraic counterpart of the real topological K-theory and it is represented by a $(8,4)$-periodic $\Omega_{\mathbb{P}^1}$-spectrum. Along this subsection we will work with different kinds of spectra ($(\mathbb{HP}^1)^{\wedge 2}$-spectra, $\mathbb{P}^1$-spectra or $\mathbb{T}^{\wedge 4}$-spectra) representing the hermitian case, and this will make as change the periodicity.

The first appearing was the Hornbostel's one, that we denote by **KO**. In his paper [Hor13] he proved the existence of a spectrum representing the Hermitian K-theory and the Witt groups in the stable and unstable motivic homotopy categories. He starts with the affine case using the definition of the hermitian K-theory for an additive category with duality $(\mathcal{A},*,\nu)$, i.e, $\mathcal{K}^h(\mathcal{C})=\mathsf{B}(i(\mathcal{A}_h)^+)$ for $(\mathcal{A},*,\nu)$, applied to the category of algebraic vector bundles over an affine scheme. And then he used the a descent process for the Nisnevich site to define the representation for general schemes (i.e, in $Sm\mid_S$) in the unstable category. He did not deduce the stable representation as Voevodsky by periodicity but from the study of $\mathcal{K}^h(R[t,t^{-1}])$, and then he deduced the periodicity. He got a $(8,4)$-periodic $\Omega_{\mathbb{P}^1}$-spectrum that we denote by **KO**.

The second one is the model developed by Panin and Walter. They use a different approach and they define three different models. Following the idea that the Grassmannian is the classifying space for vector bundles, considering the direct limit as a motivic pointed space and crossing with integers represents the motivic K-theory in the unstable case, Panin and Walter studied the properties of the quaternionic Grassmannian in [PW10c] and the hermitian K-theory in [PW10b] using these Grassmannians.

Definition 1.3.7. The *quaternionic Grassmannian* $\mathrm{HGr}\,(r,n)$ is the affine scheme parametrizing the $2r$-dimensional subspaces of all the $2n$-dimensional symplectic vector spaces with non-degenerated form. It can be consider as the open subscheme of $\mathrm{Gr}\,(2r,2n)$ parametrizing the subspaces of trivial vector bundle over $\mathcal{O}^{2n}$ of dimension $2r$ on which the standard symplectic form is nonsingular. The *quaternionic projective spaces* are $\mathbb{HP}^{n-1}=\mathrm{HGr}\,(1,n)$.

In [PW10b] they construct a different approach that Hornbostel. They define a $\mathbb{T}$-spectrum, that we denote by **BO**, representing the hermitian K-theory. Such a spectrum is a fibrant replacement of the Schlichting construction applied to the presheaves of algebraic vector bundles in $Sm\mid_S$. They obtain a cohomology theory $\mathbf{BO}^{p,q}(X_+/U_+)$ (for $X \in Sm\,|_S$ and U an open subscheme of X) which is canonically isomorphic to the functor $(X, U) \mapsto \mathsf{KO}^{[q]}_{2q-p}(X, U)$. They also prove that the spectrum representing such a cohomology theory has a structure of commutative monoid in the category $\mathsf{SH}(S)$.

They also construct three different $\mathbb{HP}^1$-spectra representing the theory $\mathbf{BO}$: $\mathbf{BO}_{geom}$ which are alternately the spaces $\mathbb{Z} \times \mathrm{RGr}$ and $\mathbb{Z} \times \mathrm{HGr}$, and $\mathbf{BO}_{fin}$ which are finite unions of finite dimensional real and quaternionic Grassmannians and $\mathbf{BO}_{\mathbb{HP}^1}$.

If $\tilde{\mathbf{KO}}^{[n]}$ is the fibrant replacement in the category of motivic spaces of the presheaves composed with the Waldhausen-like hermitian spaces for the bounded complexes of algebraic vector bundles with shifted dualities we obtain an $\mathbb{T}$-spectrum $\mathbf{BO} = \left(\tilde{\mathbf{KO}}^{[0]}, \tilde{\mathbf{KO}}^{[1]}, \tilde{\mathbf{KO}}^{[2]}, \dots\right)$.

The composition described in [PW10b, § 7], which is the product with the Thom class in the Waldhausen-schlichting construction spaces $\mathsf{KO}^{[n]}$ (definition 1.2.15), produces an isomorphism $\tilde{\mathbf{KO}}^{[n]} = \tilde{\mathbf{KO}}^{[n+1]}(-\wedge\mathbb{T})$ in $\mathcal{H}_\bullet(S)$. Using the Quillen adjoint given by the left functor $(-) \to - \wedge \mathbb{T}$ and right adjoint $F(-) \to F(- \wedge \mathbb{T})$ it follows that $\tilde{\mathbf{KO}}^{[n+1]}(- \wedge \mathbb{T})$ is fibrant while $\tilde{\mathbf{KO}}^{[n]}$ is cofibrant. Then it exists a morphism $\sigma_n^*\colon \tilde{\mathbf{KO}}^{[n]} \to \tilde{\mathbf{KO}}^{[n+1]}(- \wedge \mathbb{T})$ in $\mathsf{Spc}_\bullet(S)$. And then we take the adjoint morphism $\sigma_n\colon \tilde{\mathbf{KO}}^{[n]} \wedge \mathbb{T} \to \tilde{\mathbf{KO}}^{[n+1]}$. We can choose such a morphism so it is periodic.

Definition 1.3.8 (Def. 7.1, [PW10b])**.** The $\mathbb{T}$-spectrum $\mathbf{BO}$ consists of the sequences of pointed motivic spaces $\left(\tilde{\mathbf{KO}}^{[0]}, \tilde{\mathbf{KO}}^{[1]}, \tilde{\mathbf{KO}}^{[2]}, \dots\right)$, together with the structure maps described just before.

The $\mathbb{HP}^1$-spectrum $\mathbf{BO}_{\mathbb{HP}^1}$ is given precisely by the odd spaces of such a spectrum, i.e,

$$\mathbf{BO}_{\mathbb{HP}^1} = \left(\tilde{\mathbf{KO}}^{[0]}, \tilde{\mathbf{KO}}^{[2]}, \tilde{\mathbf{KO}}^{[4]}, \dots\right)$$

1.3.4 Bott element in the hermitian spectrum.

In this section we are going to construct a Bott element in the Hermitian motivic spectrum. This Bott element will be used in the last part of the thesis for the delooping construction. As it was said in the subsection above we work with three different models for the motivic hermitian K-theory spectrum: Our model from Section 2.5, $\mathbf{KQ}_{\mathbb{P}^1}$ (the delooping construction applied to $\mathbf{KQ}_{\mathbb{S}^1}$), the Hornbostel's one $\mathbf{KO}$, and the different models from Panin and Walter that we summarize as $\mathbf{BO}$ and we specify the model by subscripts. We are going to define this Bott element β as in the spectrum $\mathbf{BGL}$ for the usual motivic K-theory , i.e, we use the inclusions from the projective line $\mathbb{P}^1$ and the tautological bundle over it to define the Bott element of the cohomology theory $\mathbf{BGL}$ with periodicity $(2,1)$-periodicity (for more details see [Voe98]). More concretely we have the following inclusions of motivic spaces:

$$\mathbb{P}^1 = \mathrm{Gr}\,(1,2) \hookrightarrow \mathbb{P}^\infty \hookrightarrow \mathrm{Gr} \hookrightarrow \mathbb{Z} \times \mathrm{Gr} \sim \mathbf{BGL}$$

Let us consider the universal vector bundle over $\mathrm{Gr}\,(r,n)$. The class of the restriction of such vector bundle to $\mathbb{P}^1$ is $[\mathcal{O}(-1)] - [\mathcal{O}(1)]$, the tautological bundle over $\mathbb{P}^1$, and it is represented by the composition of maps above $b\colon \mathbb{P}^1 \to \mathbb{Z} \times \mathrm{Gr}$. This map is our Bott element for $\mathbf{BGL}$, it is an invertible element $\beta \in K^0\left(\mathbb{P}^1\right) = \mathrm{Hom}_{\mathsf{H}(S)}\left(\left(\mathbb{P}^1, \infty\right), \mathbb{Z} \times \mathrm{Gr}\right) = BGL^{2i,i}\left(\mathbb{P}^1\right)$ under multiplication in the cohomology ring.

Now we want to obtain a Bott element in the same way for the hermitian case. In [Hor04] Hornbostel shows the existence of periodicity in $\mathbf{KO}$ but he doesn't give any Bott element. In the other hand, in [IPa10] Panin and Walter induce periodicity objects in $\mathbf{BO}$ from the Schlichting's construction for exact categories with duality developed in [Shc10a], but still not be an explicit map. Here we get a Bott element following the same process that in the algebraic K-theory and we will move it later to our spectrum in Section 2.3.

From the models, $\mathbf{BO}_{\mathbb{HP}^1}$ is the one that we are going to use. In [PW10b] Walter and Panin construct a $\mathbb{T}$-spectrum $\mathbf{BO} = (\tilde{\mathbf{KO}}^{[0]}, \tilde{\mathbf{KO}}^{[1]}, \tilde{\mathbf{KO}}^{[2]}, \dots)$ which has as spaces the fibrant replacements of the presheaves of Schlichting-Waldhausen construction for bounded complexes of vector bundles with shifted dualities from [Shc10b]. The notation for that spaces is changed to avoid confusion from the Hornbostel's spectrum. In the same paper they construct the $\mathbb{HP}^1$-spectrum $\mathbf{BO}_{\mathbb{HP}^1}$ which has as spaces $(\tilde{\mathbf{KO}}^{[0]}, \tilde{\mathbf{KO}}^{[2]}, \tilde{\mathbf{KO}}^{[4]}, \dots)$, where $\tilde{\mathbf{KO}}^{[2i]}$ is $\mathbb{Z} \times \mathrm{HGr}$ if i is odd and $\mathbb{Z} \times \mathrm{RGr}$ if i is even.

Remember from [PW10b] that the bonding maps of the $\mathbb{HP}^1$-spectrum $\mathbf{BO}_{\mathbb{HP}^1}$ are adjoints to the maps

$$\phi_n \;=\; - \times -p_1\left(\mathcal{U}_{\mathbb{HP}^1}, \phi_{\mathbb{HP}^1}\right) : \tilde{\mathbf{KO}}^{[2n]}(-) \to \tilde{\mathbf{KO}}^{[2n+2]}\left(- \wedge \left(\mathbb{HP}^1, x_0\right)\right) \tag{1.4}$$

where $p_1\left(\mathcal{U}_{\mathbb{HP}^1}, \phi_{\mathbb{HP}^1}\right)$ is the first Pontryagin class of the tautological symplectic bundle on $\mathbb{HP}^1$.

Now we are going to work in an analogous way that before with $\mathbb{HP}^1$ instead of $\mathbb{P}^1$. $\mathbb{HP}^1$ is the quaternionic Grassmannian $\mathrm{HGr}\left(1, \mathbb{H}^2\right)$ for trivial symplectic 2 rank bundle $\mathbb{H} = \left(\mathcal{O}^{\otimes 2}, \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\right)$. For a general quaternionic Grassmannian we have the inclusion

$$\mathrm{HGr}\,(r,n) = \mathrm{HGr}\,(r,\mathbb{H}^n) \hookrightarrow \mathrm{Gr}\,(2r,2n) = \mathrm{Gr}\,(2r,\mathbb{H}^n)$$

and in particular $\mathbb{HP}^1 \hookrightarrow \mathrm{Gr}(2,4)$. Let us consider the universal bundle over $\mathrm{Gr}(2r,2n)$ and restrict it over $\mathrm{HGr}(r,n)$ via the inclusions. This give us the tautological bundle over $\mathrm{HGr}(r,n)$ that we are going to denote by $\mathcal{U}_{r,n}$. The standard symplectic form of $\mathbb{H}$ restricts to a symplectic form on $\mathcal{U}_{r,n}$ and it will be denoted by $\phi_{r,n}$. In particular, we have the symplectic bundle $(\mathcal{U}_{\mathbb{HP}^1}, \phi_{\mathbb{HP}^1})$ on $\mathbb{HP}^1$.

The sequence of inclusions

$$\mathbb{HP}^1 \hookrightarrow \mathrm{HGr} \hookrightarrow \mathbb{Z} \times \mathrm{HGr} \sim \tilde{\mathbf{KO}}^{[2]}$$

represents by definition the symplectic bundle $(\mathcal{U}_{\mathbb{HP}^1}, \phi_{\mathbb{HP}^1})$ in the hermitian K-theory of $\mathbb{HP}^1$, i.e, this is an element in $\mathbf{BO}^{(4,2)}\left(\mathbb{HP}^1\right)$. We will use this composition of maps that we will denote by $\tau\colon \mathbb{HP}^1 \to \tilde{\mathbf{KO}}^{[2]}$ to define our Bott element. To use it later we emphasize the inclusion of $\mathbb{HP}^\infty$ and we denote it by $\iota\colon \mathbb{HP}^1 \to \mathbb{HP}^\infty$.

To prove that τ is a Bott element we need to remember the fact from [PW10c] that $\left(\mathbb{HP}^1, x_0\right)$ is isomorphic to $\mathbb{T}^2$ in the unstable motivic homotopy category.

Since the periodicity is 4, as $\mathbb{P}^1$-spectra, or 2 as $\mathbb{HP}^1$-spectra, if we are considering the model $\mathbf{BO}_{\mathbb{HP}^1}$ for $\mathbf{BO}$, we smash τ with it self, and using the multiplicative structure, we get the composition

$$\mathbb{HP}^1 \wedge \mathbb{HP}^1 \xrightarrow{\tau^{\wedge 2}} \tilde{\mathbf{KO}}^{[2]} \wedge \tilde{\mathbf{KO}}^{[2]} \xrightarrow{m_{2,2}} \tilde{\mathbf{KO}}^{[4]}$$

Let us consider the unit morphism $e\colon \mathbb{1} \to \mathbf{BO}_{\mathbb{HP}^1}$ of the commutative ring $\mathbb{HP}^1$-spectrum structure in $\mathbf{BO}_{\mathbb{HP}^1}$. We will also use e to denote the map from the motivic 0-sphere to the first space of the spectrum $e\colon \mathbb{S}^{0,0} \to \tilde{\mathbf{KO}}^{[0]}$. Smashing twice with $\mathbb{HP}^1$ we get

$$\mathbb{T}^{\wedge 4} \cong \mathbb{HP}^1 \wedge \mathbb{HP}^1 \to \tilde{\mathbf{KO}}^{[0]} \wedge \left(\mathbb{HP}^1\right)^{\wedge 2} \to \tilde{\mathbf{KO}}^{[4]}$$

Following the intuitive belief of the usual Bott element, it must be given by smashing the unit map from the motivic ring structure in $\mathbf{BO}_{\mathbb{HP}^1}$ with the projective line, in our case the quaternionic projective line $\mathbb{HP}^1$. Therefore, to check that our construction with τ agrees with Bott element we must see that they give rise to the same maps. In fact, we have the following diagram

$$\begin{tikzcd}
\tilde{\mathbf{KO}}^{[2]} \wedge \tilde{\mathbf{KO}}^{[2]} \arrow[r, "m_{2,2}"] & \tilde{\mathbf{KO}}^{[4]} \\
\mathbb{HP}^1 \wedge \tilde{\mathbf{KO}}^{[2]} \arrow[u, "\tau"] \arrow[ur, "\phi_1"] & \\
\mathbb{HP}^1 \wedge \mathbb{HP}^1 \arrow[u, "\tau"] \arrow[r, "\sim"] & \mathbb{T}^{\wedge 4} \arrow[uu]
\end{tikzcd}$$

where ϕ_1 is the second bonding map from equation 1.4 of the $\mathbb{HP}^1$-spectrum, the lower map is the natural weak equivalence, and the map on the right is e smashed with $\mathbb{T}^{\wedge 4}$. So we need to see that it τ defines a bonding map.

We have that $-p_1\left(\mathcal{U}_{\mathbb{HP}^1}, \phi_{\mathbb{HP}^1}\right)$ defines the bonding maps of the spectrum $\mathbf{BO}_{\mathbb{HP}^1}$. This corresponds to the map $\mathbb{HP}^1 \to \tilde{\mathbf{KO}}^{[2]}$ defined by the inclusions, i.e, the universal Pontryagin class is exactly the given by the inclusions $\mathbb{HP}^1 \hookrightarrow \mathbb{HP}^\infty \hookrightarrow \mathbb{Z} \times \mathrm{HGr}$. By definition of $\mathcal{U}_{\mathbb{HP}^1}$ it is the restriction of $\mathcal{U}_{\mathbb{HP}^\infty}$ and since it is an inclusion we have that $\iota^*\mathcal{U}_{\mathbb{HP}^\infty} \sim \mathcal{U}_{\mathbb{HP}^1}$. This restriction is represented by $\Sigma\mathbb{1} \colon \mathbb{HP}^1 \xrightarrow{\tau} \mathbb{Z} \times \mathrm{HGr}$. So smashing with τ is the bonding maps $\mathbb{HP}^1 \wedge \tilde{\mathbf{KO}}^{[2]} \xrightarrow{\phi_1} \tilde{\mathbf{KO}}^{[4]}$ and $\tau^{\wedge 2}$ is our desired Bott element.

Such a Bott element is defined in the spectrum $\mathbf{BO}_{\mathbb{HP}^1}$. We have used it because its constructions is similar to the construction of the Bott element for $\mathbf{BGL}$. Since it is one of the spectra modeling the Hermitian K-theory we can move it to our spectrum $\mathbf{KQ}$ that we will construct in Section 2.3, or to $\mathbf{KH}$ constructed in 2.4, and apply our delooping construction in Section 2.5. To fix notation for later, we will denote this map by β in the spectrum $\mathbf{KQ}_{\mathbb{S}^1}$.

Definition 1.3.9. Given the Bott element $\tau^{\wedge 2} \colon \mathbb{HP}^1 \wedge \mathbb{HP}^1 \to \mathbf{BO}$ in the spectrum $\mathbf{BO}$. Our *Bott element* in the spectrum $\mathbf{KQ}_{\mathbb{S}^1}$ is the map $\beta \colon \mathbb{HP}^1 \wedge \mathbb{HP}^1 \to \mathbf{KQ}_{\mathbb{S}^1}$ induced by $\tau^{\wedge 2}$.

Remark. We will also denote by β the linear extension of such a map, since it produce also a Bott element. It is expected that this abuse of notation does not produce confusion.

Chapter 2

E_∞-ring structures.

In this chapter we are going to see the two infinite loop space machines appearing in this text and its application to the presheaf of bipermutative categories defined in 1.3.

The first one, in Section 2.3, is the most new. This part of the thesis appears in a join work with Hadrian Heine and Markus Spitzweck. This approach use infinity categories (here quasicategories) as main tool. To group complete in a hermitian context we define the infinity categories with duality and develop an Hermitian a direct sum K-theory for a symmetric infinity category with duality. This machine produces a $\mathbb{S}^1$-spectrum representing the direct sum K-theory. In case of existence of a second symmetric monoidal structure this machine preserves it given an E_∞-ring spectrum. We will also see how to obtain an infinity category with duality from an usual category with duality. Almost at the end of Section 2.3 it is provided a recognition principle which covers our case for the category $Vb(X)$ from Definition 1.3.1. All said before in this section will be put together in the last part, where it will be constructed our $\mathbb{S}^1$-spectrum **KQ**.

In the second approach to the infinite loop space machines we use May's infinite loop space machine. For this machine we defined the bipermutative categories and we have seen that the direct sum and the tensor product from Definition 1.3.1 produce a bipermutative structure levelwise, since these are the input categories for this machine. This is a more classic approach to infinite loop space machines (ILSM, in short). Applying this machine to our case will produce our second presheaf of spectra **KH**.

The first two section cover the necessary background for these two constructions. There is a first review of infinite loop space machines together with a motivation relating infinite loop space, spectra and cohomology theories, and the respective multiplicative structures. The second one introduce and motivate the operads, more concretely the E_∞-operads and the A_∞ operads from the point of view of loop spaces and infinite loop spaces.

Both spectra **KQ** and **KH** from these two machines are $\mathbb{S}^1$-spectra. We will obtain our a definitive $\mathbb{P}^1$-spectrum in the last section this chapter, Section 2.5, also with an E_∞-ring structure by the delooping construction explained there.

2.1 Operads and infinite loop space machines.

The two main points of this thesis are the representability of the hermitian K-theory in the motivic setting and the study of E_∞-ring structures in such a spectrum. In this short section we are going to motivate and remember some of the main definitions related the E_∞-ring structures.

In the first subsection we talk about the relation between spectra, infinite loop spaces and cohomology theories focusing attention in the infinite loop spaces and its structure for motivating the operads in the next subsection. We will talk broadly about infinite loop space machines. In Section 2.4, we will use the May's one for our bipermutative structure described in 1.3.

In the second subsection we define the operads and give some examples. We will concentrate the attention in the A_∞ and E_∞ operads.

2.1.1 Infinite loop space machines.

The cohomology theory was one of the first theories in algebraic topology, and it stills play a central role. Any generalized cohomology theory $\{h^n\}_{n\in\mathbb{Z}}$ is represented by an spectrum E, i.e, it exists and spectrum $\{E(n)\}_{n\in\mathbb{Z}}$ such that $h^n(X) = [X, E(n)]$ for X a topological space and $[-,-]$ denotes the homotopy classes of pointed maps. The most simple case are the Eilenberg-Mclane spectra (Definition 1.2.6). Any spectrum represents a generalized cohomology theory and viceversa. In contrast to the spaces, the spectra can have negative stable homotopy groups. If we restrict to spectra with negative groups equal to zero , the so-called connective spectra (definition 1.2.2), there is a one to one correspondence between these spectra and the infinite loop spaces, given by taking the zero space of the spectrum to get a infinite loop space and by taking iterated classifying spaces from the infinite loop space.

Definition 2.1.1. The *loop space of a pointed topological space* X is $\Omega X = \mathrm{Hom}_{\mathsf{Top}_*}\left(\mathbb{S}^1, X\right)$ with the compact-open topology. The *delooping* of a space A is, if it exists, a pointed space $\mathsf{B}A$ such that it exists a weak equivalence $A \simeq \Omega\mathsf{B}A$. A *loop space* is a space X, such that it has a delooping, i.e, it exists a space Y together with a weak equivalence $\Omega Y \simeq X$, an *n-iterated loop space* is an space X such that they exist spaces Y_k for $k = 0, \dots, n$ with $Y_0 = X$ and weak equivalences $Y_k \simeq \Omega Y_{k+1}$ for $k < n$, and an *infinite loop space* is an n-iterated loop space for all $n \in \mathbb{N}$.

Remark. The term delooping can give rise to confusion since we are going to used later in the delooping's section (2.5). This definition is for a space or an object and the later one is a construction. We will refer to the second one always as the delooping construction.

Definition 2.1.2. An *$\mathcal{H}$-space* is a magma in the homotopy category of topological spaces $\mathcal{H}$. An *$\mathcal{H}$-semigroup* is a semigroup in $\mathcal{H}$, an *$\mathcal{H}$-monoid* is a monoid in $\mathcal{H}$, an *$\mathcal{H}$-group* is a group in $\mathcal{H}$ and the same for other algebraic structures.

Clearly any infinite loop spaces produces an Ω-spectrum by taking iterated deloopings, and therefore a cohomology and homology theories. So, in some sense, infinite loop spaces and connective spectra have the same information.

A loop space ΩX admits a homotopy associative multiplication map by the composition of loops

$$\Omega X \times \Omega X \to \Omega X.$$

This multiplication is not associative since the scalings turn not to be compatibles, but it is associative up to homotopy and these homotopies are coherent up to higher homotopies, and so on. So this is an $\mathcal{H}$-group structure. An double loop space $\Omega^2 X$ is an abelian $\mathcal{H}$-group.

In the other hand, not all homotopy associative $\mathcal{H}$-spaces have naturally a structure of infinite loop space. Since this is an operation defined for $\mathcal{H}$ so it establish that two maps must be homotopics. One can require more. One can require that the homotopies are part of the data and then they satisfy coherent conditions. At the starts of the seventies this information was codify with the concept of operad. This concept was mainly defined by Boardman, and a classic reference Boardman-Vogt [JV73].

Segal introduced a new construction for the algebraic K-theory as an homotopy analog of the Grothendieck group completion ([Seg74]). He defined the a nice definition of a commutative monoid up homotopy, the (special) Γ-spaces, which can produce an infinite loop space and then define the K-groups as the homotopy groups of the spectrum associated to its group completion. However, Segal's approach only works for split exact sequences, not general exact sequences. For finitely generated projective modules the short exact sequences split, but not for vector bundles. This is why the Quillen construction became more successful. This machinery are the infinite loop space machines (ILSM, in short).

If we consider $A \in \mathsf{Ab}(\mathsf{Top})$ we can compute its classifying space and iterate this construction getting a sequence $A, \mathsf{B}A, \mathsf{B}^2A, \ldots$ of spaces together with the weak equivalences $\mathsf{B}^{n-1}A \simeq \Omega\mathsf{B}^n A$ for $n \geq 2$. This produces a generalized cohomology theory for the spectrum associated to the space $\Omega\mathsf{B}A$. In case we take a group instead an abelian monoid we get that the maps are equivalences for $n \geq 0$, and therefore we get a cohomology associated to A. The problem here is that all the abelian topological spaces turn out to be equivalents to products of Eilenberg-MacLane spaces.

Segal developed the Γ-spaces with the purpose of get a good definition of homotopy coherent abelian monoid in Top. It should be at least a commutative $\mathcal{H}$-monoid, such that if it is group-like it gives rise to an infinite loop space. All the infinite loop spaces must be candidates, and we should not assume the existence of deloopings.

Definition 2.1.3. Let Γ be the opposite category of the skeleton of the category of finite pointed sets and partially defined maps ($\mathcal{F}in_*^{op}$) and let $\rho_i \colon \langle n \rangle \to \langle 1 \rangle$ to be the map that send i to 1 and j to $*$ for $j \neq i$. A Γ-*space* is a functor $\mathcal{F} \colon \Gamma^{op} \to \mathsf{Top}$ satisfying the following conditions

1. $\mathcal{F}(\emptyset) \simeq *$, i.e, $\mathcal{F}(\emptyset)$ is equivalent to the point.
2. The morphisms $\mathcal{F}(n) \xrightarrow{\Pi_i \rho_i} \Pi_i \mathcal{F}(1)$ induced by the morphisms ρ_i are weak equivalences.

A *special* Γ-*space* is a functor $\mathcal{F} \colon \Gamma^{op} \to \mathsf{Top}$ such that for any finite sets S and T the morphism

$$\mathcal{F}(S \bigsqcup T) \to \mathcal{F}(S) \times \mathcal{F}(T)$$

is a homotopy equivalence.

Remark. One can replace the category Top for any category with finite products $\mathcal{C}$, and define a Γ-category as $\mathcal{F} \colon \Gamma^{op} \to \mathcal{C}$ such that it converts coproducts into products. These functors are in one to one correspondence with the abelian monoid objects in the category $\mathcal{C}$. A Γ-space is just a functor which encodes the information of an topological abelian monoid as a functor. This condition will become later the Segal condition for ∞-operads.

Remark. The space $\mathcal{F}(*)$ encodes the structure of a homotopy commutative $\mathcal{H}$-space. The composition $\mathcal{F} \colon \Gamma^{op} \to \mathsf{Top} \to \mathcal{H}$ preserves products. In some sense a Γ-space is an structure encoded in $\mathcal{F}(*)$.

The Segal's ILSM gives a functor from topological symmetric monoidal categories to spectra. We use to apply the K-theory to the category of the isomorphisms of some symmetric monoidal category, e.g, $P(A)$ for a ring A. So, this is a way to obtain the K-theory for some symmetric monoidal categories.

Propositon 1 (Segal)**.** *Let* $(\mathcal{C}, \otimes)$ *be a symmetric monoidal category. Then the nerve of the category of isomorphisms, i.e,* $N(i(\mathcal{C}))$ *is canonically an* Γ*-space.*

Example 2.1.1. Let consider the symmetric monoidal category of complex vector bundles with the direct sum product. If we take the isomorphism classes we get $(i(VB_{\mathbb{C}}(X)), \oplus)$ and applying the nerve we get the Γ-space $\coprod_{n\geq 0} \mathsf{BU}(n)$. One can apply the Segal infinite loop space machine to get the group completion of this space: $\mathbb{Z} \times \mathsf{BU}$.

The infinite loop space machines were defined as group completion constructions. They are defined for spaces which are no so nice as a monoid but almost, the Γ-spaces. We will call Grothendieck group completion the original construction that completes an abelian monoid. Before defining group completion we have to define the group-like spaces. Notice, given a homotopy commutative and homotopy associative $\mathcal{H}$-space the set $\pi_0 X$ is an abelian monoid.

Definition 2.1.4. Let X be an homotopy associative $\mathcal{H}$-space. We say that X is *group-like* if it has an homotopy inverse. To be group-like implies that $\pi_0 X$ is a group, and this will be used as definition later in Definition 2.4.7. For cellular complex the opposite is true. The *group completion* of a homotopy associative, homotopy commutative $\mathcal{H}$-space is an other $\mathcal{H}$-space Y together with a map $X \to Y$ such that the abelian monoid $\pi_0 X$ has $\pi_0 Y$ as Grothendieck group completion. Moreover, we require $H_*(Y;k) \simeq \pi_0(X)^{-1} H_*(X;k)$, where the right side is the localization of $H_*(X;k)$ by the induced map, for any commutative ring k.

Example 2.1.2. Given a topological monoid X we can take the loop space of its classifying space, i.e, the geometric realization, and get an infinite loop space $\Omega \mathsf{B} X$. The natural morphism $X \to \Omega \mathsf{B} X$ is the group completion. This can be done for example for $\mathbb{N}$. We get $\Omega \mathsf{B} \mathbb{N} \simeq \Omega \mathbb{S}^1$ and $\pi_0(\Omega \mathbb{S}^1) = \mathbb{Z}$. Other examples are $\coprod_{n\geq 0} \mathsf{B}\Sigma_n$ (which is the Γ-space coming from the Γ-category of finite sets) and $\coprod_{n\geq 0} \mathsf{BU}(n)$ (see Example 2.1.1)

Propositon 2 (Segal)**.** *If a special* Γ*-space* X *is group-like then* $\mathsf{B}^n A \simeq \Omega \mathsf{B}^{n+1} A$ *for* $n \geq 0$ *and* X *is an infinite loop space.*

One can also ask about a second operation. In the cohomology theories we have the multiplicative structure. Such a multiplicative structure also exists for the spectra, the ring spectra, and for infinite loop space machines, the multiplicative ILSM. In fact, as it was proved in [May82a] the infinite loop space machines respect the multiplicative structure. In the case of ring spectra we have to be careful with the definition. It will be given in the next section using E_∞-operads.

In the early seventies they have appear other infinite loop space machines, but all of them turned to be equivalents ([Tho78]). There is an infinite loop space machine for permutative and bipermutative categories, in case of a second operation, that we will explain and use for our case in Section 2.4. They were many generalizations of the infinite loop space machines. Among other constructions, Mandell and Elmendorf generalized these machines for multicategories and multifunctors. In [Nik13], Gepner, Groth and Nikolaus defined a direct sum K-theory (i.e, for symmetric monoidal structures) from the point of view of the infinity categories. Such a construction differs from others by using universal properties instead of explicit constructions with concrete pairs of operads. An Hermitian version of this machine will developed in Section 2.3.

2.1.2 Operads.

In this subsection we define the operads, following the ideas of the last subsection. The operads were defined at the start of the seventies and developed by Boardman, Vogt and May, among others. The actual definition differs a little of the original May's one given in [May72], which was essentially the definition of an enriched category. An operad is a way to parametrize a collection of operations of arbitrarily many arguments together with a composition. One use to differentiate between coloured operads (multicategories) and plain operads. We will refer to the second one along all this section, in fact, we will mean what is sometimes called permutative operad, since we will consider always the case of an action of symmetric groups.

As it was said in the subsection before, the motivation for the operads was the underlying structure of a loop space up to homotopy. The composition of loops in a loop space ΩX is an associative operation up to homotopy. Any way of composing a collection on n loops is equivalent to any other way to compose the same loops; they are just differently parametrized, and the equivalence turns to be canonical, at least up to homotopy. This gives to the space ΩX an operation which is not just associative up homotopy, but homotopy coherently associative. This information can be encoded in an operad, and their algebras correspond to the objects with such operations. In the special case of loop spaces these are A_∞-algebras.

If we consider the infinite loop spaces, the operation becomes commutative at all coherent levels, and they are an special case of algebras over E_∞-operads.

Morally, an operad is a collection of operations represented by arries. The concept of an operad makes sense for a general symmetric monoidal category; in any symmetric monoidal category we can define operads and algebras (or spaces) over those operads. We will consider mainly the case when the monoidal structure is closed since the internal hom allows to define algebras in a nice way and it is the case that we are going to employ later in 2.4

Definition 2.1.5. Let $(\mathcal{C}, \otimes)$ be a symmetric monoidal category. An *operad* is a sequence of objects of $\mathcal{C}$, $\mathcal{O} = (\mathcal{O}_n)_{n\in\mathbb{N}}$ (which parametrize an n-ary operation), together with a Σ_n-action ρ_n for each $\mathcal{O}_n$ and such that

1. It exists an unit map $\mu\colon \mathbf{1}_{\mathcal{C}} \to \mathcal{O}_n$ (which corresponds to the assignment of an identity map as unary operation)
2. and a composition for the operation defined by the morphisms

$$\mathcal{O}_r \otimes \mathcal{O}_{n_1} \otimes \mathcal{O}_{n_2} \otimes \cdots \otimes \mathcal{O}_{n_r} \to \mathcal{O}_{n_1+\cdots+n_r},$$

for $r \geq 0$ and $n_1, n_2, \cdots, n_r \geq 0$ and the natural equivariance, the unit, and the associative diagrams are verified (see [Fre14] for details). A *morphism between operads* $\phi\colon \mathcal{O} \to \mathcal{O}'$ is a sequences morphism $\phi_n\colon \mathcal{O}_n \to \mathcal{O}'_n$ for $r \in \mathbb{N}$, such that they commute with the Σ_n actions and preserve the unit and the compositions.

Example 2.1.3. The most easy examples are in the category of sets Set. The symmetric groups $(\Sigma_n)_{n\in\mathbb{N}}$ form an Operad in Set, where the action on each Σ is given by the left traslation, and the composition $\mu\colon \Sigma_r \otimes \Sigma_{n_1} \otimes \Sigma_{n_2} \otimes \cdots \otimes \Sigma_{n_r} \to \Sigma_{n_1+\cdots+n_r}$ is given by $(s, t_1, t_2, \ldots t_r) \mapsto s(t_1, t_2, \ldots t_r) = t_1 \oplus t_2 \cdots t_r s_*(n_1, n_2, \ldots n_r)$, where $\oplus$ denotes the direct sum permutation and s_* means s acts on. This is the *permutation operad.* There is an operad $pt(r) = pt$ in the category Set with trivial actions and the identity defining the composition unit and composition product. This is the *one-point operad.*

Example 2.1.4. For any object A in a closed symmetric monoidal category $\mathcal{C}$ we can define an operad End_A, the *endomorphism operad.* It is defined by the objects $\mathsf{End}_A(n) = \mathrm{Hom}_{\mathcal{C}}(A^{\otimes n}, A)$, which inherit a Σ_n-action by functoriality from the natural action on $A^{\otimes n}$. By adjunction, the unit $\mathbf{1}_{\mathcal{C}} \otimes A \simeq A$ gives $\mathbf{1}_{\mathcal{C}} \to \mathrm{Hom}_{\mathcal{C}}(A, A)$. Also by adjunction we can define the composition maps.

The main application of the operads are the algebras. An algebra is an object of the category $\mathcal{C}$ with an action of $\mathcal{O}$. This means, an object with the operation encoded by the operad.

Definition 2.1.6. Let $\mathcal{O}$ be an operad in the symmetric monoidal category $(\mathcal{C}, \otimes)$. An *algebra over* $\mathcal{O}$ or a $\mathcal{O}$*-algebra* is an object $A \in \mathcal{C}$ together with morphisms

$$\mathcal{O}_n \otimes A \otimes \cdots A \xrightarrow{\lambda} \otimes A$$

for $r \geq 0$, and the obvious equivariance, associativity and unit diagrams commuting. We say that $\mathcal{O}$ acts over A. A morphism between two $\mathcal{O}$-algebras A and B is a morphism in $\mathcal{C}$, $f\colon A \to B$, which preserves the $\mathcal{O}$-actions. We denote the category of $\mathcal{O}$-algebras by $\mathcal{O} - Alg$.

Remark. In the context of the symmetric monoidal closed categories the algebras can be expressed in a nicer way. A $\mathcal{O}$-algebra over an object $A \in \mathcal{C}$ is, by the adjunction defined by Hom, equivalent to a morphism of operads

$$\lambda'\colon \mathcal{O}_n \to \mathrm{Hom}_{\mathcal{C}}\left(A^{\otimes n}, A\right)$$

in $\mathcal{C}$ (see Example 2.1.4) for all $n \geq 0$.

Example 2.1.5. The category of algebras respect to the permutation operad defined in 2.1.3 is isomorphic in Set to the category of associative monoids. The category of algebras respect to the one-point operad defined in 2.1.3 is isomorphic in Set to the category of commutative monoids.

The last examples are more concentrated in the case $\mathcal{C} = \mathsf{Set}$, they corresponds to classic algebraic structures. In a general symmetric monoidal category we can define the *associative operad* as the operad the one with associative algebras as algebras, and the *commutative operad* the one with commutative algebras as algebras. We denote them Ass and Comm, respectively. In the category Set they correspond to the permutation and the one-point operads respectively.

In our case we want to work in a topological context. Here they also exist the operads Ass and Comm, but they are not really useful since they give strict structures in the algebras. We will define two weaker concepts of operads, the A_∞-operads for the associativity and the E_∞ operads for the commutativity. The subscript ∞ comes form the idea that the associativity or the commutativity are verify up to all the higher homotopies.

In this setting $\mathcal{O}_n$ are spaces, and we can consider its homotopy proprieties to define them. For example, the connectivity of these spaces is a way to define the homotopy coherence.

Definition 2.1.7. An A_∞*-operad in the category of spaces* is an operad $\mathcal{O} = (\mathcal{O}_n)_{n\in\mathbb{N}}$ such that $\pi_0(\mathcal{O}_n) = \Sigma_n$ for all n, each one of these $n!$ components, with the obvious Σ_n-action, is contractible, and the composition is defined by the composition of the symmetric groups.

For any n there is a space of $n!$ operations corresponding to the n-arries. There exit $n!$ connected components but each one of them is contractible. Because of the contractibility, once we choose an order to compose, a connected component, the n-ary turns to be unique up to contractible space. This is gives the desired level of uniqueness for the operations.

Remark. The associative operad Ass in Top is a particular A_∞-operad in Top given by the discrete spaces Σ_n and the natural composition. The A_∞-operads are the homotopy coherent generalization of Ass.

Remark. A space X is a loop space is equivalent to be an A_∞-space such that the monoid $\pi_0(X)$ is a group. So, all the loop spaces are A_∞-spaces and the A_∞-operads encode its composition of loops up to group-like condition. The group completion of an A_∞-space is a loop space. In the other hand the loop spaces are not necessarily an algebra for Ass.

Example 2.1.6. The little n-discs operad D_n was introduced by Boardman, Vogt and May, to model n-loop spaces. The spaces $\mathcal{O}_r$ correspond to the collections $D_n(r)$ of little n-dimensional discs with disjoint interiors in the unitary disc of dimension n. The symmetric groups act on $D_n(r)$ by permutation of the indices of the discs. The composition is given by the embeddings of the discs. One can find a draw of such composition law in [Fre14], to better understand. The case $n = 1$ is an A_∞-operad in spaces.

In other categories different of Top the condition of being contractible must be replaced by a suitable property. But the general definition is the following.

Definition 2.1.8. A A_∞*-operad in a symmetric monoidal category* $\mathcal{C}$ is a free resolution of the standard associative operad Ass in $\mathcal{C}$.

If the A_∞-operads encode homotopy coherently associative operations, the loop spaces are natural A_∞-spaces and they are a homotopy resolution of Ass, the E_∞-operads encode homotopy coherently associative and commutative operations, the infinite loop space are natural E_∞-space and they are a homotopy resolution of Comm.

Definition 2.1.9. An E_∞*-operad in the category of spaces* is an operad $\mathcal{O}$ such that all the spaces $\mathcal{O}_n$ are contractibles for all $n \in \mathbb{N}$, and the Σ_n-actions are free. A E_∞*-operad in a symmetric monoidal category* $\mathcal{C}$ is a free resolution of the standard commutative operad Comm in $\mathcal{C}$.

This second part of the definition can be a little vague. The idea is these operads must give the commutative monoids up higher homotopies, so they are weak equivalents to Comm in the model category of operads. For us, it is just necessary the first definition since it is the one that we are going to use in Section 2.4. To know more one can see the book [Fre14], or [SS96].

Remark. There exist the E_n-operads and any n-loop space is an E_n-space. The E_1-operads are the A_∞-operads and the E_2-spaces are the homotopy commutative A_∞-spaces. And all the possible values for n correspond to the an n-level of commutativity. There exit the A_n-operads corresponding to the different levels of associativity: the A_1-spaces are the pointed spaces, the A_2-spaces are the $\mathcal{H}$-spaces and so on.

Example 2.1.7. Some authors do not require a free action, in such a case the one-point operad is an example of E_∞-operad. The little n-discs operad operad D_n when $n \to \infty$ is other example. The most practice example is the Barratt-Eccles operad E_n. This is an specific realization for E_∞-operads, in Top is given by $\mathsf{E}_n = E\Sigma_n$, the universal principal bundles for the Σ_n, and in Set by the nerve of the action groupoid $\mathsf{E}_n = N\left(\Sigma_n//\Sigma_n\right)$.

We have talk before about the relation between infinite loop spaces and spectra. Based on what has been said, a spectrum inherits always a structure of abelian group up to homotopy. Morally, a ring spectrum E is a spectrum with an unit map $\mathbb{S}^0 \to E$ and a multiplicative map $E \wedge E \to E$ at some homotopy level. Again, to require for a strict structure was to hard and to require just up to homotopy to weak, but we want also to remember the higher coherence. Let us remember that there are many suitable models for spectra, so the next definition will be according to any of these models.

Definition 2.1.10. An *associative ring spectrum* or an *A_∞-ring spectrum* is an A_∞-algebra in any of the suitable categories of spectra. An *commutative ring spectrum* or an *E_∞-ring spectrum* is an E_∞-algebra in any of the suitable categories of spectra.

2.2 Higher category background.

In this section we are going to provide the frames in infinity categories needed for the construction of the Hermitian direct sum K-theory and its preservation of E_∞-ring structures in Section 2.3. We will cover the main tools and indicate the relevant references for the reader who wants a more profound understanding. At the start, subsection 2.2.1, they are motivated and defined the infinity categories and the model that we are going to use (the quasicategories). The second part, subsection 2.2.2, provide the basic tools in homotopy theory in this setting. The third one, subsection 2.2.3, defines the different morphisms in this framework, the fibrations. The next two parts, subsections 2.2.4 and 2.2.5, explain the ∞-operads and the maps between them, which are the main tool in the section below. The last three parts, subsections 2.2.6, 2.2.7 and 2.2.8, cover the a slightly specific background for our concrete case, thus they are defined the presentable infinity categories, the homotopy fixed points and the stable categories, respectively.

2.2.1 ∞-categories.

In this section we are going to go over the higher theory. The goal of the algebraic topology is to study the algebraic invariants over topological spaces. Over time, they have appeared many models for topological spaces that make more simple to work with. One of this models are the simplicial sets. There is an adjoint between topological spaces and simplicial sets given by $| - | \dashv Sing\,(-)$ where $| - |$ associates to each simplicial set its geometric realization and $Sing\,(-)$ which gives the simplicial set consisting of the singular maps, the singular simplicial complex. $Sing\,(X)$, for any X in sSet, verifies the extension property, the Kan condition, and the simplicial sets with such conditions are called Kan complex.

Definition 2.2.1. A *Kan complex* is a simplicial set $K \in \mathsf{sSet}$ such that for any $0 \leq i \leq n$ and any diagram:

$$\begin{tikzcd} \Lambda^n_i \arrow[r, "g"] \arrow[d, hook] & K \\ \Delta^n \arrow[ur, dotted, "f"] & \end{tikzcd}$$

there exists f making the diagram commute. Here $\Lambda^n_i \subset \Delta^n$ is the ith horn of the standard n-simplex. This condition imposed to the Kan complexes is called *Kan condition.*

In the other hand the geometric realization of any simplicial set will be a CW complex. The CW complex are a good model for topological spaces since any topological space can be replaced by a CW complex up to a weak homotopy equivalence. At the homotopy level this adjunction induces a Quillen equivalence between the model structure of topological spaces and the usual one on simplicial sets. Thus the simplicial sets suppose a good model for homotopy theory in spaces.

One of the most useful theories for algebraic topologists appeared in the middle of the last century, which has become a basic tool, was the category theory. They were defined many higher versions of the categories and the groupoids. They generalizes the idea of objects and morphisms in a category adding 2-morphisms, 3-morphisms and so on, requiring the associativity and identity conditions to these higher morphisms and invertibility after some level. This can be done in a strict, but not really useful, way or in a weak way. For higher theory one typically finds the notation (n, k)-category for a higher category where all the r-morphisms are trivial for $r > n$ and they are invertibles if $r > k$. We are going to restrict our attention to $(\infty, 1)$-categories, or ∞-categories in short. There are various models but we are going to work with weak Kan complex or quasicategories which, as it has been proven the last years, provides a convenient model. It was defined and used by Boardman and Voght ([JV73]) and by Joyal ([Joy05]), but Lurie developed in depth the theory and tools in recent years ([Lur14], [Lur07b]).

Definition 2.2.2. An ∞-*category* is a simplicial set K such that for any $0 < i < n$ and any map $f_0 \colon \Lambda^n_i \to K$ admits an extension $f \colon \Delta^n \to K$. The correct morphisms between ∞-categories are the morphisms between the underling simplicial sets, so an ∞-*functor* will be just a morphism in sSet.

The higher category theory provides a generalization of the homotopy theory and the category theory at the same time. Let us remember that it exists a functor $N(-) : \mathsf{Cat} \to \mathsf{sSet}$ from the category of small categories such that to every category $\mathcal{C}$ gives the nerve of such category, a simplicial set where the objects are consider as vertices and the morphisms as edges.

Definition 2.2.3. Given a category $\mathcal{C}$ its *nerve* is $N(\mathcal{C}) \in \mathsf{sSet}$ such that for any $n \geq 0$

$$N(\mathcal{C})_n = Map_{\mathsf{sSet}}(\Delta^n, N(\mathcal{C}))$$

is the set of composable arrows

$$C_0 \xrightarrow{f_0} C_1 \xrightarrow{f_1} C_2 \xrightarrow{f_2} C_3 \xrightarrow{f_3} \cdots \xrightarrow{f_{n-1}} C_n$$

having length n.

The definition of ∞-category includes the nerve of any category and the image of any topological space by $Sing(-)$. It is clear from the definition of Kan complex that it is more restrictive that the definition of ∞-category. As one can see in the following lemma the nerve of a category is a special case of ∞-category.

Lemma 5 (Prop. 1.1.2.2, [Lur07b])**.** Let $K \in \mathsf{sSet}$ be a simplicial set. The following conditions are equivalents.

1. It exists a category $\mathcal{C}$ and an isomorphism $K \simeq N(\mathcal{C})$ in sSet.

2. For any i such that $0 < i < n$ and each diagram

$$\begin{tikzcd} \Lambda_i^n \arrow[r, "g"] \arrow[d, hook] & K \\ \Delta^n \arrow[ur, dotted, "f"] & \end{tikzcd}$$

   there exists an unique f making the diagram commute.

Remark. Generally it is not true that 2 holds for $i \in \{0, n\}$, but it is true when $\mathcal{C}$ is a groupoid. The nerve of a groupoid is a Kan complex.

One of the tools of this definition of ∞-category is that it is a special type of simplicial set inside of sSet. When thinking about an ∞-category like a higher category the vertices corresponds to vertices, the edges to morphisms, the 2-simplex to 2-morphisms, and so on. The homotopy point of view is that the 2-simplex identify one of the morphisms with the composition of the other two morphisms. This identification is the homotopy between a morphism and the composition of the other two. The simplices of higher dimension represent the higher homotopy equivalence. For a general simplicial set there is no general way to compose the morphisms, but the extension property imposed for ∞-categories makes this composition unique up to homotopy. For a more detailed review let see the first chapter of [Lur07b].

Notation. We will denote by Cat_∞ the $(\infty, 2)$-category of the small infinity categories. One can consider the maximal sub quasicategory of Cat_∞, since for many purpose one only cares about invertible ∞-functors. In this case we will use the same notation during this section, but not in the following. Its full subcategory ∞-groupoids will be denoted by $\infty Grpd$. $\widehat{\mathsf{Cat}}_\infty$ will be the category of not necessarily small infinity categories. We can work analogously with Cat for small categories and $\widehat{\mathsf{Cat}}$ for not necessary small categories.

2.2.2 The homotopy category.

In this subsection we are going to define some of the basic tool that we are going to employ later to work with ∞-operads. We are going to denote by Cat_Δ the category of simplicial categories, i.e, the categories enriched in simplicial sets. Clearly these categories are closely related with simplicial sets, there is a simplicial nerve functor $N_\Delta(-)$ and a left adjoint functor, that we will denote by $\mathbb{C}$. The usual nerve can be characterized by the formula $Hom_{\mathsf{sSet}}(\Delta^n, N(\mathcal{C})) = Hom_{\mathsf{Cat}}([n], \mathcal{C})$. We are going to replace $[n]$ by a simplicial category and define the simplicial nerve with an analogous property.

Definition 2.2.4 (Def. 1.1.5.1, [Lur07b]). 1. First we define the functor $\mathbb{C}\colon \Delta \to \mathsf{Cat}_\Delta$. For S a finite nonempty linearly ordered set we define the simplicial category $\mathbb{C}[\Delta^S]$ with objects the elements of S and morphisms defined as follows: If $i, j \in S$, then:

$$Map_{\mathbb{C}[\Delta^S]}(i,j) = \begin{cases} \emptyset, & \text{if } j < i \\ N(P_{i,j}), & \text{if } j \leq i \end{cases}$$

for $P_{i,j}$ the partial ordered set $\{S' \subseteq S\colon\ (i,j \in S') \wedge (\forall k \in S')\,[i \leq k \leq j]\}$. If $i_o \geq i_1 \geq \cdots \geq i_n$ the the composition

$$Map_{\mathbb{C}[\Delta^S]}(i_0,i_1) \times \cdots \times Map_{\mathbb{C}[\Delta^S]}(i_{n-1},i_n) \to Map_{\mathbb{C}[\Delta^S]}(i_0,i_n)$$

is induced by the map of partially ordered sets

$$\begin{array}{ccc} P_{i_0,i_1} \times \cdots \times P_{i_{n-1},i_n} & \to & P_{i_0,i_n} \\ (I_1, I_1, \ldots, I_n) & \mapsto & I_1 \bigcup I_1 \bigcup \cdots \bigcup I_n. \end{array}$$

2. The *Simplicial nerve* $N(\mathbb{C})_\Delta$ of a category $\mathbb{C}$ is defined by

$$Hom_{\mathsf{sSet}}(\Delta^n, N_\Delta(\mathcal{C})) = Hom_{\mathsf{Cat}_\Delta}(\mathbb{C}[\Delta^n], \mathcal{C})$$

Remark. 1. The simplicial nerve and the usual one do not need to agree for general simplicial category, but, if we consider a small category as a simplicial category by taking each of the simplicial sets $Hom(X,Y)$ to be constant, they must agree ([Lur07b, Example 1.1.5.8.]).

2. The functor $\mathbb{C}\colon \Delta \to \mathsf{Cat}_\Delta$ can be extended uniquely to simplicial sets: $\mathbb{C}\colon \mathsf{sSet} \to \mathsf{Cat}_\Delta$, which is the left adjoint of N_Δ. From now on, $\mathbb{C}$ will denote this last functor.

3. If all the map spaces of a simplicial category are Kan complex then the simplicial nerve will be an ∞-category.

4. The adjunction $|-| \dashv Sing(-)$ between simplicial sets and topological spaces can be extended to Cat_Δ and $\mathsf{Cat}_{\mathsf{Top}}$ (topological categories; Top-enriched categories). Let $\mathcal{C}$ be a category in $\mathsf{Cat}_{\mathsf{Top}}$, then the topological nerve $N_{\mathsf{Top}}(\mathcal{C}) := N_\Delta(Sing(\mathcal{C}))$ will be an ∞-category.

The homotopy category of an ∞-category $\mathcal{C}$ is the decategorification of $\mathcal{C}$ by identifying the 1-morphisms which are connected by 2-morphisms. The last definitions and remarks allow us to go from simplicial sets to simplicial categories and also move information to topological categories. The archetypical homotopy category is the well-know homotopy category of topological spaces or CW complex, $\mathcal{H}$. Any homotopy category of a simplicial/topological category is an $\mathcal{H}$-enriched category.

Definition 2.2.5.
1. The *homotopy category of a simplicial set* S is defined as $hS := h\mathbb{C}[S]$. This is an $\mathcal{H}$-enriched category.
2. Given a morphism of simplicial sets $f\colon S \to T$ we will say that it is a *categorical equivalence* if the induced functor $hf\colon hS \to hT$ is an equivalence of $\mathcal{H}$-enriched categories.

Definition 2.2.6. Let $\mathcal{C}$ be an ∞-category and $(h\mathcal{C})' \subseteq h\mathcal{C}$ a subcategory of $h\mathcal{C}$. We will say that $\mathcal{C}'$ is the *subcategory of* $\mathcal{C}$ *spanned by* $(h\mathcal{C})'$ if the following commutative diagram in simplicial sets exists

$$\begin{tikzcd} \mathcal{C}' \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ N(h\mathcal{C})' \arrow[r] & N(h\mathcal{C}) \end{tikzcd}$$

Now, we are going to define in the higher context some of the common definitions in category theory. They use to be just a new conception that verifies the old properties in the category theory, or analogous ones. We are going to start with the join of two simplicial sets (which endows sSet with a monoidal structure that will be useful later for limits) and just after the undercategories and the overcategories. The overcategories verify the relation $Hom\left(\mathcal{C}', \mathcal{C}_{/X}\right) \simeq Hom_X\left(\mathcal{C}' \star [0], \mathcal{C}\right)$ in the categorical case, and will use the equivalent property in the higher case to define them.

Definition 2.2.7. Let S and S' be two simplicial sets. For any nonempty finite linearly ordered set I we define the *join*, $S \star S'$, as follows

$$\left(S \star S'\right)(I) = \coprod_{I = J \cup J'} S(J) \times S'\left(J'\right)$$

for all decomposition of I in J and J' such that $j < j'$ for all $j \in J$, $j \in J$. In the event of either I or I' being empty by convention we get $S(\emptyset) = S'(\emptyset) = *$.

Remark. If S and S' are ∞-categories the join $S \star S'$ is an ∞-category.

Definition 2.2.8. Let K be a simplicial set. We define the *left cone* of K as $K^{\triangleleft} = \Delta^0 \star K$. Analogously the *right cone* of K is $K^{\triangleright} = K \star \Delta^0$

Definition 2.2.9. 1. Let $\mathcal{C}$ and K be simplicial sets, and $f\colon K \to \mathcal{C}$ a morphism in sSet. It exists always a simplicial set $\mathcal{C}_{/f}$ such that $Hom_{\mathsf{sSet}}\left(Y, \mathcal{C}_{/f}\right) = Hom_{/f}\left(Y \star K, \mathcal{C}\right)$, where $Hom_{/f}$ means that we consider just the morphisms $g\colon Y \star K \to \mathcal{C}$ such that $g_{|K} = f$. If $\mathcal{C}$ is an ∞-category then $\mathcal{C}_{/f}$ is the *overcategory of* $\mathcal{C}$ *over* f.

2. Similarly, the *undercategory* $\mathcal{C}_{f/}$ *of an infinity category* $\mathcal{C}$ *over* $f\colon K \to \mathcal{C}$ is the simplicial set given by the property that $Hom_{\mathsf{sSet}}\left(Y, \mathcal{C}_{f/}\right) = Hom_{f/}\left(K \star Y, \mathcal{C}\right)$, where $Hom_{f/}$ denotes the maps $g\colon K \star Y \to \mathcal{C}$ such that $g_{|K} = f$.

Remark. Both definitions exit always and they are unique up to categorical equivalence.

To finish this subsection we are going to define the core of a category. It will be an useful tool later for us in K-theory, playing the role of the isomorphism classes in a category.

Definition 2.2.10. For a given category $\mathcal{C} \in \mathsf{Cat}$ we define the *core of* $\mathcal{C}$, and we will denote it by $Core\left(\mathcal{C}\right)$, as the groupoid obtained by preserving all the objects and all the isomorphisms, but no other morphisms. In other words, this subcategory of $\mathcal{C}$ is the maximal groupoid.

For $\mathcal{C} \in \widehat{\mathsf{Cat}}_\infty$ (where $\widehat{\mathsf{Cat}}_\infty$ is the infinity category of not necessarily small infinity categories) we define the *core of* $\mathcal{C}$ as the infinity groupoid consisting only of equivalences at each level, i.e, the maximal Kan complex inside it.

Remark. This association defines a functor $Core\colon \mathsf{Cat} \to Grpd$ which is the right adjoint of the forgetful functor $U\colon Grpd \to \mathsf{Cat}$. To see that it is an adjoint one just have to realize that any functor from a groupoid to a category factors along the core of the category. Analogously $(-)^{\sim}\colon \mathsf{Cat}_\infty \to \infty Grpd$ is the right adjoint respect to the inclusion $\mathsf{Spc} \hookrightarrow \mathsf{Cat}_\infty$. The inclusion is a functor between two presentable infinity categories which preserves small colimits, thus it has a right adjoint (the adjoint functor theorem for the higher case can be found in [Lur07b, Cor. 5.5.2.9]).

Notation. We will denote interchangeably the notation $Core(-)$ and $(-)^{\sim}$ to refer the core in the category setting and the higher one.

2.2.3 Fibrations.

Now we are going to define many of the different kinds of fibrations in simplicial sets. For categories one has concepts as Grothendieck fibration and as a special case a fibration fibered in groupoids. The analogous role will be played by the Cartesian fibrations and the right/left fibrations for ∞-categories. This concepts are closely related with the ones in Cat via the nerve, and it use to happens that the concept in Cat corresponds to the analogous one via the nerve.

Definition 2.2.11. Let $f\colon X \to Y$ be a morphism in sSet.

1. We will say that f is a *trivial fibration* if it has the right lifting property respect to all the inclusions $\delta\Delta^n \subset \Delta^n$.

2. Let's remember that f is a *Kan fibration* if it has the right lifting property respect to all the horn inclusions $\Lambda^n_i \subset \Delta^n$.

3. We will say that f is a *left fibration* if it verifies the right lifting property respect to all the horn inclusions $\Lambda^n_i \subset \Delta^n$ for $0 \leq i < n$. And it will be called *right fibration* if it verifies the right lifting property respect to all the horn inclusions $\Lambda^n_i \subset \Delta^n$ for $0 < i \leq n$.

4. We will say that f is a *inner fibration* if it verifies the right lifting property respect to all the horn inclusions $\Lambda^n_i \subset \Delta^n$ for $0 < i < n$.

Remark.

1. If $p\colon X \to S$ is an inner fibration in sSet then each fiber is an ∞-category. Moreover, for a given morphisms $f\colon s \to s'$ in S there is an induced morphism between fibers $f^*\colon X_{s'} \to X_s$ in X. One can think about an inner fibration as a family of ∞-categories.

2. All the types of fibrations above and below are inner fibrations. To be an inner fibration is the minimum requirement to lift morphisms on the fibers, but such a lifts are not necessarily uniquely determined. In the definition below we are going to define the Cartesian fibration which are the correct type of fibrations to make lifts and get a functorial family of ∞-categories.

3. The left fibrations are stable under pullbacks and each fiber is a Kan complex. Moreover, we get a map $f_!\colon X_s \to X_{s'}$ for any morphism $f\colon s \to s'$ in S and the association $s \in S \mapsto X_s$, $f \in S_1 \mapsto f_!$ induce a functor $hS \to H$. Morally, to giving a left fibration is equivalent to giving a functor from S to the infinity category of spaces Spc.

To make the lift being determined up to equivalence we are going to require to the lift to be a Cartesian morphism. Considering a map $\mathcal{C} \to \mathcal{D}$ in Cat, if for a given morphism in $\mathcal{D}$ and a given lift of its target there is an universal lift, such a lift in $\mathcal{C}$ will called Cartesian morphism. The analogous concept in higher theory are the p-Cartesian morphisms.

Definition 2.2.12. 1. Let $p\colon X \to Y$ be an inner fibration in sSet and $f\colon x_1 \to x_2$ an edge in X. We say that f is a *p-Cartesian morphism* or a *p-Cartesian edge* if the induced morphism

$$X_{/f} \to X_{/x_2} \times_{/p(x_2)} Y_{/p(f)}$$

is an acyclic Kan fibration.

2. We will say that a morphism $p\colon X \to S$ in sSet is a *Cartesian fibration* if it is an inner fibration and moreover for any edge $f\colon x \to y$ of S and every vertex $\tilde{y}$ of X with $p(\tilde{y}) = y$ there exists a p-Cartesian edge $\tilde{f}\colon \tilde{x} \to \tilde{y}$ with $p(\tilde{f}) = f$.

3. A morphism $p\colon S \to S'$ is a *categorical fibration* if it has the right lifting property respect to all the maps which are categorical equivalence and cofibrations (monomorphisms in sSet) at the same time.

Remark. 1. Being p-Cartesian ensures that its lift is determined up to equivalence, and to be a Cartesian fibration implies that it happens for any lift.

2. There is a dual definition for the p-Cartesian edges and Cartesian fibrations, the p-coCartesian edges and coCartesian fibrations, i.e, the opposite map verifies the properties.

3. The Cartesian fibrations suppose the equivalent in Cat_∞ to the Grothendieck fibrations in Cat.

4. The right fibrations are a special case of Cartesian fibrations. A right fibration is a Cartesian fibration where all the edges are p-Cartesian. The right fibrations play the role of the Grothendieck fibrations fibered in groupoids. The $(\infty, 0)$-Grothendieck construction says that the right fibrations over an ∞-category $\mathcal{C}$ are isomorphic to the ∞-presheaves over $\mathcal{C}$.

5. There is a model structure on simplicial sets given by monomorphisms as cofibrations, categorical fibrations (isofibrations) as fibrations and categorical equivalences. The fibrant objects are the ∞-categories. Moreover, this model structure is combinatorial and left proper. This model structure is usually called Joyal's model structure to differentiate from the usual one, i.e, monomorphisms, weak equivalences after applying the nerve and Kan fibrations.

One of the useful properties of an infinity category $\mathcal{C}$ is that the maps between two objects $x, y \in \mathcal{C}$ is again a space, meaning a simplicial set. A morphism $f\colon x \to y$ is nothing other than $f\colon \Delta^1 \to \mathcal{C}$ such that $f_{|0} = x$ and $f_{|1} = y$. If we denote by $\Delta^{i_0, i_1, \ldots, i_k} \subseteq \Delta^n$ the k-simplex of Δ^n spanned by the given vertices $i_0, i_1, \ldots, i_k$, we can extend this point of view for the 2-morphisms and so on. An n-morphism from x to y is $\sigma\colon \Delta^n \to \mathcal{C}$ such that $\sigma_{|\Delta^{0,1\ldots,n}} = x$ and $\sigma_{|\Delta^{n+1}} = y$. For varying n, this defines a simplicial set.

Definition 2.2.13. The simplicial set of the construction above is the *Mapping space* of x and y, and it is denoted by $\mathrm{Map}_{\mathcal{C}}(x, y) \in \mathsf{sSet}$.

Remark.
1. The mapping space $\mathrm{Map}_{\mathcal{C}}(x, y)$ of two objects in an infinity category is always a Kan complex. We can think about $\mathrm{Map}_{\mathcal{C}}(x, y)$ as a Kan complex among as an object of $\mathcal{H}$, the homotopy category of spaces.

2. There are many possible definitions for the mapping spaces. The one used here is easiest with the covered background, but there is other with marked simplicial sets, sometimes more convenient for computations ([Lur07b, § 3.1.3]).

2.2.4 ∞-operads.

Now we are going to study the generalization of operads in the higher category setting, meaning the generalization of the colored operads. They are a generalization of the notion of an ∞-category, as the colored operads are a generalization of a category, and they are defined intuitively as the category of operators associated to the operad (Def. 2.4.4). Morally, an ∞-operad consist of a collection of objects together with a space of operations for every finite collection, restricted to a coherence multiplication law. We are going to define also the correct definition of a maps between two ∞-operads, which inherit an ∞-category structure. When the source ∞-operad is the commutative ∞-operad and the target a symmetric monoidal ∞-category we obtain the ∞-category of commutative algebras over the target ∞-operad.

Before, let us remember the definition and fix the notation of the skeleton of the category of pointed finite sets.

Definition 2.2.14. The *skeleton of the category of finite pointed sets*, $\mathcal{F}in_*$, is defined as follows:

1. The objects are the sets $\langle n\rangle = \{*, 1, 2, 3, \ldots, n\}$, i.e, the pointed set obtained by pointing the set $\langle n\rangle^\circ = \{1, 2, 3, \ldots, n\}$.

2. A morphism $\alpha\colon \langle m\rangle \to \langle n\rangle$ is a pointed preserving map, i.e, $\alpha(*) = *$.

We are going to denote $\rho^i\colon \langle n\rangle \to \langle 1\rangle$, the maps defined by $\rho^i(j) = 1$ if $i = j$ and $\rho^i(j) = *$ otherwise.

We shall also pay increased attention to two kinds of morphisms. We will say that $f\colon \langle m\rangle \to \langle n\rangle$ is an *innert morphisms* if $\forall i \in \langle n\rangle^{\circ}$, $f^{-1}(i)$ has exactly one object. And a morphism $g\colon \langle m'\rangle \to \langle n'\rangle$ will be called *active morphisms* if $g^{-1}(*) = *$.

Remark. 1. Every morphism in $\mathcal{F}in_*$ admits a factorization in innert and active morphisms, so restrict our attention to such morphisms is enough to work with $\mathcal{F}in_*$.

2. Every innert morphism $\alpha\colon \langle m\rangle \to \langle n\rangle$ induce an injective map $\alpha\colon \langle m\rangle^{\circ} \to \langle n\rangle^{\circ}$.

The symmetric monoidal categories are a simple special case of algebra for an ∞-operad, so first we reformulate such a definition to see how it would be in the higher setting. Given $(\mathcal{C}, \otimes)$ a symmetric monoidal category rather than think about it as a bifunctor $\otimes\colon \mathcal{C}\times\mathcal{C} \to \mathcal{C}$ we consider an n-tuple $(C_1, C_2, C_3, \dots, C_n)$, for $C_i \in \mathcal{C}$, and a collection of morphism $C_1 \otimes \cdots \otimes C_n \to \mathcal{D}$ and defining how they are composed. In the categorical context the identities do not hold as equalities, but up to isomorphism. In the higher case the equations satisfied by isomorphism must themselves hold only up to isomorphism. We define a new category $\mathcal{C}^{\otimes}$ as follows:

1. Objects: finite sequences of elements of $\mathcal{C}$, $[C_1, C_2, \dots, C_n]$.

2. Morphisms: $f\colon [C_1, C_2, \dots, C_n] \to [C'_1, C'_2, \dots, C'_m]$ consists of a subset $S \in \{1, 2, \dots, n\}$, a map of finite sets $\alpha\colon S \to \{1, \dots, m\}$, and a collection of morphisms $\{f_j\colon \bigotimes_{\alpha(i)=j} C_i \to C'_j\}_{1\leq j\leq m}$ in $\mathcal{C}$.

3. Composition: Given $f\colon [C_1, C_2, \dots, C_n] \to [C'_1, C'_2, \dots, C'_m]$ with $\alpha\colon S \to \{1, \dots, m\}$ and

   $g\colon [C_1, C_2, \dots, C'_m] \to [C''_1, C''_2, \dots, C''_l]$ with $\beta\colon T \to \{1, \dots, l\}$ the composition $g \circ f$ has set $U = \alpha^{-1}T$, the map of finite sets $\beta\circ\alpha\colon U \to \{1, \dots, l\}$ and the collection

$$\bigotimes_{(\beta\circ\alpha)(i)=k} C_i \xrightarrow{\sim} \bigotimes_{\beta(j)=k} \bigotimes_{\alpha(i)=j} C_j \to \bigotimes_{\beta(j)=k} C'_j \to C''_k$$

   for $1 \leq k \leq l$.

Remark. 1. This category comes with a forgetful functor $\mathcal{C}^{\otimes} \xrightarrow{p} \mathcal{F}in_* : [C_1, C_2, C_3, \dots, C_n] \to \langle n\rangle$. In fact, all the information that we will need is encoded in this functor.

2. In case of $\mathcal{C}^{\otimes}$ with an unique morphism and an unique object we get an isomorphism of categories: $\mathcal{C}^{\otimes} \to \mathcal{F}in_*$. Later the nerve of such a map will be the commutative operad.

Remark. 1. This functor has two features which are useful for the understanding of ∞-operads

(a) $p\colon \mathcal{C}^{\otimes} \to \mathcal{F}in_*$ is an op-fibration of categories. As a consequence every morphism in $\mathcal{F}in_*$ induce a functor between fibers.

$$\mathcal{C}^{\otimes}_{\langle m\rangle} \to \mathcal{C}^{\otimes}_{\langle n\rangle}$$

(b) $\mathcal{C}^{\otimes}_{\langle 1\rangle}$ is equivalent to $\mathcal{C}$ and $\mathcal{C}^{\otimes}_{\langle n\rangle}$ to an n-fold product of copies of $\mathcal{C}$. This equivalence is induced by the functor associated to the maps $\{\rho^i\colon \langle n\rangle \to \langle 1\rangle\}_{1\leq i\leq n}$ in $\mathcal{F}in_*$.

2. A symmetric monoidal structure is completely determined by the category $\mathcal{C}^{\otimes}$ and the functor p. In fact, giving a functor $p\colon \mathcal{D} \to \mathcal{F}in_*$ with the two properties (a) and (b) just above define a symmetric monoidal category $\mathcal{D}^{\otimes}$ with $\mathcal{D}^{\otimes}_{\langle 1\rangle}$ a classic symmetric monoidal category equivalent to $\mathcal{D}$ up to canonical isomorphism.

Definition 2.2.15. A *symmetric monoidal ∞-category* is a ∞-category $\mathcal{C}^{\otimes}$ together with a functor p such that:

1. $p\colon \mathcal{C}^{\otimes} \to \mathcal{F}in_*$ is a coCartesian fibration (Def. 2.2.12) in sSet.

2. $\forall\, n \leq 0$, $\{\rho^i\}_{1\leq i\leq n}$ induce functors $\rho^i_!\colon \mathcal{C}^{\otimes}_{\langle n\rangle} \to \mathcal{C}^{\otimes}_{\langle 1\rangle}$ which determines an equivalence $\mathcal{C}^{\otimes}_{\langle n\rangle} \simeq \left(\mathcal{C}^{\otimes}_{\langle 1\rangle}\right)^n$.

Relaxing the assumptions in the last remark and definition we are going to define now the ∞-operads. To any ∞-operad we can associate a theory of $\mathcal{O}$-monoidal ∞-categories. We will recover the symmetric monoidal ∞-category definition for the commutative ∞-operad case.

Definition 2.2.16. An ∞-*operad* is a functor $p\colon \mathcal{O}^{\otimes} \to N\left(\mathcal{F}in_*\right)$ between ∞-categories such that:

1. $\forall f\colon \langle m\rangle \to \langle n\rangle$ innert, $\forall C \in \mathcal{O}^{\otimes}_{\langle m\rangle}$, $\exists \bar{f}\colon C \to C'$ coCartesian morphism lifting f. In particular, for every f we get $f_!\colon \mathcal{O}^{\otimes}_{\langle m\rangle} \to \mathcal{O}^{\otimes}_{\langle n\rangle}$.

2. Let $C \in \mathcal{O}^{\otimes}_{\langle m\rangle}$, $C' \in \mathcal{O}^{\otimes}_{\langle n\rangle}$, $f\colon \langle m\rangle \to \langle n\rangle$ and $Map^f_{\mathcal{O}^{\otimes}}\left(C, C'\right)$ be the union of the connected components of $Map_{\mathcal{O}^{\otimes}}\left(C, C'\right)$ which lies over f. Then:

$$Map^f_{\mathcal{O}^{\otimes}}\left(C, C'\right) \to \prod_{1\leq i\leq 0} Map^{\rho^i\circ f}_{\mathcal{O}^{\otimes}}\left(C, C'_i\right)$$

is a homotopy equivalence.

3. For any collection $C_1, C_2, C_3, \ldots, C_n \in \mathcal{O}^{\otimes}_{\langle 1\rangle}$, there exists $C \in \mathcal{O}^{\otimes}_{\langle n\rangle}$ and a collection of p-coCartesian morphism $C \to C_i$ covering ρ^i.

Remark. 1. p is in fact a categorical fibration.

2. It is implicit in the third point of the definition that there is a canonical equivalence

$$\mathcal{O}^{\otimes}_{\langle n \rangle} \simeq \mathcal{O}^{n}$$

2.2.5 Maps of ∞-operads.

Now we are going to see which are the correct morphisms between ∞-operads, concretely the fibrant morphisms. We are going to define morphisms in the source of an ∞-operad using the maps induced by the innert and active morphisms in $\mathcal{F}in_*$.

Notation. Let $\mathcal{O}^{\otimes} \xrightarrow{p} N(\mathcal{F}in_*)$ be an ∞-operad.

1. f is innert if $p(f)$ is innert and f is p-coCartesian.
2. f is active if $p(f)$ is active.

Remark. For every ∞-operad the collection of innert and active morphisms determines a factorization system.

Now we are going to define the maps between ∞-operads and fibrations. Just below it will be defined an $\mathcal{O}$-monoidal ∞-category. And then, as an extension of the definition of maps between ∞-operads we will define the algebras over an operad.

Definition 2.2.17. $\mathcal{O}^{\otimes} \xrightarrow{f} \mathcal{O}'^{\otimes}$ in sSet is a ∞-*operad map* if:

1. the following diagram commutes

$$\begin{tikzcd} \mathcal{O}^{\otimes} \arrow[rr, "f"] \arrow[dr] & & \mathcal{O}'^{\otimes} \arrow[dl] \\ & N(\mathcal{F}in_*) & \end{tikzcd}$$

2. and f preserves innert morphisms.

We say that a ∞-operad map $\mathcal{O}^{\otimes} \xrightarrow{f} \mathcal{O}'^{\otimes}$ is a *fibration of* ∞-*operads* if f is a categorical fibration.

Definition 2.2.18. Given $\mathcal{O}^{\otimes}$ an ∞-operad, $\mathcal{C}^{\otimes}$ an ∞-category and $p\colon \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ a coCartesian fibration of ∞-categories, we will say that p exhibits $\mathcal{C}^{\otimes}$ as a *$\mathcal{O}$-monoidal ∞-category* or $p\colon \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ is a *coCartesian fibration of ∞-operads* if p satisfies any of the following equivalent conditions:

1. The composition $q\colon \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes} \to N(\mathcal{F}in_*)$ exhibits $\mathcal{C}^{\otimes}$ as ∞-operad.

2. $\forall\, T \simeq T_1 \oplus T_2 \oplus \cdots \oplus T_n \in \mathcal{O}^{\otimes}_{\langle n \rangle}$, the innert morphism $T \to T_i$ induce an equivalence of ∞-categories:
$$\mathcal{C}^{\otimes}_T \to \prod_{1 \leq i \leq 0} \mathcal{C}^{\otimes}_{T_i}$$

Remark.

1. If p is a coCartesian fibration of ∞-operads then it is a map of ∞-operads, in fact, a fibration of ∞-operads.

2. A symmetric monoidal ∞-category is an ∞-category $\mathcal{C}^{\otimes}$ with a coCartesian fibration of ∞-operads $\mathcal{C}^{\otimes} \to N(\mathcal{F}in_*)$.

Definition 2.2.19.

1. Let $p\colon \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a fibration of ∞-operads, and $\alpha\colon \mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$ a map of ∞-operads. We define $Alg_{\mathcal{O}'/\mathcal{O}}(\mathcal{C})$ as the full subcategory of $\mathcal{F}un_{\mathcal{O}}\left(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes}\right)$ spanned by the maps of ∞-operads.

2. $Alg_{\mathcal{O}'/\mathcal{O}}(\mathcal{C})$ is also the fiber over the vertex α of the categorical fibration $Alg_{\mathcal{O}'}(\mathcal{C}) \to Alg_{\mathcal{O}'}(\mathcal{O})$.

3. When $\mathcal{O}'^{\otimes} = \mathcal{O}'^{\otimes}$ then we get $Alg_{/\mathcal{O}}(\mathcal{C})$.

4. We call *∞-category of commutative algebra objects of* $\mathcal{C}$ to $Alg_{\mathcal{O}'/\mathcal{O}}(\mathcal{C})$ when $\mathcal{O}'^{\otimes} = \mathcal{O}^{\otimes} = N(\mathcal{F}in_*)$, and will be denoted by $C.Alg(\mathcal{C})$ (or $E_{\infty}(\mathcal{C})$).

Definition 2.2.20. Let $\mathcal{O}^{\otimes}$ be an ∞-operad, $p\colon \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ and $q\colon \mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes}$ coCartesian fibration of ∞-operads. An ∞-operad map $f \in Alg_{\mathcal{C}}(\mathcal{D})$ is a *$\mathcal{O}$-monoidal functor* if it carries a p-coCartesian morphisms to q-coCartesian morphisms.

2.2.6 Reflective localizations and presentable ∞-categories.

Now we are going to revise the reflective localizations and the (locally) presentable categories. We have seen before the localization of a category $\mathcal{C}$ respect to a set of morphisms W. This provides a new category $\mathcal{C}[W^{-1}]$ where the morphisms in W have become isomorphisms in an universal way via the functor $L_W\colon \mathcal{C} \to \mathcal{C}[W^{-1}]$. We are going to work with a special type of localizations, the reflective ones.

Definition 2.2.21. 1. We say that a localization $L_W\colon \mathcal{C} \to \mathcal{C}[W^{-1}]$ is a *reflective localization* if L_W has a full and faithful right adjoint $L_W \dashv i$.

2. We say that $\mathcal{C}$ is a *reflective subcategory* of $\mathcal{D}$ if the inclusion functor $i\colon \mathcal{C} \to \mathcal{D}$ has left adjoint T, called the reflector.

Remark. 1. In what follows we are going to work with reflective localizations. In this case, the localizations can be thinking on as just left adjoints respect to the inclusions.

2. The advantage of the reflective localizations is that $\mathcal{C}[W^{-1}]$ is equivalent to the category of W-local objects, which does not necessarily happens for normal localizations.

In classic category theory the (locally) presentable categories are those which contains a set S of small objects such that every object is a colimit over S, i.e, they are generated by a small data. Using the reflective localizations, a presentable category $\mathcal{C}$ is a reflective localization $\mathcal{C} \to PSh\,(S)$ of the category of presheaves over S. The category of presheaves over a category is the free cocompletion of a category, thus the presentables categories, as reflective localizations, are those categories which admit a presentation by generators and relations over S.

The presentable categories play the role of the finitely generated modules for categories, but the finitude here is too strict as requirement and we just require to be generated by small data. To play this role will give rise to an idea of linearity and the existence of a tensor product for (locally) presentable ∞-categories. Morally, a ∞-category $\mathcal{C}$ is (locally) presentable if it has all the small ∞-colimits and its objects are presented under ∞-colimits by a small set of small objects. The definition in a precise fashion is the following.

Definition 2.2.22. An ∞-category $\mathcal{C}$ is called *(locally small) presentable* if

1. it is an accessible ∞-category, i.e, $\mathcal{C}$ is locally small and it exists a regular cardinal κ such that $\mathcal{C}$ has all the κ-filtered colimits, the full ∞-subcategory $\mathcal{C}^\kappa \hookrightarrow \mathcal{C}$ of κ-compact objects is an essentially small ∞-category and $\mathcal{C}^\kappa$ generates $\mathcal{C}$ under κ-filtered ∞-colimits.

2. it has all small ∞-colimits.

One of the advantages offered by the locally presentable categories is that the Freyd adjoint functor theorem is simplified, since the solution set condition is verify automatically, and the theorem's requirement is just the preservation of limits and colimits. This extends to the ∞-category case, practically unchanged. The left adjoints between ∞-presentable categories agree with the small colimit preserving functor, while the right adjoints are those small limit preserving functors which are also accessibles. Thus the correct notion of morphisms between two presentable ∞-categories are the adjoints.

Definition 2.2.23. 1. We define as Pr^L the category with objects the presentable ∞-categories and morphisms the ∞-functors that preserve small ∞-colimits. They are precisely the left adjoints in the case of the presentable ∞-categories ([Lur07b, Cor. 5.5.2.9], adjoint functor theorem).

2. The ∞-category Pr^R will denote the category with objects the presentable ∞-categories and morphisms the ∞-functors that preserve small ∞-limits and are accessible (right adjoints).

Remark. Pr^L and Pr^R are subcategories of $\widehat{\mathsf{Cat}}_\infty$.

2.2.7 Homotopy fixed points.

The Hermitian K-theory was generalized to the category setting for categories with duality. A duality in a category allow us to define the equivalent to symmetric forms over a field in a categorical way. Since we are going to use higher category theory we must interpretate the categories with duality in this context. Our ∞-categories with duality are defined as homotopy fixed points of a C_2-action. In this subsection we are going to review the homotopy fixed points. With this purpose in mind we will see the limits, the actions, the way to think about them, comparing the algebraic and the topological point of view, the different ways to express such actions and its invariants. For fulfilling this task, we are going to present the ideas first in a categorical way and below in a higher setting.

One of the most basic tools in category theory are the limits. There are many ways to define them, always with the idea in mind of having an universal property respect to a diagram $F\colon C \to D$. The most intuitive way is to think about it like the universal cone respect to the diagram F. Since the cones are objects with morphisms to the diagram in such a way that the whole diagram commute it can be also expressed as an object in the comma category $\Delta \downarrow F$, where Δ denotes the constant functor, with an universal property. A third and common way is to state it as the right adjoint respect to the constant functor Δ. From these three ways to express the limits, maybe, the most useful in a higher setting is to define it as a terminal object in the over category.

Definition 2.2.24. Given two infinity categories $\mathcal{C}$ and $\mathcal{D}$, and an ∞-functor between them $F\colon \mathcal{C} \to \mathcal{D}$, i.e, a morphism in sSet, the *limit of* F, it it exists, is the terminal object in the overcategory of F:

$$\varprojlim F = Terminalobject\left(\mathcal{C}_{/F}\right)$$

Remark.

1. This definition is well defined up to contractible space of choice.
2. It exists in the higher setting a global definition, i.e, the limit can be defined as the right ∞-adjoint respect to the constant ∞-functor.
3. As one would hope, the definition of colimit is a limit in the opposite category.

The idea of an action of a group G in a set X is that for any element $g \in G$ we get an automorphism in X, i.e, there exists a morphism in groups $\rho\colon G \to Aut\,(X)$. If we define the action as a function $\rho'\colon G \times X \to X$ in *Set* we must require to ρ' that it respects the algebraic structure of G. This requirement can be encoded in a functor and, in fact, it is the most usual way to think about actions in a category. Before we must define the delooping of a group or a monoid. The term delooping will appears with different meanings but for a group will always mean the following one.

Definition 2.2.25. Given a monoid M the *delooping of* M is the category BM with an unique object $*$ and an endomorphism for every element of the monoid, such that the composition in BM is given by the operation in the monoid. The definition of the *delooping of a group* G is the same respect to the product in the group G.

Definition 2.2.26. Given a category $\mathcal{C}$ the *action of a group* G *on an object of* $\mathcal{C}$ is a functor $\rho\colon BG \to \mathcal{C}$. Analogously an G-action over an infinity category K is nothing but a functor $\rho'\colon BG \to K$.

Remark. The delooping defines a functor $B\colon Grp \to Grpd$.

Here the action of G on an object $x \in C$ corresponds to the functor that sends the unique object in BG to x, this induces an automorphism for each morphism in BG, recovering the all definition of an action: $\rho\colon G \to Aut\,(x)$.

Normally, in a classic action $G \circlearrowleft X$, the set X has another algebraic structure, as for example, the action of a commutative ring R over an abelian group M. This gives rise to the concept of module R-module. There is a correspondence between algebra and topology that allows us to think in a more topological way. The ring R plays the role of the ring of functions in a topological spaces, the R-modules corresponds to the sections of a vector bundle in such a space, and the extension of scalars, f^*, induced from a ring morphism f corresponds to the pullback of vector bundles along the dual map of spaces. So it is natural to think about objects with an action as some special kind of vector bundles, or objects in an overcategory, or its sections.

To think about actions as functors from the delooping of a group allows us to employ the Grothendieck construction. Then the information of an action of G on a category C is encoded in a fiber sequence $C \to C//G \to BG$, where $C//G$ is some homotopy quotient of the orbits. If one think on BG as a classifying space of G-universal principal bundle, then this bundle $C/G \to BG$ is the associated G-bundle of the universal bundle with fiber C.

To facilitate our work later we are going to define the actions in the higher setting using a fiber sequence way. One use to denote by C/G the set of orbits of a G-action over C. If we consider the objects in the same orbit as just isomorphics and not equals we get $C//G$. We can think about it as a resolution of the usual quotient. When the action is not free it is useful to work with $C//G$ instead of C/G.

Definition 2.2.27. Given an classic action of G, i.e, a functor $\rho\colon BG \to Sets$, we define the *action groupoid* of ρ as the 2-colimit of ρ: $S//G \simeq colim_{BG}\, \rho$. It can also be defined as the pullback respect to the universal bundle in sets $Set_* \to Set$ via the action map $\rho\colon BG \to Set$, i.e, $S//G$ is the pullback:

$$\begin{tikzcd} S//G \arrow[r] \arrow[d] & Sets_* \arrow[d, "U"] \\ BG \arrow[r, "\rho"] & Sets \end{tikzcd}$$

where the functor U is the canonical forgetful functor, the universal bundle in Set.

Definition 2.2.28. Let G be a group and $\rho\colon BG \to \mathsf{Cat}_\infty$ an action of G over an infinity category $\mathcal{C}$, $\rho(*) = \mathcal{C}$. We can define the *action groupoid of* $\mathcal{C}$ as the ∞-colimit $\mathcal{C}//G \coloneqq colim_{BG}\, \rho$. Alternatively, it can be also expressed as the pullback:

$$\begin{tikzcd} \mathcal{C}//G \arrow[r] \arrow[d, "\tilde{\rho}"] & Z \arrow[d] \\ BG \arrow[r, "\rho"] & \mathsf{Cat}_\infty \end{tikzcd}$$

where the functor $Z \to \mathsf{Cat}_\infty$ is the universal bundle in the ∞-Grothendieck construction and $\mathcal{C}//G \xrightarrow{\tilde{\rho}} BG$ is the $\mathcal{C}$-associated ∞-bundle.

Remark. We can extend this to a fiber sequence by adjoining the ∞-pullback along the morphism $* \to BG$:

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d] & \mathcal{C}//G \arrow[r] \arrow[d] & Z \arrow[d] \\ * \arrow[r] & BG \arrow[r, "\rho"] & \mathsf{Cat}_\infty \end{tikzcd}$$

This gives rise to a fiber sequence, and the action is encoded in such a fiber sequence.

As we said before the action over an infinity category can be encoded in a functor from the delooping of the group but for our purposes it is better to express it as a fiber sequence. The following definition is a special case of an ∞-action in a ∞-topos, but we are not going to need such a generalization. The topoid implicit here is sSet, and the model structure on sSet is the Joyal's one defined above.

Definition 2.2.29. Given an ∞-category $\mathcal{C}$ and a group G, an *G-action ρ in $\mathcal{C}$* is a fiber sequence of the form

$$\begin{tikzcd} \mathcal{C} \arrow[r] & \mathcal{C}//G \arrow[d, "\tilde{\rho}"] \\ & BG \end{tikzcd}$$

The ∞-category of the G-actions in Cat_∞ is the slice ∞-category $(\mathsf{Cat}_\infty)_{/BG}$. In a more general setting the *∞-category of objects in $\mathcal{C} \in \mathsf{Cat}_\infty$ with a G-action* will be denoted $\mathcal{C}[G]$.

The homotopy fixed points are an invariant of an action, which is given via the depend product. We are going to apply this invariant to the fiber sequence above to define the homotopy fixed points. The idea behind is to consider a morphism $g\colon A \to B$ as bundle, B is a family indexed by A.

Definition 2.2.30. Let $g\colon B \to A$ be a morphism in C such that the pullbacks along it exists.

$$g^*\colon C_A \to C_B$$

Then *the dependent product of* g, if it exists, is the right adjoint to $g^* \dashv \prod_g\colon C_B \to C_A$.

Remark. If we consider just a morphism as a bundle in a category $\mathcal{C}$ then, if $(\mathcal{C}, [\ ,\ ])$ is cartesian closed with all limits, the dependent product defines the sections of such a bundle via the terminal map $f\colon X \to *$. We get the adjoint

$$\begin{tikzcd} \mathcal{C}_{/X} \arrow[r, bend left, "\Pi_f"] & \mathcal{C} \arrow[l, bend left, "f^*"] \end{tikzcd}$$

If we have the "bundle" $P \to X$, $\prod_f$ gives the objects of sections of such a bundle $\Pi_{x\in X} P_x \simeq \Gamma_x(P) := [X,P] \times_{[X,X]} id$.

Definition 2.2.31. The *homotopy fixed points* of a G-action ρ over an ∞-category $\mathcal{C}$ are the sections of the morphism $\mathcal{C}//G \xrightarrow{\tilde{\rho}} BG$ of the fiber sequence of the action (Def. 2.2.29)

$$(\ \mathcal{C}\ )^{hG} := Inv(\mathcal{C}) = \Pi_{BG\to *}\ (\mathcal{C}//G \to BG)$$

The homotopy fixed points can also be interpreted as the G-equivariant maps from the point with trivial action and $\mathcal{C}$:

$$(\ \mathcal{C}\ )^{hG} := G - Act(*, \mathcal{C})$$

Remark. A good reference for ∞-bundles and fixed points on them is [SS12], where one can find the comparison between the different models.

2.2.8 Stable ∞-categories.

In the next section we are going to use many kinds of structures over ∞-categories which are mainly an axiomatization in the higher context of ideas already existing in category theory and topology.

In the classic setting the main subject in homotopy theory were the homotopy types of topological spaces, wherein the information was encoded in the homotopy category $Ho(Top)$, or more generally $Ho(\mathcal{C})$ for a general category with a model structure. Thus the stable homotopy theory studies the stable homotopy types, i.e, to look the homotopy category of spectra, $Ho(Sp(Top))$, which should be thought as the stabilization under reduced suspension and loop space operations of the classical homotopy category. Among other properties, this category is triangulated and additive. The stable categories refers to the category with such kind of features. Its higher categorical definition is focused on doing the fiber sequences and the cofiber sequence to be the same.

Definition 2.2.32.
1. A *zero object* in an ∞-category $\mathcal{C}$ is an object which is initial and final at the same time, and we will denote it just like 0. We say that $\mathcal{C}$ is *pointed* if it has an zero objects.
2. The natural map defined by the zero object, $Map_{\mathcal{C}}(X,0) \times Map_{\mathcal{C}}(0,Y) \to Map_{\mathcal{C}}(X,Y)$, has contractible domain, thus the induced map is well-defined in $h\mathcal{C}$ and we call it *zero map*.

3. Given an ∞-category $\mathcal{C}$ a *triangle* is a diagram $\Delta^1 \times \Delta^1 \to \mathcal{C}$:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r] & Z \end{tikzcd}$$

   We will say that such a triangle is a *fiber sequence* if it is a pullback square and a *cofiber sequence* if it is a pushout square.

4. Let $f\colon X \to Y$ be a morphism in $\mathcal{C}$. A *fiber of f* is a fiber sequence

$$\begin{tikzcd} W \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ 0 \arrow[r] & Y \end{tikzcd}$$

   And a *cofiber of f* is a fiber sequence

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r] & W \end{tikzcd}$$

5. We will say that an ∞-category $\mathcal{C}$ is a *stable ∞-category $\mathcal{C}$* if it verifies the following conditions:

   (a) It has a zero object.

   (b) All the morphisms $\mathcal{C}$ admit a fiber and a cofiber.

   (c) A triangle is a fiber sequence if and only if it is a cofiber sequence.

Remark. 1. In any ∞-category $\mathcal{C}$ with initial object and homotopy pullbacks we can define the *loop space object of X* as the homotopy pullback:

$$\begin{tikzcd} \Omega X \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & X \end{tikzcd}$$

In any ∞-category $\mathcal{C}$ with final object and homotopy pushouts we can define the *suspension object of X* as the homotopy pushout:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & \Sigma X \end{tikzcd}$$

2. For a pointed ∞-category $\mathcal{C}$ the above construction can be extended to get two functors. We denote by $\mathcal{M}^{\Sigma}$ the ∞-category of coCartesian squares of the form

$$\begin{tikzcd} X \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & Y \end{tikzcd}$$

where X and Y are objects of $\mathcal{C}$. If $\mathcal{C}$ admits cofibers for all morphisms one can see that the functor $e\colon\ \mathcal{M}^{\Sigma} \to \mathcal{C}$ given by evaluation over the first vertex X is a trivial fibration. Thus we can take a section $s\colon\ \mathcal{C} \to \mathcal{M}^{\Sigma}$ and compose with e given rise to $\Sigma := e \circ s\colon \mathcal{C} \to \mathcal{C}$. This functor has a right adjoint which is the loop space functor $\Omega\colon \mathcal{C} \to \mathcal{C}$.

3. The idea behind making the fiber and cofiber sequences equals is to make the loop space functor $\Omega\colon \mathcal{C} \to \mathcal{C}$ and the suspension functor $\Sigma\colon \mathcal{C} \to \mathcal{C}$ equivalences.

Example 2.2.1. 1. The basic example is the classic category of spectra Sp, where the objects are the Ω-spectra , i.e, the sequences of spaces $\{X_i\}_{i\in\mathbb{Z}}$ such that $\Omega X_{n+1} \simeq X_n$ with the obvious diagrams as morphisms.

2. Let $\mathcal{C}$ be an pointed ∞-category. A functor $X\colon \mathbb{Z}\times\mathbb{Z} \to \mathcal{C}$ is called prespectrum object in $\mathcal{C}$ if for all integers $i \neq j$ $X(i,j) = *$, where $*$ is a zero object in $\mathcal{C}$. This gives rise to a homotopy commutative diagram

$$\begin{tikzcd} X(n,n) \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & X(n+1,n+1) \end{tikzcd}$$

Taking the suspension functor in the lower right corner and the loop space functor in the upper left corner we get the morphisms $\alpha_n\colon \Sigma X(n,n) \to X(n+1,n+1)$ and $\beta_n\colon X(n,n) \to \Omega X(n+1,n+1)$. X is called *spectrum object* if β_m is an equivalence for all $m \in \mathbb{Z}$, and suspension spectrum if so α_m. The full ∞-sub-category of $Fun(\mathbb{Z}\times\mathbb{Z},\mathcal{C})$ spanned by the spectrum objects is the ∞-category category of spectra in $\mathcal{C}$, $\mathrm{Sp}(\mathcal{C})$.

3. Given a ∞-category we can always take its pointed ∞-category $\mathcal{C}_*$. One defines the stabilization of an ∞-category $\mathcal{C}$ as $Stab(\mathcal{C}) := Sp(\mathcal{C}_*)$.

The homotopy category in the classic setting has many desired properties to work with. These exist in the higher setting. Let's remember those that we will use later.

Definition 2.2.33. A *preadditive ∞-category* is a pointed ∞-category $\mathcal{C}$ such that finite products an finite coproduct exist and they are equivalents. In consequence, it exists a canonical morphism $X \coprod Y \to X \prod Y$ which is an equivalence for any $X, Y \in \mathcal{C}$. We will denote the resulting biproduct by $X \oplus Y$.

Remark. There are different and equivalent definitions of a preadditive infinity category, e.g, if its homotopy category $h\mathcal{C}$ is preadditive in the classic way. One of the properties of the preadditive categories is that every object admits a monoid structure, and moreover there are equivalences between $C.Alg(\mathcal{C})$, $Mon_{Comm}(\mathcal{C})$, $Mon_{E_\infty}(\mathcal{C})$ and $\mathcal{C}$. The E_∞-algebra structure of $C \in \mathcal{C}$ is given by the fold map $\nabla \colon C \oplus C \to C$ (see [Lur14, § 2.4.3]). There is also a map called shear map $s = (pr_1, \nabla)$

Definition 2.2.34. Given a preadditive ∞-category $\mathcal{C}$ we say that it is *additive* if the shear map of each object is an equivalence.

2.3 Infinity categories with duality and Hermitian multiplicative infinite loop space machines.

In this section we are going prove the existence of an E_∞-ring structure in the presheaf of Hermitian K-theory spectra presented in the Section 1.3.1. To carry it out, we are going to work in the higher category setting, which has been proven to be an useful tool for topologists in the recent years. Before, we have already talked about the K-theory and the Hermitian K-theory in section 1.2. As stated earlier, the K-theory started as process to complete an abelian monoid adding the inverses in a universal way. That idea was extend later to the topology, more concretely to vector bundles, then to finitely generated modules over a field since they have a similar behavior, and to other contexts, e.g, the non-degenerated symmetric forms (or to quadratic forms, in the event of being working with rings where 2 is invertible) giving rise to the Hermitian K-theory (Section 1.2.3). Moreover, one of the ideas coming from the extension to the topological case is that the K_0-group can be extended, giving rise to a cohomology theory. In the algebraic setting we get a homology theory for rings and a cohomology theory for varieties. These generalized cohomology theories use to be represented by a connective spectrum or an infinite loop space, and the K-theory constructions focus on collecting them. Our construction in this section is the group completion of an infinity category with duality. We will obtain a commutative group in the infinity category of spaces and then the associated spectrum. This will produce our presheaf of $\mathbb{S}^1$-spectra $\mathbf{KQ}$ presented in Section 1.3.1. The bipermutative structures defined there are not going to be used in this Section, but we use the bimonoidal ring structure given by the direct sum and the tensor product.

The develop of the Hermitian K-theory followed the one of the classic K-theory with the special treatment for rings with involution or categories with duality, as explained in 1, and the infinite loop space machines can be used to get it. In this section we are going to develop a new way to define the K-theory of a category with duality and the construction will be focused more precisely in the multiplicative structures. Morally, the infinite loop space machines give the group completion of an E_∞-space (Section 2.1), and then the associated spectrum. This is encoded in a functor

$$E_\infty\left(Spc\right) \to \mathrm{Sp}.$$

In case of a multiplicative infinite loop space machine we get a functor

$$\mathrm{Rig}_{E_\infty}\left(Spc\right) \xrightarrow{Gp} E_\infty\left(\mathrm{Sp}\right).$$

They were many generalizations of the infinite loop space machines, but mainly along combinatorial constructions with explicit pairs of operads evolved on them. In [Nik13], Gepner, Groth and Nikolaus defined a direct sum K-theory (i.e, for symmetric monoidal structures) from the point of view of the infinity categories. Such a construction differ from others by using universal properties.

In this section we are going to follow this idea but in the Hermitian case. First we are going to define the infinite categories with duality. A category with duality is a category with a functor from it to its opposite category such that apply it twice is naturally isomorphic to the identity. Therefore we can detect them with a C_2-action, where the zero corresponds to the identity and the non-trivial element of C_2 gives the duality. So we are going to see that it exists such a C_2-action in Cat_∞ and we will define the ∞-categories with duality as the homotopy fixed points respect to such a C_2-action. Then, we will define the direct sum K-theory of a symmetric monoidal infinite category with duality. In short, the Hermitian K-theory of a given a category with duality $\mathcal{C}$ is obtained by getting first the Hermitian category (the non-degenerated bilinear forms in the classic context), $\mathcal{C}_h$, which is a symmetric monoidal respect to the orthogonal sum (direct sum), then consider the symmetric monoidal groupoid of $(\mathcal{C}_h, \perp)$, group complete it and get the associated spectrum. This agrees with common definitions in the case of $\mathcal{P}(R)$ (see [Sch04] and [Sch12, Theorem A.1] if 2 is invertible in R and [HM15, Theorem A] in general). To get the Hermitian category is going to be encoded inside of our Hermitian direct sum functor. Our Hermitian K-theory gives rise to a functor

$$K_h \colon \mathsf{SymMonCat}_\infty^{hC_2} \to \mathrm{Sp}.$$

This functor is in fact a lax symmetric monoidal functor, so it refines to a functor

$$\mathrm{Rig}_{E_\infty}(\mathsf{Cat}_\infty)^{hC_2} \to E_\infty(\mathrm{Sp}).$$

Once we have construct our ILSM, we will see how to interpretate the usual categories with duality as infinity categories with duality. Then, we will prove a Recognition Principle for preadditive categories. We will construct a functor from the symmetric monoidal preadditive infinity categories with duality to symmetric monoidal infinity categories with duality. This Recognition principle together with the interpretation of the usual categories with duality as infinity categories with duality will allow us to introduce our ring category from Section 1.3.1 as an input in the machine. This process will be described in a short section at the end and it will give rise to our presheaf of spectra **KQ**.

This work on developing the Hermitian case disgorged in the pre-print [Spi16] by Markus Spitzweck, Hadrian Heine and the author. The first part of this paper presents the main results of this section. In the present Section, they will extracted and explained these parts of the paper.

Notation. Before starting with the develop of the theory we fix some notation that will be used. In some cases we will use different notation for the same concept since the contexts induces us to do it, for instance, the notation for the E_∞-algebras will be preserved from [Nik13]. Given an infinity category $\mathcal{C}$: $E_\infty(\mathcal{C})$ is the ∞-category of E_∞-algebras in $\mathcal{C}$, $Mon_{E_\infty}(\mathcal{C})$ is the ∞-category of commutative monoids in $\mathcal{C}$, $Grp_{E_\infty}(\mathcal{C})$ is the ∞-category of commutative groups in $\mathcal{C}$, $Spt(\mathcal{C})$ is the ∞-category of spectra in $\mathcal{C}$. $\mathcal{H}$ will represent the homotopy category of topological spaces.

2.3.1 Good faithful functors.

In this subsection we are going to define the concepts appearing in the infinity categories with duality and fix its notation. As we said in Section 2.2.3 the infinity categories are enriched over spaces, i.e, in Kan complexes, and they play the same role that *Set* in the classic category theory. There are many ways to define the ∞-category of spaces. The category of compactly generated weakly Hausdorff topological spaces, Top, is Quillen equivalent to the category of the CW complexes, so to do homotopy theory over CW complexes is the same that do it in Top. And this one turns to be Quillen equivalent to the Kan complexes in sSet.

Let Kan be the category spanned by the Kan complexes in sSet, which is clearly simplicial and we can take its homotopy coherent nerve. We define the ∞-category of spaces as $\mathsf{Spc} = N(\mathsf{Kan})$. The maps between two objects in Kan is again a Kan complex, so Spc is an infinity category by [Lur07b, Prop. 1.1.5.10].

Given two simplicial sets $\mathcal{C}$ and $\mathcal{D}$ we denote by $\mathrm{Fun}(\mathcal{C},\mathcal{D})$ the simplicial set $\mathrm{Map}_{\mathsf{sSet}}(\mathcal{C},\mathcal{D})$. We are going to use this notation along this subsection just when the target simplicial set is an ∞-category. The simplicial set $\mathcal{C}$ is usually an ∞-category, but not always.

One of the key points of the recognition principle in Section 2.3.6 is that all the categories are presentables and the functors that will define the K-theory will be endofunctors in Pr^L, in fact, they are localizations (smashing localizations) with the preadditive (additive, stable or pointed) categories as local objects. In particular, Spc and Cat_∞ are in Pr^L.

Definition 2.3.1.

1. An ∞-functor $F : \mathcal{C} \to \mathcal{D}$ between two ∞-categories is said to be *full and faithful* if for all objects $x, y \in \mathcal{C}$ the induced map $hF_{x,y}\colon \mathrm{Map}_{h\mathcal{C}}(x, y) \to \mathrm{Map}_{h\mathcal{D}}(Fx, Fy)$ is an equivalence on the mapping spaces, i.e, an isomorphism in $\mathcal{H}$.

2. We will say that a faithful ∞-functor is *good faithful* if for any $x, y \in C$ every component of $\mathrm{Map}_{\mathcal{D}}(Fx, Fy)$ on an equivalence is hit by an equivalence in $\mathrm{Map}_{\mathcal{C}}(x, y)$.

3. The *good image* of a good faithful ∞-functor $F : \mathcal{C} \to \mathcal{D}$ is the subcategory $F_G \subset \mathcal{D}$ with vertices those which are equivalent to vertices in the image of F and those edges that lie up to equivalence in the image of F.

Remark.

1. A good faithful ∞-functor $F : \mathcal{C} \to \mathcal{D}$ between infinity categories factorizes as $\mathcal{C} \xrightarrow{\sim} F_G \hookrightarrow \mathcal{D}$, an equivalence followed by an inclusion.

2. A full and faithful ∞-functor $F : \mathcal{C} \to \mathcal{D}$ exhibits $\mathcal{C}$ as a full ∞-subcategory of $\mathcal{D}$. Moreover, the full and faithful ∞-functors are always good faithful ∞-functors, and the essential image coincides with the good image.

3. A good fully faithful ∞-functor is precisely a monomorphism in Cat_∞ or $\widehat{\mathsf{Cat}}_\infty$, where a map $F\colon X \to Y$ is a monomorphism in an infinity category $\mathcal{C}$ (in our case in Cat_∞) if the square

$$\begin{tikzcd} X \arrow[r, "Id_X"] \arrow[d, "Id_X"] & X \arrow[d, "F"] \\ X \arrow[r, "F"] & Y \end{tikzcd}$$

is cartesian. One can argued the faithfulness by the definition of the hom space as the pullback

$$\begin{tikzcd} \mathrm{Map}_{\mathcal{C}}(x, y) \arrow[r] \arrow[d] & \mathcal{C} \times \mathcal{C} \arrow[d] \\ \mathcal{C}^{\Delta^1} \arrow[r] & * \end{tikzcd}$$

This induces the diagram for the monomorphisms in hom spaces, so the $F_{x,y}$ is a monomorphism of spaces and therefore F is faithful. Using that the core functor is a right adjoint and preserves pullbacks we can check the goodness. The opposite direction follows directly from the definition of good faithful ∞-functor.

4. The good fully faithful functors are exactly the faithful functors which induce fully faithful functors on core groupoids.

Propositon 3. *Let I be a small ∞-category and let f be a monomorphism in Cat_∞^I. Then $\lim_I f$ is again a monomorphism whose good image can be characterized as follows: An object belongs to the good image if and only if its components lie in the good images of all the f_i, $i \in I$, and a morphisms g in I.*

Proof. The existence is given by [Lur07b, Corollary 5.1.2.3]. The limits in functor categories are given objectwise so the second assertion follows from the definition of good faithful. $\square$

Propositon 4. *Let $\varphi\colon \mathcal{C} \to D$ be a good faithful functor between ∞-categories and let I be an ∞-category. Then the induced functor $\mathrm{Fun}(I, \mathcal{C}) \to \mathrm{Fun}(I, D)$ is a good faithful functor whose good image can be characterized as follows: A functor $I \to D$ belongs to the good image if and only if it factors through the good image of φ, and similarly for morphisms.*

Proof. As in Proposition 3 to be a monomorphism can be expressed as a limit property, thus if it is verify objectwise it is also true on the functor category, and the second assertion is immediate. $\square$

2.3.2 Infinity categories with duality.

Our goal in this subsection is to interpretate the classic categories with dualities in a new sense more suitable for the higher category setting. We want to see that the categories with duality can be expressed as homotopy fixed points of a C_2-action in Cat_∞. In the first step we are going to define the infinity categories with duality. To that end, first we are going to see that Cat_∞ is canonically an object of Pr^L with a C_2-action. So we can define the infinity categories with duality as homotopy fixed points respect to C_2. In the second we see that this action induce an action at the level of spaces, and moreover such an action is trivial. In the next section it will be proved that the classic categories with duality are just a particular case of the infinity categories with duality.

Let G be a discrete group and $\mathcal{C}$ an infinity category, we will denote by $\mathcal{C}[G]$ the ∞-category of objects of $\mathcal{C}$ with a G-action. Seeing the actions as functors from the delooping of G it corresponds to $\mathrm{Fun}\,(BG, \mathcal{C})$, where we abuse of terminology by identifying BG with its nerve. Pulling back along the morphism $BG \to *$ we get a morphism $\mathcal{C} \to \mathcal{C}[G]$, whose essential image is the full ∞-subcategory of objects with a trivial G-action.

In subsection 2.2.7 they were defined the homotopy fixed points, X^{hG}, for $X \in \mathcal{C}[G]$ when the BG-shaped limits exit in $\mathcal{C}$. In what follows we are going to focus our attention in the group $C_2 \simeq \mathbb{Z}/2$. For an infinity category $\mathcal{C}$ we write $Aut(\mathcal{C})$ for the full sub-∞-category of $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$ spanned by the auto-equivalences. In the following theorem Cat_∞ is consider as an infinity category (Notation 2.2.1). The following theorem canonically exhibits Cat_∞ as an object of $\mathsf{Pr}^L[C_2]$. And then, using this fact, we present the definition of infinity category with duality.

Theorem 2.3.1. *The (group-like A_∞-) space of autoequivalences of Cat_∞ in Pr^L is (equivalent to) the discrete space $\mathbb{Z}/2$. $Aut(\mathsf{Cat}_\infty)$ is equivalent to $\{Id, op\}$, where the autoequivalence induced by the non-trivial element of C_2 sends a $C \in \mathsf{Cat}_\infty$ to C^{op}.*

Proof. This is [Toë05, Thm. 6.3] (see also [BSP11, Thm. 8.12]). □

Definition 2.3.2. The ∞*-category of small* ∞*-categories with duality* is defined to be the homotopy fixed points of the C_2-action on Cat_∞, i.e. $\mathsf{Cat}_\infty^{hC_2} \in \mathsf{Pr}^L$. An ∞*-category with duality* is an object of such a ∞-category.

Theorem 2.3.2. *The C_2-action on Cat_∞ induces a C_2-action on the full subcategory $\mathsf{Spc} \subset \mathsf{Cat}_\infty$. This action is trivial.*

Proof. Let consider M the maximal Kan subcomplex of $\mathrm{Fun}(\mathsf{Cat}_\infty, \mathsf{Cat}_\infty)$ on those vertices which are equivalences, i.e autoequivalences. The composition in $\mathrm{Fun}(\mathsf{Cat}_\infty, \mathsf{Cat}_\infty)$ induces simplicial monoid structure in M. Since an infinity category $\mathcal{C}$ is an ∞-groupoid if $C^{op} \in \mathsf{Cat}_\infty$ is an ∞-groupoid M preserves $\mathsf{Spc} \subset \mathsf{Cat}_\infty$ by the Theorem 2.3.1, establishing the first assertion. The space of autoequivalences of Spc in Pr^L is contractible, establishing the second assertion. □

From the last theorem we can deduce the behavior of this C_2-action in the case of the spaces Spc, where the inclusion induces at the level of the homotopy fixed points full and faithful maps in Pr. For $\mathcal{C}$ an

∞-category, $X \in \mathcal{C}$ and $K \in \mathsf{Spc}$ we denote by $X^K \in \mathcal{C}$ the co-tensor if it exists, i.e, the limit of the constant functor diagram on X.

The trivial G-action over $\mathcal{C}$ corresponds to the fiber sequence $\mathcal{C} \to \mathcal{C} \times BG \to BG$. Clearly when the action of G over $X \in \mathcal{C}[G]$ is trivial we get $X^{hG} \simeq X^{BG}$. In the case of the category of spaces we get the following equivalences $\mathsf{Spc}^{hC_2} \simeq \mathsf{Spc}^{BC_2} \simeq \mathsf{Spc}[C_2]$. The map $\mathsf{Spc} \hookrightarrow \mathsf{Cat}_\infty$ in $\mathsf{Pr}^L[C_2]$ has a retraction in $\mathsf{Pr}^L[C_2]$, its left adjoint. So we have canonical fully faithful maps $\mathsf{Spc}[C_2] \to \mathsf{Cat}_\infty^{hC_2}$ in Pr^R and Pr^L. This can be deduced also from the argument that a limit of full and faithful maps in $\widehat{\mathsf{Cat}}_\infty$ is again a fully faithful map.

Remark. The full subcategory $\mathsf{Cat} \subset \mathsf{Cat}_\infty$ of ∞-categories which are equivalent to (small) categories is stable under the C_2-action giving a full embedding $\mathsf{Cat}^{hC_2} \to \mathsf{Cat}_\infty^{hC_2}$ in Pr^R.

2.3.3 From categories with duality to infinity categories with duality.

In this section we are going to see that the Hermitian K-theory functor, that we are going to develop in the following subsections, can be also applied to usual categories with duality.

Let denote by Cat^1 te category of small categories to remark that we are going to consider it as a an usual 1-category and not like an infinity category. The category $\widehat{\mathsf{Cat}}^1$ is the 1-category of not necessarily small categories. Cat^1 is not an small category so it is an object of $\widehat{\mathsf{Cat}}^1$.

Taking the nerves of these categories we get two infinity categories (by Lemma 5) and they exist natural infinity functors $S\colon \mathrm{N}\mathsf{Cat}^1 \to \mathsf{Cat}$ and $\widehat{S}\colon \mathrm{N}\widehat{\mathsf{Cat}}^1 \to \widehat{\mathsf{Cat}}$.

There is a third infinity functor induced by considering Cat^1 as an object of $\widehat{\mathsf{Cat}}^1$ and apply $\widehat{S}$ to it. This is the infinity functor $S'\colon \widehat{S}(\mathsf{Cat}^1) \to \mathsf{Cat}$ in $\widehat{\mathsf{Cat}}_\infty$.

Even if we are interested in C_2-actions over categories and infinity categories, the next lemma is true for a general group G.

Lemma 6. Let G be a group. Let $\mathcal{C}$ be a (1-) category with G-action (in the strict sense, i.e. an object of $\widehat{\mathsf{Cat}}^1[G] = \mathrm{Fun}(BG, \widehat{\mathsf{Cat}}^1)$). Then there is a natural equivalence

$$\widehat{S}(\mathrm{Fun}^G(EG, \mathcal{C})) \simeq \widehat{S}(\mathcal{C})^{hG}$$

in $\widehat{\mathsf{Cat}}$ (here EG is the translation groupoid of G).

There is a natural C_2-action in the category Cat^1 given by the identity functor and the functor that associates to each category its opposite category $\mathcal{C} \mapsto \mathcal{C}^{\mathrm{op}}$, for $\mathcal{C} \in \mathsf{Cat}^1$. This induces a C_2-action on $\widehat{S}(\mathsf{Cat}^1)$.

Lemma 7. The infinity functor S' is compatible with the C_2-actions.

Proof. This follows from the fact that the nerve functor $\mathrm{N}\colon \mathsf{Cat}^1 \to \mathsf{sSet}$ is strictly compatible with the C_2-actions. $\square$

Let I be the category with two elements and an isomorphism between them as unique not trivial morphism (the interval groupoid). For the concrete case of the C_2-actions we get $I \cong EC_2$. We get following sequence of functors.

$$\widehat{S}(\mathrm{Fun}^{C_2}(I, \mathsf{Cat}^1)) \simeq \widehat{S}(\mathsf{Cat}^1)^{hC_2} \to \mathsf{Cat}^{hC_2} \to \mathsf{Cat}_\infty^{hC_2}$$

in $\widehat{\mathsf{Cat}}_\infty$. The first one is the equivalence of Lemma 6 for $G = C_2$, the second is induced by the preservation of the C_2-action of Lemma 7 and the last one use the inclusion as infinity subcategory of $\widehat{\mathsf{Cat}}$ in $\widehat{\mathsf{Cat}}_\infty$.

Since the infinity functor $\widehat{S}$ consists in applying the nerve this composition can be equivalently viewed as

$$\epsilon\colon \mathrm{NFun}^{C_2}(I, \mathsf{Cat}^1) \to \mathsf{Cat}_\infty^{hC_2}$$

in $\widehat{\mathsf{Cat}}_\infty$.

Like last step we are going to see that $\mathrm{Fun}^{C_2}(I, \mathsf{Cat}^1)$ corresponds to the usual categories with duality.

Remark. It can be shown that the functor

$$\mathrm{NFun}^{C_2}(I, \mathsf{Cat}^1) \to \mathsf{Cat}^{hC_2}$$

is a localization (in the naive sense, e.g. this functor has not a fully faithful right adjoint) at those natural transformations which are objectwise equivalences.

Now we are going to work in the 2-category setting. As in the higher category case there are many ways to define them depending on the focus, but always with idea in mind of a category enriched over Cat. As usual, we will let the prefix 2- reserved to the strict case, but nevertheless, it is also used prompt to mean any of the definitions, since they use to be equivalents. We will pay more attention to the prefix 2- for the functors, since it is more relevant when working with. A direct generalization of the categorical case gives rise to the 2-categories, i.e, the same definition but with objects, 1-morphisms and 2-morphisms. We will refer to these 2-categories as *strict 2-categories* or just 2-categories, calling by a different name the other kind of 2-categories. More specifically, these are categories enriched over Cat in the strict way, thus the morphisms between two objects are categories and the compositions of 1-cells and 2-cells are strictly associativity and unital. If we do the enrichment in a weak way we get the so-called *bicategory*, where the associativity and unity diagrams of the enrichment hold only up to coherent isomorphism. This use to be the most useful definition, enough weak to work with, and moreover, any bicategory is equivalent to a strict 2-category. There is an alternative way to define the two categories via the higher categories. We can define a 2-category as an ∞-category such that every 3-morphism is an equivalence, and all parallel pairs of i-morphisms are equivalent for $i \geq 3$.

The three definitions of 2-categories described just above give rise to 'equivalent' theories for 2-categories. One can take a look to any introduction paper as the Lack's paper [Lac07] or the Leinster's one [Lei98] Hereafter we are going to define the possible functors between 2-categories, which are going to play a more significant role in what follows.

Between two bicategories there are different possible correct definitions of 2-functor, but unlike the category case one need to take into account in addition the two cells, as we did before for the 2-category definitions. Since we are working with 2-categories the 2- prefix use to be understood, at least for the strict case. In the bicategory case we can use the strict and the weak definition for 2-functors, so we will do a distinction between 2-functors (extending the definition of functor between categories in a strict way) and pseudofunctors (the weak version). Moreover, there is a even weaker version that pseudofunctor, the lax functors, that as its name suggests from the lax case for monoidal categories, it just verifies the associativity and identity diagrams without require invertibility in the comparison maps. Even if this last definition preserves all the properties preserved by the lax monoidal functors but in the 2- category setting we will use the pseudofunctors, since we want to preserve the adjoints given by the 2-cells.

Definition 2.3.3. Given two bicategories $\mathcal{C}$ and $\mathcal{D}$, a *lax functor* between them is $F\colon \mathcal{C} \to \mathcal{D}$ such that:

1. It sends every object $a \in \mathcal{C}$ to an object $F(a) \in \mathcal{D}$
2. For each pair of objects $a, b \in \mathcal{C}$ there is a functor $\mathcal{C}(a,b) \to \mathcal{D}(F(a), F(b))$, thus preserving 2-cells in a strict way.
3. And for each pair of morphisms f, g in $\mathcal{C}$ they exists comparison maps $\phi\colon Fg \circ Ff \to F(g \circ f)$ and $\phi_0\colon 1_{F(a)} \to F(1_a)$.

And in addition it verifies some associativity and identify coherent conditions which are equal to those for monoidal categories. We say that F is a *pseudofunctor* if the comparison maps ϕ and ϕ_0 are invertible. In case these morphisms are identities we call it *strict functor*.

From the definition of pseudofunctor one can deduce that it preserves adjoints. The fact that the comparison maps are invertibles allows us to define by composition the unit and counit natural transformations of an adjoint in the target category coming from an adjoint in the source category. This is an important property for us which is not shared by the lax functors between bicategories.

Given two categories $\mathcal{C}$ and $\mathcal{D}$ with a G-action, an equivariant pseudofunctor between them is a pseudofunctor that preserves the action. More specifically, if for an element $g \in G$ the G-action induces automorphisms ϕ_g and φ_g, for $\mathcal{C}$ and $\mathcal{D}$ respectively, to be G-equivariant for a pseudofunctor $F\colon \mathcal{C} \to \mathcal{D}$ means that for every $c \in \mathcal{C}$ we get that $F(\phi_g(a)) = \varphi_g(F(a))$.

Lemma 8. Let I be the category with two elements and an isomorphism between them as unique not trivial morphism (the interval groupoid), Cat_D the category of categories with (strong) duality, and $2PseudoFun_{C_2}(I, \mathsf{Cat})$ the equivariant pseudofunctors between I and Cat, where C_2 acts in the obvious way on I. Then, there is an isomorphism between the category of categories with duality and the equivariant pseudofunctors between I and Cat, i.e:

$$\mathsf{Cat}_D \simeq PseudoFun_{C_2}(I, \mathsf{Cat})$$

Proof. Let $(\mathcal{C}, \eta, *)$ be a category with strong duality, i.e, a category $\mathcal{C}$, a functor $(-)^* : \mathcal{C} \to \mathcal{C}^{op}$, and a natural transformation $\eta\colon Id_{\mathcal{C}} \Rightarrow (-)^{**}$ such that $\eta^*_X \circ \eta_{X^*} = id_{X^*}$. This gives rise to an adjunction $(*, *^{op}, \eta, \eta^{op}) : \mathcal{C} \to \mathcal{C}^{op}$. In addition, each adjunction $(F, G, \eta, \varepsilon) : \mathcal{C} \to \mathcal{C}^{op}$ between a category and its opposite such that the $\eta^{op} = \varepsilon$ and $F^{op} = G$ defines a category with duality automatically by the unit and counit formulas. So we consider as categories with duality this special case of adjunction.

The fact that the category $PseudoFun_{C_2}(I, \mathsf{Cat})$ are actually these adjunctions is based on two observations: first, the pseudofunctors are precisely the 2-functor which preserve adjoints, and second, the C_2-action in Cat assign to each category its opposite category and to each functor its opposite functor. So the diagrams generated by I in Cat corresponds to the adjoints in Cat and the equivariance restricts to the desired adjunctions, the categories with duality. □

We can work analogously but without pseudofunctors. To do it we need to restrict to strict categories with duality but this fact doesn't make us lose information since every category with a strong duality can be strictify in such a way that the resulting Hermitian K-theory is the same.

Definition 2.3.4. *Strictification*: Given $(\mathcal{C}, *, \mu)$ a category with strong duality, we can define a new category with a strict duality, $S(\mathcal{C}, *, \mu)$, as follows:

- The objects of $S(\mathcal{C}, *, \mu)$ are (c, c', f) such that $f\colon c' \to c^*$ is an isomorphism.
- A morphism $g\colon (c, c', f) \to (d, d', f')$ is a pair of maps $(g\colon c \to d, h\colon c' \to d')$ in $\mathcal{C}$ such that

$$\begin{tikzcd} c' \arrow[r, "f"] & c^* \\ d' \arrow[u, "h"] \arrow[r, "f'"] & d^* \arrow[u, "g^*"'] \end{tikzcd}$$

  commutes. The composition is by component.

The duality on $S(\mathcal{C}, *, \mu)$ associate to every map $f\colon c' \to c^*$ the composition $c \xrightarrow{\mu_c} c^{**^{op}} \xrightarrow{f^*} c'^*$ and $(g\colon c \to d, h\colon c' \to d')$ to $(h\colon c' \to d', g\colon c \to d)$.

The original category and the strict one obtained after the strictification are equivalents. We can define a duality preserving functor $(I, i) : (\mathcal{C}, *, \mu) \to S(\mathcal{C}, *, \mu)$ by $I(c) = (c, c^*, id_{c^*})$, $I(f) = (f, f^*)$ and $i_c = (id_{c^*}, \mu_c)$. This functor is an equivalence of categories with duality with inverse the functor (K, k) defined by $K(c, c', f) = c$, $K(g, h) = g$ and $k_{(c,c',f)} = f$. These two functors define weak equivalence after taken the isomorphic classes and apply the geometric realization of the nerve, so the resulting Hermitian K-theory are equivalents. Moreover, the strict case has the advantage that the resulting space has a natural C_2-action (see Moi's thesis [Moi14] for more details). We can also perform the strictification in the other side of the equivalence. This is a particular case of the general strictification construction by Power in [Pow89].

Corollary 2.3.3. $\mathrm{Fun}^{C_2}(I, \mathsf{Cat}^1)$ *is (isomorphic to) the category of small categories with strict duality and duality preserving functors in the sense of [Shc10b, § 2.1].*

2.3.4 Symmetric monoidal structures.

In this section we will see that the C_2-action on Cat_∞ is compatible with the natural symmetric monoidal structure in Cat_∞, the given by the cartesian product. First, we are going to define ∞-categories with symmetric monoidal structures, the normal ones, the cartesian monoidal ∞-categories and the ∞-categories with finite products, and remember that the last two are equivalents. Then, it will be defined by liftings from the ∞-categories above a functor from BC_2 to $E_\infty(\mathsf{Pr}^L)$, which exhibits Cat_∞ as an object of $E_\infty(\mathsf{Pr}^L)[C_2]$. Whereupon, we are going to precompose this functor with the smashing localizations appearing in [Nik13], and hence build the group completion in the case of the ∞-categories with duality in the following subsection.

The E_∞-algebras over the ∞-category Cat_∞ are precisely the symmetric monoidal small ∞-categories and together with the symmetric monoidal ∞-functors they form the ∞-category $Alg_{E_\infty}(\mathsf{Cat}_\infty)$, but we will denote it by $\mathsf{SymMonCat}_\infty$. The ∞-category of not necessarily small symmetric monoidal ∞-categories will be denoted $\widehat{\mathsf{SymMonCat}}_\infty$. We are going to consider two special cases inside of $\mathsf{SymMonCat}_\infty$ and $\widehat{\mathsf{SymMonCat}}_\infty$ to work with. The first are the cartesian monoidal categories, which in the normal category setting they are the symmetric monoidal categories where the product is the monoidal product. The second one is an abstraction of the last one, the infinity categories with finite products, but in fact the forgetful functor gives rise to an equivalence.

Definition 2.3.5. [Lur14, Def. 2.4.0.1] Let $\mathcal{C} \in \widehat{\mathsf{SymMonCat}}_\infty$, we will say that its symmetric monoidal structure is *cartesian* if the following conditions are satisfied.

1. The unit object $\mathbb{1}_\mathcal{C}$ is final.

2. For every pair of objects $C, D \in \mathcal{C}$, the canonical maps:

$$C \simeq C \otimes 1_\mathcal{C} \leftarrow C \otimes D \rightarrow 1_\mathcal{C} \otimes D \simeq D$$

   exhibits $C \otimes D$ as a product of C and D in $\mathcal{C}$.

In what follows, $\mathsf{SymMonCat}_\infty^\times \subset \mathsf{SymMonCat}_\infty$ will be the full subcategory spanned by the small Cartesian symmetric monoidal ∞-categories and $\widehat{\mathsf{SymMonCat}}_\infty^\times \subset \widehat{\mathsf{SymMonCat}}_\infty$ will means the no necessarily small case. Let $\mathsf{Cat}_\infty^{\mathrm{Cart}}$ be the subcategory of Cat_∞ of small ∞-categories with finite products and product-preserving functors, and similarly for $\widehat{\mathsf{Cat}}_\infty^{\mathrm{Cart}}$ for the no necessarily small case.

Theorem 2.3.4. *The canonical forgetful functors*

$$\mathsf{SymMonCat}_\infty^\times \rightarrow \mathsf{Cat}_\infty^{\mathrm{Cart}}$$

and

$$\widehat{\mathsf{SymMonCat}}_\infty^\times \rightarrow \widehat{\mathsf{Cat}}_\infty^{\mathrm{Cart}}$$

are equivalences.

Proof. This is [Lur14, Cor. 2.4.1.9]. $\square$

It was already defined the ∞-category of presentable ∞-categories and ∞-colimit preserving functors Pr^L in subsection 2.2.6. For $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \mathsf{Pr}^L$ an infinity functor $\mathcal{C} \times \mathcal{D} \to E$, from the cartesian product of $\mathcal{C}$ and $\mathcal{D}$ to $\mathcal{E}$, is bilinear if it preserves ∞-colimits in each variable. It exists an universal such bilinear form, the tensor product $\mathcal{C} \times \mathcal{D} \to \mathcal{C} \otimes \mathcal{D}$ in Pr^L ([Lur07a, Thm. 4.1.4]). Such a tensor product defines a symmetric monoidal ∞-category structure in Pr^L. An E_∞-algebra in Pr^L is a symmetric monoidal ∞-category whose underlying category is presentable and such that the tensor product preserves colimits separately in each variable. These E_∞-algebras will be denoted by $E_\infty\left(\mathsf{Pr}^L\right)$. We get a natural good faithful ∞-functor

$$E_\infty(\mathsf{Pr}^L) \to \widehat{\mathsf{SymMonCat}}_\infty.$$

We let $E_\infty^\times(\mathsf{Pr}^L)$ be the full subcategory of $E_\infty(\mathsf{Pr}^L)$ on the cartesian symmetric monoidal presentable ∞-categories. Taking into account the cartesian case and the logic inclusions we complete a commutative diagram

$$\begin{tikzcd}
E_\infty(\mathsf{Pr}^L) \arrow[r] & \widehat{\mathsf{SymMonCat}}_\infty & \\
E_\infty^\times(\mathsf{Pr}^L) \arrow[u] \arrow[r, "\varphi"] & \widehat{\mathsf{SymMonCat}}_\infty^\times \arrow[u] \arrow[r, "\sim"] & \widehat{\mathsf{Cat}}_\infty^{\mathrm{Cart}} .
\end{tikzcd}$$

Using this diagram we are going to lift the C_2-action on $\widehat{\mathsf{Cat}}_\infty^{\mathrm{Cart}}$ to get a C_2-action on Cat_∞. The C_2-action in $\widehat{\mathsf{Cat}}_\infty$ can be viewed as a functor $BC_2 \to \widehat{\mathsf{Cat}}_\infty$ which factors through $\widehat{\mathsf{Cat}}_\infty^{\mathrm{Cart}}$.

Thus we naturally get (up to a contractible space of choices) a functor $\psi\colon BC_2 \to \widehat{\mathsf{SymMonCat}}_\infty^\times$ which lifts (up to a contractible space of choices) to a functor $BC_2 \to E_\infty^\times(\mathsf{Pr}^L)$, since ψ factors through the good image of φ. Its prolongation

$$\psi'\colon BC_2 \to E_\infty(\mathsf{Pr}^L)$$

exhibits Cat_∞ as an object of $E_\infty(\mathsf{Pr}^L)[C_2]$.

We are going argue in a analogous way that in [Nik13] but taking in consideration the duality, they will be consider bimonoidal structures with duality. In [Nik13] were defined the infinite loop space machines by the use of universal properties instead of combinatorial models as in May's work. They proved that all the functors appearing in the group completion of a symmetric monoidal infinity category are smashing localizations in Pr^L. The local objects respect to such a localizations have an universal closed symmetric monoidal structure.

Lemma 9. Let $\mathcal{C}^{\otimes}$ be a closed symmetric monoidal presentable ∞-category, $\mathcal{C}^{\otimes} \in E_\infty(\mathsf{Pr}^L)$. The ∞-categories $\mathcal{C}_*$, $\mathrm{Mon}_{E_\infty}(\mathcal{C})$, $\mathrm{Grp}_{E_\infty}(\mathcal{C})$ and $\mathrm{Sp}(\mathcal{C})$ admit closed symmetric monoidal structures, which are uniquely determined by the requirement that the respective free functors from $\mathcal{C}$ are symmetric monoidal. Moreover, each of the following free functors also extends uniquely to a symmetric monoidal functor

$$\mathcal{C} \mapsto \mathcal{C}_*,\ \mathcal{C} \mapsto \mathrm{Mon}_{E_\infty}(\mathcal{C}),\ \mathcal{C} \mapsto \mathrm{Grp}_{E_\infty}(\mathcal{C}),\ \mathcal{C} \mapsto \mathrm{Sp}(\mathcal{C})$$

in Pr^L.

Proof. This is [Nik13, Thm. 5.1]. □

Remark. This is an special case of [Nik13, Prop. 3.9]. The main idea of the proof is that these functors are smashing Bousfield localizations, from Pr^L to Pr^L. This kind of localizations in Pr^L induce a unique symmetric monoidal structure in the local objects. And the infinity categories $\mathcal{C}_*$, $\mathrm{Mon}_{E_\infty}(\mathcal{C})$, $\mathrm{Grp}_{E_\infty}(\mathcal{C})$ and $\mathrm{Sp}(\mathcal{C})$ are the local objects.

Remark. In this lemma appear two different monoidal structures, the first one which is the cartesian monoidal structure used to define the E_∞-monoids in $\mathcal{C}$ and the second one is the closed symmetric monoidal structure in $\mathcal{C}$ which induces a closed symmetric monoidal structure in $\mathrm{Mon}_{E_\infty}(\mathcal{C})$.

Let us denote by $\mathcal{L}$ any of these we get induced functors

$$E_\infty(\mathsf{Pr}^L) \xrightarrow{\mathcal{L}} E_\infty(\mathsf{Pr}^L)$$

which are localizations.

Precomposing with ψ' yields C_2-objects $\mathcal{L}(\mathsf{Cat}_\infty)$ in $E_\infty(\mathsf{Pr}^L)$, in particular

$$\mathsf{SymMonCat}_\infty \simeq \mathrm{Mon}_{E_\infty}(\mathsf{Cat}_\infty) \in E_\infty(\mathsf{Pr}^L)[C_2].$$

Remark. As we did with Cat_∞ one can exhibit any cartesian closed presentable ∞-category with C_2-action as object of $E_\infty(\mathsf{Pr}^L)[C_2]$, in particular $\mathsf{Cat} \in E_\infty(\mathsf{Pr}^L)[C_2]$. Applying $\mathcal{L}$ to Cat yields $\mathsf{SymMonCat} \in E_\infty(\mathsf{Pr}^L)[C_2]$.

To work with E_∞-semiring structures we are going to consider the E_∞-algebras on $\mathsf{SymMonCat}$ and to see that it still have a C_2-action, and that such an action commutes with the symmetric monoidal structures.

For $\mathcal{C} \in E_\infty(\mathsf{Pr}^L)$ we let $E_\infty(\mathcal{C}) := \mathrm{Alg}_{E_\infty}(\mathcal{C})$, the E_∞-algebras over such a presentable infinity category. The assignment $\mathcal{C} \mapsto E_\infty(\mathcal{C})$ defines a functor $e\colon E_\infty(\mathsf{Pr}^L) \to \mathsf{Pr}^L$. Precomposing this functor with the functor $BC_2 \to E_\infty(\mathsf{Pr}^L)$ corresponding to the C_2-action on $\mathsf{SymMonCat}_\infty$ yields

$$\mathrm{Rig}_{E_\infty}(\mathsf{Cat}_\infty) := E_\infty(\mathsf{SymMonCat}_\infty) \in \mathsf{Pr}^L[C_2].$$

Therefore the ∞-category of the E_∞-semirings infinity categories is a presentable category with a C_2-action. The ∞-category $E_\infty(\mathsf{Pr}^L)$ has (small) limits so we can take homotopy fixed points. Thus we get an object

$$\mathsf{SymMonCat}_\infty^{hC_2} \in E_\infty(\mathsf{Pr}^L).$$

The following proposition tells us that the C_2-action respects the second monoidal structure, i.e, taking homotopy fixed points from the E_∞-semirings infinity categories is equivalent to take the homotopy fixed points and then the second monoidal structure.

Propositon 5. *There is a natural equivalence*

$$E_\infty(\mathsf{SymMonCat}_\infty^{hC_2}) \simeq \mathrm{Rig}_{E_\infty}(\mathsf{Cat}_\infty)^{hC_2}$$

in Pr^L.

Proof. This follows from the fact that e preserves limits: Note first that it is sufficient to show that the functor $e'\colon E_\infty(\widehat{\mathsf{Cat}}_\infty) \to \widehat{\mathsf{Cat}}_\infty$, $\mathcal{C} \mapsto E_\infty(\mathcal{C})$, preserves limits, since the inclusions $\mathsf{Pr}^L \to \widehat{\mathsf{Cat}}_\infty$ and $E_\infty(\mathsf{Pr}^L) \to E_\infty(\widehat{\mathsf{Cat}}_\infty)$ preserve limits.

The functor e' can be factored as

$$E_\infty(\widehat{\mathsf{Cat}}_\infty) \to \widehat{\mathrm{Op}}_\infty \to \widehat{\mathsf{Cat}}_\infty,$$

where $\widehat{\mathrm{Op}}_\infty$ is the ∞-category of not necessarily small ∞-operads, the first functor the natural inclusion and the second functor assigns to an ∞-operad $\mathcal{O}$ its ∞-category of commutative algebra objects in $\mathcal{O}$. But both functors have left adjoints, the first the symmetric monoidal envelope functor and the second the functor $\mathcal{C} \mapsto \mathcal{C} \otimes \mathrm{Comm}$, where $\otimes$ denotes the natural action of ∞-categories on ∞-operads and Comm the commutative operad. This shows that e preserves limits. $\square$

Remark. Similarly to Proposition 5 we have an equivalence

$$\mathsf{SymMonCat}_\infty^{hC_2} \simeq E_\infty(\mathsf{Cat}_\infty^{hC_2})$$

in Pr^L, which can be promoted to an equivalence in $E_\infty(\mathsf{Pr}^L)$, since

$$E_\infty(\mathsf{Cat}_\infty^{hC_2}) \simeq \mathrm{Mon}_{E_\infty}(\mathsf{Cat}_\infty^{hC_2})$$

, where $\mathsf{Cat}_\infty^{hC_2}$ has the cartesian symmetric monoidal structure.

As the group completion appearing in the subsection 2.3.5 use the special case of Spc, we will see that the inclusion in Cat_∞ induces an ∞-functor at the level of the E_∞-semirings with a C_2-action. The constructions done in this section can also be done with Spc instead of Cat_∞, now giving Spc as an object of $E_\infty(\mathsf{Pr}^L)[C_2]$. And these resulting constructions are compatibles with the inclusion $\mathsf{Spc} \hookrightarrow \mathsf{Cat}_\infty$ in Pr^L.

In particular there is a map $\mathsf{Spc} \to \mathsf{Cat}_\infty$ in $E_\infty(\mathsf{Pr}^L)[C_2]$, yielding (by applying the smashing localization $\mathrm{Mon}_{E_\infty}(-)$) a map

$$\mathrm{Mon}_{E_\infty}(\mathsf{Spc}) \to \mathsf{SymMonCat}_\infty$$

in $E_\infty(\mathsf{Pr}^L)[C_2]$.

Applying again the functor $e\colon E_\infty(\mathsf{Pr}^L) \to \mathsf{Pr}^L$, $C \mapsto E_\infty(C)$ we get a map

$$\mathrm{Rig}_{E_\infty}(\mathsf{Spc}) = E_\infty(\mathrm{Mon}_{E_\infty}(\mathsf{Spc})) \to \mathrm{Rig}_{E_\infty}(\mathsf{Cat}_\infty)$$

in $\mathsf{Pr}^L[C_2]$, which finally gives by taking limits in Pr^L a map

$$\mathrm{Rig}_{E_\infty}(\mathsf{Spc})^{hC_2} \to \mathrm{Rig}_{E_\infty}(\mathsf{Cat}_\infty)^{hC_2}$$

in Pr^L.

Remark. We note that there are equivalences

$$\mathrm{Rig}_{E_\infty}(\mathsf{Spc}[C_2]) \simeq E_\infty(\mathrm{Mon}_{E_\infty}(\mathsf{Spc})[C_2]) \simeq \mathrm{Rig}_{E_\infty}(\mathsf{Spc})[C_2] \simeq \mathrm{Rig}_{E_\infty}(\mathsf{Spc})^{hC_2}.$$

2.3.5 Group completion and spectra.

Here we define the K-theory functor for the Hermitian case. Its definition is simple using the functors induced by the core and the last subsection. We are going see that such Hermitian K-theory functor is lax symmetric monoidal and thus it respects the multiplicative structures that we are interested on, refining to a functor from E_∞-semiring infinity categories with dualities to E_∞-ring spectra:

$$K_h\colon \mathrm{Rig}_{E_\infty}(\mathsf{Cat}_\infty)^{hC_2} \to E_\infty(\mathrm{Sp})$$

In Definition 2.2.10 we defined the core for categories and infinity categories. The functor $(-)^{\sim}\colon \mathsf{Cat}_\infty \to \mathsf{Spc}$ remains the right adjoint to the map $\mathsf{Spc} \to \mathsf{Cat}_\infty$ in Pr^L. They will be denoted by the same symbol all the induced functors after taking E_∞-algebras, e.g, the right adjoint to $\mathrm{Mon}_{E_\infty}(\mathsf{Spc}) \to \mathsf{SymMonCat}_\infty$.

The direct sum K-theory functor ([Nik13, Def. 8.3]) is the composition

$$K\colon \mathsf{SymMonCat}_\infty \xrightarrow{(-)^{\sim}} \mathrm{Mon}_{E_\infty}(\mathsf{Spc}) \to \mathrm{Grp}_{E_\infty}(\mathsf{Spc}) \to \mathrm{Sp},$$

which is lax symmetric monoidal [Nik13, Thm. 8.6].

Its effect on rig-categories (E_∞-objects in $\mathsf{SymMonCat}_\infty$) can be described by the composition

$$\mathrm{Rig}_{E_\infty}(\mathsf{Cat}_\infty) \xrightarrow{(-)^{\sim}} \mathrm{Rig}_{E_\infty}(\mathsf{Spc}) \to \mathrm{Ring}_{E_\infty}(\mathsf{Spc}) \to E_\infty(\mathrm{Sp}),$$

where $\mathrm{Ring}_{E_\infty}(\mathsf{Spc}) \coloneqq E_\infty(\mathrm{Grp}_{E_\infty}(\mathsf{Spc}))$.

We also denote by $(-)^{\sim}$ a right adjoint of the map

$$\alpha\colon \mathrm{Mon}_{E_\infty}(\mathsf{Spc})[C_2] \simeq \mathrm{Mon}_{E_\infty}(\mathsf{Spc})^{hC_2} \to \mathsf{SymMonCat}_\infty^{hC_2}$$

in Pr^L.

Definition 2.3.6. We define the direct sum Hermitian K-theory functor as the composition

$$K_h\colon \mathsf{SymMonCat}_\infty^{hC_2} \xrightarrow{(-)^{\sim}} \mathrm{Mon}_{E_\infty}(\mathsf{Spc})[C_2] \xrightarrow{(-)^{hC_2}}$$
$$\mathrm{Mon}_{E_\infty}(\mathsf{Spc}) \to \mathrm{Grp}_{E_\infty}(\mathsf{Spc}) \to \mathrm{Sp}.$$

Theorem 2.3.5. *K_h is lax symmetric monoidal and it refines to a functor*

$$\mathrm{Rig}_{E_\infty}(\mathsf{Cat}_\infty)^{hC_2} \to E_\infty(\mathrm{Sp})$$

Proof. Since α is in fact a map in $E_\infty(\mathsf{Pr}^L)$, the first map in the definition of K_h is lax symmetric monoidal. The second map is the right adjoint to the symmetric monoidal functor

$$\mathrm{Mon}_{E_\infty}(\mathsf{Spc}) \to \mathrm{Mon}_{E_\infty}(\mathsf{Spc})[C_2],$$

which sends an object to the object endowed with the trivial C_2-action. The third and fourth maps in the definition of K_h are symmetric monoidal. $\square$

Remark. Note that the effect of K_h on E_∞-objects (semiring categories with duality) can be described by the composition

$$\mathrm{Rig}_{E_\infty}(\mathsf{Cat}_\infty)^{hC_2} \xrightarrow{(-)^{\sim}} \mathrm{Rig}_{E_\infty}(\mathsf{Spc})[C_2] \xrightarrow{(-)^{hC_2}}$$
$$\mathrm{Rig}_{E_\infty}(\mathsf{Spc}) \to \mathrm{Ring}_{E_\infty}(\mathsf{Spc}) \to E_\infty(\mathrm{Sp}).$$

2.3.6 Recognition principle.

In this subsection we have defined the hermitian K-theory functor for a rig category with duality and we can apply such result for any object of $\mathrm{Rig}_{E_\infty}(\mathsf{Cat}_\infty)^{hC_2}$. In this subsection we are going to give a variant of the recognition principle [Nik13, Thm. 8.8]. This section is heavily inspired by the proof of [Nik13, Thm. 8.8]. Our recognition principle states that any symmetric monoidal infinity category with duality $\mathcal{C} \in E_\infty(\mathsf{Cat}_\infty)^{hC_2}$ such that its underling category $\mathcal{C}_h$ is preadditive and such that the monoidal product $\otimes_h \colon \mathcal{C}_h \times \mathcal{C}_h \to \mathcal{C}_h$ preserves finite coproducts separately in each variable, is canonically an object of $\mathrm{Rig}_{E_\infty}(\mathsf{Cat}_\infty)^{hC_2}$ and therefore we can apply our functor K_h. In short, denoting by $E_\infty^{\mathrm{preadd}}(\mathsf{Cat}_\infty)^{hC_2}$ the infinity categories verifying the requirements just above, we get a functor

$$E_\infty^{\mathrm{preadd}}(\mathsf{Cat}_\infty)^{hC_2} \to E_\infty(\mathrm{Sp}).$$

As we will see in the next subsection this covers our case in the motivic setting.

Remark. The symbol $\otimes_h$ denotes the second symmetric monoidal structure, while $\coprod_{\mathcal{C}}$ is the first symmetric monoidal structure.

We denote by $\mathsf{Cat}_\infty^\Sigma$ the subcategory of Cat_∞ of ∞-categories which admit finite coproducts and coproduct preserving functors. $\mathsf{Cat}_\infty^\Sigma$ is presentable and has a closed symmetric monoidal structure such that the left adjoint to the inclusion $\mathsf{Cat}_\infty^\Sigma \subset \mathsf{Cat}_\infty$ is symmetric monoidal (see the references in the proof of [Nik13, Thm. 8.8] to [Lur14]). $\mathsf{Cat}_\infty^\Sigma$ is preadditive (proof of [Nik13, Thm. 8.8]), and the right adjoint to the composition

$$\beta \colon \mathsf{SymMonCat}_\infty \simeq \mathrm{Mon}_{E_\infty}(\mathsf{Cat}_\infty) \to \mathrm{Mon}_{E_\infty}(\mathsf{Cat}_\infty^\Sigma) \simeq \mathsf{Cat}_\infty^\Sigma$$

is the full embedding

$$\mathsf{Cat}_\infty^\Sigma \to \mathsf{SymMonCat}_\infty$$

which sends an ∞-category with coproducts to the corresponding coCartesian symmetric monoidal ∞-category (loc. cit.). Thus β is a symmetric monoidal localization.

The map β and its right adjoint are not compatible with the C_2-action on $\mathsf{SymMonCat}_\infty$, therefore we introduce another localization.

We let $i \colon \mathsf{Cat}_\infty^{\mathrm{preadd}} \subset \mathsf{Cat}_\infty^\Sigma$ be the full subcategory of preadditive categories, i.e. those categories which also have finite products, are pointed and in which for any two objects X and Y the natural map $X \coprod Y \to X \times Y$ is an equivalence.

Propositon 6. *The functor i has both a left and a right adjoint. The right adjoint is informally given by the assignment $\mathcal{C} \mapsto \mathrm{Mon}_{E_\infty}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$.*

Proof. We first note that the assignment $\mathcal{C} \mapsto \mathrm{Mon}_{E_\infty}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$ can be refined to a functor

$$r\colon \mathsf{Cat}_\infty^\Sigma \to \mathsf{Cat}_\infty^{\mathrm{preadd}}$$

, where we use in particular that $\mathrm{Mon}_{E_\infty}(\mathcal{C}^{\mathrm{op}})$ is preadditive for $\mathcal{C} \in \mathsf{Cat}_\infty^\Sigma$ by [Nik13, Cor. 2.5 (i)].

Moreover $r \circ i$ is naturally equivalent (say via φ) to the identity by [Nik13, Prop. 2.3 (iv)]. For $C \in \mathsf{Cat}_\infty^\Sigma$ and $\mathcal{D} \in \mathsf{Cat}_\infty^{\mathrm{preadd}}$ we have a sequence of equivalences of ∞-categories

$$\mathrm{Fun}^{\amalg}(\mathcal{D}, r(\mathcal{C})) \simeq \mathrm{Fun}^{\amalg}(\mathcal{D}, \mathrm{Mon}_{E_\infty}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}) \simeq \mathrm{Fun}^{\Pi}(\mathcal{D}^{\mathrm{op}}, \mathrm{Mon}_{E_\infty}(\mathcal{C}^{\mathrm{op}}))^{\mathrm{op}}$$

$$\simeq \mathrm{Fun}^{\Pi}(\mathcal{D}^{\mathrm{op}}, \mathcal{C}^{\mathrm{op}})^{\mathrm{op}} \simeq \mathrm{Fun}^{\amalg}(\mathcal{D}, \mathcal{C}),$$

where the third equivalence follows from [Nik13, Cor. 2.5 (iii)]. Here $\mathrm{Fun}^{\amalg}$ resp. Fun^{Π} denotes finite coproducts resp. finite products preserving functors. It follows that φ exhibits r as a right adjoint to i.

Thus r is a co-localization, in particular the co-local objects in $\mathsf{Cat}_\infty^\Sigma$, i.e. $\mathsf{Cat}_\infty^{\mathrm{preadd}}$, are closed under colimits, thus $\mathsf{Cat}_\infty^{\mathrm{preadd}}$ has all small colimits and they are computed in $\mathsf{Cat}_\infty^\Sigma$.

Note that r preserves filtered colimits ($(-)^{\mathrm{op}}$ and $\mathrm{Mon}_{E_\infty}(-)$ preserve them). It follows that $\mathsf{Cat}_\infty^{\mathrm{preadd}}$ is presentable, thus, since i preserves colimits and limits (note that a limit of preadditive categories in $\mathsf{Cat}_\infty^\Sigma$ is preadditive), i also has a left adjoint. □

Let us denote a left adjoint of i by l.

Propositon 7. *The localization $i \circ l$ of $\mathsf{Cat}_\infty^\Sigma$ is symmetric monoidal.*

Proof. We have to check that the tensor product of a local equivalence f in $\mathsf{Cat}_\infty^\Sigma$ with an object $\mathcal{C} \in \mathsf{Cat}_\infty^\Sigma$ is again a local equivalence. By mapping into local objects this is equivalent to the fact that for each preadditive $\mathcal{Z} \in \mathsf{Cat}_\infty^\Sigma$ the internal Hom $\underline{\mathrm{Hom}}(\mathcal{C}, \mathcal{Z})$ is preadditive, which can be verified directly. □

Thus $\mathsf{Cat}_\infty^{\mathrm{preadd}}$ is closed symmetric monoidal and l and $l \circ \beta$ are also symmetric monoidal localizations.

This time $l \circ \beta$ is compatible with the C_2-action on $\mathsf{SymMonCat}_\infty$ since its right adjoint

$$\mathsf{Cat}_\infty^{\mathrm{preadd}} \to \mathsf{SymMonCat}_\infty$$

is.

Theorem 2.3.6. *The functor*

$$l \circ \beta \colon \mathsf{SymMonCat}_\infty \to \mathsf{Cat}_\infty^{\mathrm{preadd}}$$

is naturally a map in $E_\infty(\mathsf{Pr}^L)[C_2]$.

Proof. Apply [Lur14, Lemma 2.2.4.11] to the BC_2-coCartesian family

$$p \colon S^\otimes \to BC_2 \times \mathrm{N}\mathsf{Fin}_*$$

corresponding to the functor $BC_2 \to \widehat{\mathsf{SymMonCat}}_\infty$ encoding the C_2-action on the symmetric monoidal ∞-category $\mathsf{SymMonCat}_\infty$. $\square$

Corollary 2.3.7. *There is a natural symmetric monoidal functor*

$$\mathsf{SymMonCat}_\infty^{hC_2} \to (\mathsf{Cat}_\infty^{\mathrm{preadd}})^{hC_2}$$

in Pr^L *with lax symmetric monoidal right adjoint*

$$(\mathsf{Cat}_\infty^{\mathrm{preadd}})^{hC_2} \to \mathsf{SymMonCat}_\infty^{hC_2}$$

in Pr^R.

We denote by $E_\infty^\Sigma(\mathsf{Cat}_\infty)$ the following subcategory of $E_\infty(\mathsf{Cat}_\infty)$: The objects are symmetric monoidal ∞-categories such that the underlying ∞-categories have finite coproducts and the given tensor products preserve these coproducts separately in each variable, and the maps are symmetric monoidal functors such that the underlying functors preserve finite coproducts.

We denote by $E_\infty^{\mathrm{preadd}}(\mathsf{Cat}_\infty) \subset E_\infty^\Sigma(\mathsf{Cat}_\infty)$ the full subcategory of those symmetric monoidal categories in $E_\infty^\Sigma(\mathsf{Cat}_\infty)$ whose underlying categories are preadditive.

Propositon 8. *The functor*

$$E_\infty(\mathsf{Cat}_\infty^\Sigma) \to E_\infty(\mathsf{Cat}_\infty)$$

factors through $E_\infty^\Sigma(\mathsf{Cat}_\infty)$*, the functor*

$$E_\infty(\mathsf{Cat}_\infty^{\mathrm{preadd}}) \to E_\infty(\mathsf{Cat}_\infty)$$

factors through $E_\infty^{\mathrm{preadd}}(\mathsf{Cat}_\infty)$*, and in the naturally induced commutative diagram*

$$\begin{tikzcd}
E_\infty(\mathsf{Cat}_\infty) \arrow[d, "\|"] & E_\infty(\mathsf{Cat}_\infty^\Sigma) \arrow[l] \arrow[d, "\wr"] & E_\infty(\mathsf{Cat}_\infty^{\mathrm{preadd}}) \arrow[l] \arrow[d, "\wr"] \\
E_\infty(\mathsf{Cat}_\infty) & E_\infty^\Sigma(\mathsf{Cat}_\infty) \arrow[l] & E_\infty^{\mathrm{preadd}}(\mathsf{Cat}_\infty) \arrow[l]
\end{tikzcd}$$

the vertical maps are equivalences.

Proof. The first statements are clear, the middle vertical equivalence is [Lur14, Rmk. 4.8.1.9] and the right vertical equivalence follows from this equivalence. □

Propositon 9. *The functors*

$$E_\infty(\mathsf{Cat}_\infty) \to E_\infty(\mathsf{SymMonCat}_\infty) \to E_\infty(\mathsf{Cat}_\infty^{\mathrm{preadd}})$$

are naturally maps in $\mathsf{Pr}^L[C_2]$*, and its right adjoints*

$$E_\infty(\mathsf{Cat}_\infty^{\mathrm{preadd}}) \to E_\infty(\mathsf{SymMonCat}_\infty) \to E_\infty(\mathsf{Cat}_\infty)$$

are naturally maps in $\mathsf{Pr}^R[C_2]$.

Proof. The first statement follows from the fact that the functors

$$\mathsf{Cat}_\infty \to \mathsf{SymMonCat}_\infty \to \mathsf{Cat}_\infty^{\mathrm{preadd}}$$

are naturally maps in $E_\infty(\mathsf{Pr}^L)[C_2]$, and the second statement follows from the first. □

Propositon 10. *The* C_2*-action on* $E_\infty(\mathsf{Cat}_\infty)$ *induces a* C_2*-action on the* ∞*-category* $E_\infty^{\mathrm{preadd}}(\mathsf{Cat}_\infty)$*, and the induced functor*

$$E_\infty^{\mathrm{preadd}}(\mathsf{Cat}_\infty)^{hC_2} \to E_\infty(\mathsf{Cat}_\infty)^{hC_2}$$

is a good faithful functor whose good image consists of those objects and morphisms in $E_\infty(\mathsf{Cat}_\infty)^{hC_2}$ *whose underlying objects and morphisms in* $E_\infty(\mathsf{Cat}_\infty)$ *are in* $E_\infty^{\mathrm{preadd}}(\mathsf{Cat}_\infty)$.

Propositon 11. *The map*

$$E_\infty(\mathsf{Cat}_\infty^{\mathrm{preadd}}) \to E_\infty(\mathsf{Cat}_\infty)$$

in $\mathsf{Pr}^R[C_2]$ *factors via an equivalence*

$$E_\infty(\mathsf{Cat}_\infty^{\mathrm{preadd}}) \xrightarrow{\sim} E_\infty^{\mathrm{preadd}}(\mathsf{Cat}_\infty)$$

in $\mathsf{Pr}^R[C_2]$ *and* $\mathsf{Pr}^L[C_2]$.

Theorem 2.3.8. *Let $\mathcal{C} \in E_\infty(\mathsf{Cat}_\infty)^{hC_2} \simeq \mathsf{SymMonCat}_\infty^{hC_2}$, such that the underlying ∞-category $\mathcal{C}_u$ is preadditive and such that the monoidal product*

$$\otimes\colon \mathcal{C}_u \times \mathcal{C}_u \to \mathcal{C}_u$$

preserves finite coproducts separately in each variable. Then $\mathcal{C}$ is canonically an object of $\mathrm{Rig}_{E_\infty}(\mathsf{Cat}_\infty)^{hC_2}$, i.e. a rig-category (or semiring ∞-category or bimonoidal ∞-category) with duality.

Proof. By Proposition 10 $\mathcal{C}$ is naturally an object of $E_\infty^{\text{preadd}}(\mathsf{Cat}_\infty)^{hC_2}$, so by Proposition 11 also an object of $E_\infty(\mathsf{Cat}_\infty^{\text{preadd}})^{hC_2}$. Mapping $\mathcal{C}$ further along the functor

$$E_\infty(\mathsf{Cat}_\infty^{\text{preadd}})^{hC_2} \to E_\infty(\mathsf{SymMonCat}_\infty)^{hC_2}$$

(see Proposition 9) in Pr^R yields the desired object. $\square$

Combining Theorem 2.3.8 with the last composition in Section 2.3.4 yields a map

$$E_\infty^{\text{preadd}}(\mathsf{Cat}_\infty)^{hC_2} \to E_\infty(\mathrm{Sp}).$$

2.3.7 The spectrum KQ.

In this short subsection we are going to assemble what was said along this section until now with our construction for the Hermitian K-theory in the motivic setting. Remember from Section 1.3 that we are interested in the group completion of the bipermutative category with duality of algebraic vector bundles. As it was said our construction is able just for affine schemes, but we are going to assume the Theorem 1.3.1 implicitly.

Let write $Vb(X) = (Vb(X), \mathrm{Hom}_{\mathcal{O}_X}(-, \mathcal{O}_X), \nu_X)$ for our category with duality. This category has a bipermutative structure described in Section 1.3.1. Moreover, this is an additive category, concretely an additive category with duality (Definition 1.2.13). In subsection 2.3.3 we have seen that usual categories with duality can be interpretate as infinity categories with duality. We can consider the category $Vb(X)$ as an object of $\mathrm{Fun}^{C_2}(I, \mathsf{Cat})$ by Lemma 2.3.3, then we can apply the nerve to such a category and the functor

$$\epsilon\colon \mathrm{NFun}^{C_2}(I, \mathsf{Cat}^1) \to \mathsf{Cat}_\infty^{hC_2}$$

Our category is in fact a symmetric bimonoidal category, so we start with a rig additive category with duality and then we apply ϵ. If we remember how was it constructed the functor ϵ this is nothing else that the composition of the nerve followed by the inclusion of $\widehat{\mathsf{Cat}}$ as a full subcategory of $\widehat{\mathsf{Cat}}_\infty$. This preserves our monoidal structures, and the additive structure. So the resulting infinity category is an object of $E_\infty^{\text{preadd}}(\mathsf{Cat}_\infty)^{hC_2}$ and we can apply the Recognition Principle in 2.3.6 followed by the Hermitian direct sum K-theory in the multiplicative case

$$E_\infty^{\text{preadd}}(\mathsf{Cat}_\infty)^{hC_2} \to \mathrm{Rig}_{E_\infty}(\mathsf{Cat}_\infty)^{hC_2} \xrightarrow{(-)^\sim} \mathrm{Rig}_{E_\infty}(\mathsf{Spc})[C_2] \xrightarrow{(-)^{hC_2}}$$

$$\mathrm{Rig}_{E_\infty}(\mathsf{Spc}) \to \mathrm{Ring}_{E_\infty}(\mathsf{Spc}) \to E_\infty(\mathrm{Sp})$$

Remark. If we do not consider the multiplicative structure, when we have applied all the functors until $\mathrm{Grp}_{E_\infty}(\mathsf{Spc})$, before get the associated spectrum, we get the same object that Hornbostel with the plus construction, $(i\mathcal{C}_h)^+$.

The resulting $\mathbb{S}^1$-spectrum to each bipermutative structure in the presheaf of bipermutative structures defined in Section 1.3.1 produces the presheaf of E_∞-ring spectra

$$\mathbf{KQ}\colon\ (Sm\,|_S)^\circ \ \longrightarrow\ E_\infty\text{-rings}$$

Here it is omitted again the pass from affine schemes to schemes in $Sm\,|_S$. The presheaf should be defined from affine schemes but we use subsection 1.3.2 implicitly, more concretely Theorem 1.3.1.

Remark. This spectrum is connective and has an E_∞-ring structure, this is a difference with the models for motivic hermitian K-theory described in Section 1.3.3. But it is not our definitive spectrum since it is a $\mathbb{S}^1$-spectrum. We will obtain the E_∞-ring spectrum $\mathbf{KQ}_{\mathbb{P}^1}$ in Section 2.5.

2.4 From bipermutative categories to E_∞-ring spectra: The spectrum KH.

In this section we are going to construct our presheaf **KH**. This is a presheaf of E_∞-rings of connective $\mathbb{S}^1$-spectra. We will revise the construction of E_∞-rings for a given bipermutative category. Remember form definition 1.3.1 that we are considering the special case of the bipermutative structure given by the direct sum and the tensor product over algebraic vector bundles. This section is mainly a review of the May's theory for such categories. Therefore, some parts of the construction will be just sketched since it is known construction and we just need to apply it to our case. We will use frequently the background about operads in Section 2.1, but with the special treatment for topological categories explained in subsection 2.4.1.

For a permutative category we can define its K-theory spectrum via the corresponding E_∞-space, obtaining its group completion and getting the associated spectrum. This is an additive infinite loop space machine, and since it is part of the construction for bipermutative categories it will reviewed in advance in subsection 2.4.1, while the bipermutative case will be treat in 2.4.2. There are other more general constructions nowadays as [EM09] or [EM06] among others, but will do not need such generality. In our case we start with a bipermutative category with two different operations and we want to preserve the multiplicative structure. Then the way to construct the Hermitian K-theory spectrum will be via the corresponding E_∞-ring space, and next obtaining a ringlike E_∞*-ring space* which gives rise to the desired E_∞-ring connective spectrum. This spectrum will be denoted by **KH**. Unlike the subsection 2.1.2, now we are going to work with reduced operads, the exact definitions for E_∞*-spaces* and E_∞*-ring spaces* in our case will be given in subsection 2.4.1. In the seventies they were developed different infinite loop space machine theories, the most renowned are due to Boardman and Vogt, Segal, and May. We will use the work of the last one to carry out our Hermitian K-theory spectrum construction in a combinatorial way in Section 2.4.2. Using [May09a] we will construct the associated E_∞*-ring space* for the bipermutative category of algebraic vector bundles over X with direct sum and the tensor product. To procure the ring completion for this E_∞*-ring space* we will use [May09b], and we will use it also to get connective spectrum from it, obtaining the corresponding Hermitian K-theory presheaf of spectra over $Sm\,|_S$

$$\mathbf{KH}\colon\ (Sm\,|_S)^\circ\ \longrightarrow\ E_\infty\text{-rings}$$

Remark. As for the presheaf **KQ** in Section 2.3.7 this is a presheaf of $\mathbb{S}^1$-spectra and even if it has an E_∞-ring structure this not our definitive presheaf. The delooping construction in Section 2.5 will produce our desired presheaf of $\mathbb{P}^1$-spectra with an E_∞-ring structure.

2.4.1 The E_∞-space of a permutative category.

Before performing the following construction we must to specify what we mean by E_∞ operads, E_∞*-spaces* and E_∞ ring spaces. In particular, when changing from bipermutative categories to E_∞ ring spaces, we are going to use topological categories. Remember from subsection 2.1.2 that Comm denotes the operad which has by algebras the commutative monoids. This operad is usually the one with all its objects the tensor unit (the point in the category of sets, Ex. 2.1.3). The idea behind the E_∞ operads is that its algebras should be the commutative monoids up to all higher homotopies. In the topological context an E_∞ operad is an operad which is isomorphic to Comm in the homotopy category of topological operads. Then, an E_∞ operad in topological spaces is an operad which is contractible (Def. 2.1.9). Normally it is also required to be cofibrant, so in particular, this imposes the condition that the action of the symmetric group in each degree is free. Hereon we will use the May's definition.

Definition 2.4.1. An operad $\mathcal{O}$ is an E_∞ *operad* if $\mathcal{O}(j)$ is contractible for all $j \in \mathbb{N}$, Σ_j acts freely and moreover it is a reduced operad, i.e, $\mathcal{O}(0)$ is a point. Therefore, an E_∞*-space* is nothing but an algebra over an E_∞ operad.

To define an E_∞*-ring space* we need to consider two actions from two different E_∞ operads at the same time. Let $\mathcal{G}$ and $\mathcal{P}$ be two E_∞ operads. We want to get the algebras over the pair $(\mathcal{G}, \mathcal{P})$ of operads, such that one should think of $\mathcal{G}$ as parametrizing addition and $\mathcal{P}$ as parametrizing multiplication. We have to interrelate both induced operations. To this end, May defined the action of an operad on other.

Definition 2.4.2. An action of $\mathcal{P}$ on $\mathcal{G}$ consists of maps:

$$\lambda : \mathcal{P}(k) \times \mathcal{G}(j_1) \times \mathcal{G}(j_k) \to \mathcal{G}(j_1 \ldots j_k)$$

for $k \geq 0$ and $j_i \geq 0$ that satisfy equivariant, unit, and distributivity properties.

Then, an E_∞*-ring space* is nothing but an algebra over a pair $(\mathcal{G}, \mathcal{P})$ of E_∞ operads such that there is an action of $\mathcal{P}$ over $\mathcal{G}$. It should be noted that confusion could arise since the name "ring" is not completely honest because we don't have negatives; we didn't do the completion yet and maybe it should be called E_∞*-semiring* space. We will continue to use E_∞*-ring space* and we hope that this may no cause confusion.

The idea behind the construction of E_∞ ring spaces from bipermutative categories is the same that the construction for permutative categories and E_∞*-spaces.* The difference with the case of permutative categories lies in the fact that we have to take into consideration two E_∞ operads and their respective actions at the same time. For E_∞*-spaces*, the May's construction introduce an intermediary between the $\mathcal{F}$*-spaces* (which are the classic Segal's Γ*-spaces* from 2.1) and the E_∞*-spaces*, the category of operators for an operad. Below, we are going to give the main steps in the permutative case, and later for bipermutative. However, we have to revise some definitions before. We will use the notation given in [May09a].

Definition 2.4.3. Let $\mathcal{F}$ be the category of finite pointed sets ($\mathcal{F}in_*$) and Π the subcategory of based functions $\phi\colon \mathbf{m} \to \mathbf{n}$ such that $\mid \phi^{-1}(j) \mid \leq 1$ for $1 \leq j \leq n$. For an injective map $\phi\colon \mathbf{m} \to \mathbf{n}$ in $\mathcal{F}$ we will denote by Σ_ϕ the subgroup $\Sigma_\phi \subset \Sigma_n$ such that $\sigma(Im(\phi)) = Im(\phi)$.

During the proof May uses the category of operators of an operad, and he compares the special spaces over them. The definition of special spaces will be remember below. Let us remember the definition of category of operators given in May's paper [May09a].

Definition 2.4.4. We say that a topological category $\mathcal{D}$ with objects the pointed sets is a *category of operators* if the natural functor $\Pi \hookrightarrow \mathcal{F}$ factors through an inclusion $\Pi \hookrightarrow \mathcal{D}$ and a surjection $\epsilon \colon \mathcal{D} \to \mathcal{F}$, both of which restrict to the identity on objects. Moreover, we require the maps $\mathcal{D}(\mathbf{p}, \mathbf{m}) \to \mathcal{D}(\mathbf{p}, \mathbf{n})$ to be Σ_ϕ-cofibrant, for $\phi \colon \mathbf{m} \to \mathbf{n}$ injective. The maps between categories of operators are given by the continuous functors over $\mathcal{F}$ and under Π.

Remark. Every operad $\mathcal{O}$ gives rise a category of operators $\hat{\mathcal{O}}$, where the objects are the based sets $\mathbf{n}$ for $n \geq 0$ and the space of morphisms $\mathbf{m} \to \mathbf{n}$ are given by

$$\hat{\mathcal{O}}(\mathbf{m}, \mathbf{n}) = \coprod_{\phi \in \mathcal{F}(\mathbf{m}, \mathbf{n})} \prod_{1 \leq j \leq n} \mathcal{O}\left(\mid \phi^{-1}(j) \mid \right)$$

The categories of operators coming from operads is the main tool in the May's construction. The usefulness of these categories comes from two important fact: first, the equivalence between two of these categories induce an equivalence in the special spaces over these categories, and second, at the homotopy level these spaces correspond to the spaces over the original operads. Let us denote by $\mathcal{U}$ the category of compactly generated unbased spaces. We have the following definitions from [May09a].

Definition 2.4.5. Let $\hat{\mathcal{O}}$ be the category of operators given by the operad $\mathcal{O}$. We say that Y is a *$\hat{\mathcal{O}}$-space in the category $\mathcal{U}$* if it is a continuous functor $\hat{\mathcal{O}} \to \mathcal{U}$, $\mathbf{n} \mapsto Y_n$. We call it *special* if the next two conditions are satisfied:

1. Y_0 is aspherical, i.e, equivalent to a point.
2. The maps $\delta \colon Y_n \to Y_1^n$ induced by the n projection $\delta_i \colon \mathbf{n} \to \mathbf{1}$, $\delta_i(j) = \delta_{i,j}$, in the category of pointed finite sets and projections, are equivalences.

As we are not going to consider *$\hat{\mathcal{O}}$-spaces* in other category different of $\mathcal{U}$ the clarification of $\mathcal{U}$ will be omitted in what follows and henceforth, the operad $\mathcal{O}$ will be always an E_∞ operad. By [May09a, Thm. 3.10], for Σ-free operads there is an equivalence between the homotopy category of special *$\hat{\mathcal{O}}$-spaces* and the homotopy category of *$\mathcal{O}$-spaces*. Then, to obtain an *E_∞-space* from a permutative category we need only a $\hat{\mathcal{O}}$-space. In [May09a, Thm. 1.8] states that an equivalence between categories of operators induce an equivalence between the homotopy categories of special spaces in theses categories of operators. As an special case we have that the *$\mathcal{F}$-spaces* are equivalents to *$\hat{\mathcal{O}}$-spaces*.

To construct an *E_∞-space* from a permutative category it is enough to set up a $\mathcal{F}$-space (or a Γ-space in Segal's notation). This is lead to the infinite loop space machine defined for a permutative category in

[May78] and we will just sketch it out. We don't need to use the entire construction but we will stop in the intermediate category of $\mathcal{F}$*-spaces*. The idea is like in the Segal's construction.

We define a special kind of functors from $\mathcal{F}$ to the category of permutative categories, let us say $\mathcal{PC}$. We will call these functors $\mathcal{F}$-permutative categories. A $\mathcal{F}$-permutative category is a functor $\mathcal{Z}$ such that it must fulfill that $\mathcal{Z}_0$ is equivalent to the trivial category and $\mathcal{Z}_n \to \mathcal{Z}_1^n$ is an equivalence in $\mathcal{PC}$. With the naked eye, this reminds us the definitions of a Segal's Γ-space or our special spaces in the category of operators. Given a permutative category, let us call it $\mathcal{A}$, we can construct a $\mathcal{F}$-permutative category, $\bar{\mathcal{A}}$. The composition with the two-sided Bar construction gives rise to $B\bar{\mathcal{A}}$, the desired $\mathcal{F}$-space.

To summarize, this last construction explained in [May78] gives an $\mathcal{F}$-space from a permutative category, this $\mathcal{F}$-space is equivalent to an $\hat{\mathcal{O}}$*-space*, for $\mathcal{O}$ an E_∞ operad, and this one is equivalent to an algebras over $\mathcal{O}$, i.e, to an E_∞*-space*. Then, by the procedure described just above we obtain an E_∞*-space* from a permutative category.

2.4.2 The E_∞-ring spectrum of a bipermutative category.

To work in the bipermutative case we have to consider two E_∞ operads and combine theirs operations in a correct way. With this purpose May defined in [May09a] the category $\left(\mathcal{K} \int \mathcal{K}'\right)$ for a pair of categories of operators $(\mathcal{K}, \mathcal{K}')$, with an action of $\mathcal{K}$ on $\mathcal{K}'$. In this category, the pair is coalesce into a single wreath product category.

Definition 2.4.6. Let us consider the pair of categories of operators $(\mathcal{K}, \mathcal{K}')$ with an action λ of $\mathcal{K}$ on $\mathcal{K}'$. The category $\left(\mathcal{K} \int \mathcal{K}'\right)$ has by objects the n-tuples of finite based sets $(n; S)$ with $S = (s_1, s_2, s_3, \ldots, s_n)$ for $n \geq 0$. When $n = 0$ there is a single object $(0; *)$. The morphisms between two objects $(n; S)$ and $(m; T)$ are defined by:

$$\coprod_{\phi \in \mathcal{F}(n;m)} \varepsilon^{-1}(\phi) \times \prod_{1 \leq j \leq m} \mathcal{K}' \left(\bigwedge_{\phi(i)=j} s_i, t_j \right) \quad \varepsilon \colon \mathcal{K} \to \mathcal{F}$$

where the empty smash product is 1. The morphisms are denoted by $(f; d)$, where $f \in \mathcal{K}(n; m)$ and $d = (d_1, d_2, \ldots, d_m)$. If $\varepsilon(f) = \phi$, then $d_j \in \mathcal{K}' \left(\bigwedge_{\phi(i)=j} s_i, t_j \right)$. For an other morphism $(g; e) : (m; T) \to (p; U)$ the composition is given by

$$(g; e) \circ (f; d) = (g \circ f; e \circ \lambda(g)(d) \circ \sigma(g; f))$$

Let us take the case when the categories of operators are induced by operads, since this is the case in which we are interested on. For two operads $\mathcal{G}$ and $\mathcal{P}$ with an action of one on the other $\mathcal{G} \circlearrowleft \mathcal{P}$, this action give rise to an action $\hat{\mathcal{G}} \circlearrowleft \hat{\mathcal{P}}$. In fact, there is a one to one correspondence between operad actions $\mathcal{G} \circlearrowleft \mathcal{P}$ and actions in the associated categories of operators (see [May09a, Prop. 4.7]). In this case, for a pair of operads $(\mathcal{G}, \mathcal{P})$ with an action $\mathcal{G} \circlearrowleft \mathcal{P}$, the construction from $(\mathcal{G}, \mathcal{P})$-*spaces* to $\left(\hat{\mathcal{G}} \int \hat{\mathcal{P}}\right)$-*spaces* is analogous to the construction from $\mathcal{D}$-*spaces* to $\hat{\mathcal{D}}$-*spaces*, just with an intermediary step in $\left(\hat{\mathcal{G}}, \hat{\mathcal{P}}\right)$-*spaces* from which we will coalesce the product given rise to the $\left(\hat{\mathcal{G}} \int \hat{\mathcal{P}}\right)$-*spaces*.

In [May09a] there are defined two pairs of adjoints, one between $(\mathcal{G}, \mathcal{P})$-*spaces* and $\left(\hat{\mathcal{G}}, \hat{\mathcal{P}}\right)$-*spaces*, and other between $\left(\hat{\mathcal{G}}, \hat{\mathcal{P}}\right)$-*spaces* and $\left(\hat{\mathcal{G}} \int \hat{\mathcal{P}}\right)$-*spaces*. Then it is shown that for E_∞ operads these adjoints induce equivalences at the homotopy level (see [May09a, Thm. 8.6 and Thm. 10.6]). As in 2.4.1, in what follows we will restrict our attention to E_∞ operads and more precisely we will consider the special case of $(\mathcal{F} \int \mathcal{F})$-*spaces* which is equivalent to a pair of E_∞ operads $(\mathcal{G}, \mathcal{P})$ with an action $\mathcal{G} \circlearrowleft \mathcal{P}$.

To obtain an $(\mathcal{F} \int \mathcal{F})$-space from a bipermutative category we use the construction explained in [May80, § 3.4]. In short, for every category $\mathcal{G}$ there is a functor from the category of lax monoidal functors $\mathcal{G} \to \mathsf{Cat}$ and lax natural transformations to the category of functors $\mathcal{G} \to \mathsf{Cat}$ and natural transformations, with a comparison between the input and the output up to homotopy. Then, for a bipermutative category $\mathcal{A}$ we just need to construct a functor $\mathcal{F} \int \mathcal{F} \xrightarrow{\overline{\mathcal{A}}} \mathsf{Cat}$. This is easy to do and there is a way to do it explained in [May09a, § 13] or in [May82b, § 3]. In our case we are going to consider the bipermutative category $Vb(X) = (Vb(X), \oplus, \otimes)$ from definition 1.3.1, so we get a $\mathcal{F} \int \mathcal{F}$-category $\overline{Vb(X)}$.

As in subsection 2.4.1, applying the two-sided bar construction to this $\mathcal{F} \int \mathcal{F}$-category $\overline{Vb(X)}$ we obtain the desired $\mathcal{F} \int \mathcal{F}$-*space*. This final $\mathcal{F} \int \mathcal{F}$-*space* coming from a bipermutative category $Vb(X)$ is equivalent to an $(\mathcal{G}, \mathcal{P})$-*space* for $\mathcal{G}$ and $\mathcal{P}$ E_∞-operads with an action $\mathcal{G} \circlearrowleft \mathcal{P}$, which is nothing but an E_∞-*ring space*. Our next goal will be the ring completion of this E_∞-*ring space* since the multiplicative operation still not have negatives.

Remark. In what follows we will denote the E_∞-*ring space*, its group completion and the associated E_∞-*ring spectrum* for the $\mathcal{F} \int \mathcal{F}$-category $\overline{Vb(X)}$ by $Vb(X)$ to simplify.

To perform this completion and ensure the consistency with what has been said above we will use the May's paper [May09b], concretely the sections 8 and 9 where they are developed the recognition principle and the multiplicative infinite loop space machines. First we will look for a good candidate for the group completion, or ring completion for multiplicative structures. Next, with the ring completion of our E_∞-*ring* *space* $Vb(X)$ we will see that the ringlike E_∞*-ring spaces* are equivalents to connective E_∞*-ring spectra*, resulting our desired spectrum. Served only as a reminder, a *connective spectrum* is a spectrum such that its negative homotopy groups are 0. Before we have seen that the name E_∞*-ring space* is not completely honest, since it does not have inverses for the multiplication; this is the point at which ringlike E_∞*-ring spaces* are useful. These concepts were already defined in 2.1.4 but let us remember what exactly is a group/ring completion for us.

Definition 2.4.7. We say that an $\mathcal{H}$-monoid X (Def. 2.1.2) is *group-like* if $\pi_0(X)$ is a group under the multiplication and a ring space Y is *ringlike* if it is group-like under its additive $\mathcal{H}$-monoid structure. Let $f\colon X \to Y$ be a morphism of $\mathcal{H}$-monoids. We say that Y is the *group completion of* X if the next requirements are satisfy:

1. Y is group-like.
2. $\pi_0(Y)$ is the Grothendieck group completion of $\pi_0(X)$
3. For any commutative ring of coefficients, the morphism $f_* H_*(X) \to H_*(Y)$ of graded commutative rings is a localization at the monoid $\pi_0(X)$ of $H_0(X)$ (i.e, $H_*(Y)$ is $H_*(X)[\pi_0(X)^{-1}]$). We say that a map $f\colon X \to Y$ is a *ring completion* if Y is a ringlike space and f is a group completion of the additive structure.

Now we are going see the pass from spaces to spectra, but we will start working with a general adjoint and its associated monad. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories, with an adjoint pair (F, G) between them. We denote $\nu\colon Id \to GF$ and $\epsilon\colon FG \to Id$ the corresponding unit and counit maps. We may think about this adjoint as $(\Sigma^\infty, \Omega^\infty)$ with $\mathcal{C}$ and $\mathcal{D}$ corresponding to the spaces and the spectra. As in any adjoint, (F, G) gives rise to a monad (GF, μ, ν) on $\mathcal{C}$ with $\mu = G\epsilon F$. Suppose we have also an other monad C on $\mathcal{C}$ and a map of monads $\alpha\colon C \to GF$. Now our goal is to find an object $\mathbb{E}X$ in $\mathcal{D}$ for a C-algebra X in $\mathcal{C}$ such that $G\mathbb{E}X$ is weakly equivalent to X as a C-algebra.

$\mathrm{B}(F, C, X)$ is a good candidate for $\mathbb{E}X$, where $\mathrm{B}(F, C, X)$ denotes the two-sided monadic bar construction (see [May72]). We follow 3 steps to prove that it is the correct choice. First step: show that for suitable objects $X \in \mathcal{C}$, $\alpha\colon CX \to GFX$ is a weak equivalence. Second step: for appropriate simplicial objects X_* and Y_* in $\mathcal{D}$, if $f_*\colon X_* \to Y_*$ is a map such that $\forall n$ f_n is a weak equivalence, then $\mid f_* \mid$ is a weak equivalence, and analogously for $\mathcal{C}$. Third step: for appropriate simplicial objects Y_* in $\mathcal{D}$, the canonical map $\zeta\colon \mid GY_* \mid_{\mathcal{C}} \to G \mid Y_* \mid_{\mathcal{D}}$ is a weak equivalence and a morphism of C-algebras.

We will restrict our attention from now on to the case with (Σ^n, Ω^n) as (F, G) and C the monad associated to the little n-cubes operad C_n (Def. 2.1.6). In the definition of α we use the Steiner operad. This operad was defined in [Ste79] by Steiner and it gives rise to the canonical E_∞ operad pair that we want to work with. Its definition and properties can be found in [May09b, § 3]. Let $\alpha\colon C \to GF$ be the composition $\theta \circ C_n\nu\colon C_nX \to \Omega^n\Sigma^nX$, where θ denotes the action of the Steiner operad over an n-fold loop space as described in [May09b, § 7]. First of all, by [May76] for $n \leq 2$, α_n is a group completion for all C_n*-spaces* and a weak equivalence for all group-like. We are going to restrict our attention to the case $(\Sigma^\infty, \Omega^\infty)$, with $\mathcal{C} = Top$, $\mathcal{D} = Spt$, C the monad associated to the Steiner operad and α the map of monads defined in [May09b, § 7]. The first two steps are easy to check for spaces and spectra, let see [May72, Chapter 11 13] and [May09a, Appendix] for spaces and [May97] for spectra. The last step is a little more delicate but it can be proven, as it is explained in [May09b, § 9]. In short, we use the results [May72, Thm. 12.3 and Thm. 12.4] to prove the case (Σ^n, Ω^n) and then we take colimits to prove the case $(\Sigma^\infty, \Omega^\infty)$. Let us denote $\mathcal{S}$ the Steiner operad in the next theorem. The next theorem encapsulates the last work for the additive infinite loop space machines.

Theorem 2.4.1 (Thm. 9.3[May09b])**.** *For a $\mathcal{S}$-space X, define $\mathbb{E}X = \mathrm{B}(\Sigma^\infty, C, X)$. Then $\mathbb{E}X$ is a connective spectrum and there is a natural diagram in the category of C-spaces*

$$X \xleftarrow{\ \epsilon\ } \mathrm{B}(C, C, X) \xrightarrow{\mathrm{B}(\alpha, id, id)} \mathrm{B}(\Omega^\infty\Sigma^\infty, C, X) \xrightarrow{\ \zeta\ } \Omega^\infty\mathbb{E}X$$

such that ϵ is a homotopy equivalence with natural homotopy inverse ν, ζ is an equivalence, and $\mathrm{B}(\alpha, id, id)$ is a group completion. The composite $\nu\colon X \to \Omega^\infty\mathbb{E}X$ is a group completion and weak equivalence in case that X is group-like. Given a spectrum Y, there is a composite natural map of spectra

$$\epsilon\colon \mathbb{E}\Omega^\infty Y \xrightarrow{\mathrm{B}(id, \alpha, id)} \mathrm{B}(\Sigma^\infty, \Omega^\infty\Sigma^\infty, \Omega^\infty Y) \xrightarrow{\ \epsilon\ } Y$$

and the induced maps of $\mathcal{S}$-spaces

$$\Omega^\infty\epsilon\colon \Omega^\infty\mathbb{E}\Omega^\infty Y \xrightarrow{\Omega^\infty\mathrm{B}(id, \alpha, id)} \Omega^\infty\mathrm{B}(\Sigma^\infty, \Omega^\infty\Sigma^\infty, \Omega^\infty Y) \xrightarrow{\Omega^\infty\epsilon} \Omega^\infty Y$$

are weak equivalences.

With this theorem we get a group completion for a $\mathcal{S}$-space, as well as a concrete adjoint, $(\mathbb{E}, \Omega^\infty)$, between group-like E_∞*-spaces* and connective spectra which induces an equivalence at the homotopy level. It should be noted that this theorem covers the Steiner operad case and we would like to state this result for general

E_∞ operads. With an operad's trick, this theorem can be widespread for any E_∞ operad and not just for the Steiner one. For $\mathcal{O}$ an E_∞ operad we can define a new operad $\mathcal{P} = \mathcal{S} \times \mathcal{O}$, taking the pullback of an *$\mathcal{O}$-space* along the second projection it is a *$\mathcal{P}$-space* and Ω^∞ is a right P-functor (for P the monad associated to $\mathcal{P}$) by the pullback along the first projection. Using also the projections we can see that C can be replace by P.

The last theorem can be extended to the multiplicative case too, in fact, the homotopy properties depend only on the additive one. The next theorem is our desired result for the ring completion and the construction of the corresponding spectrum for $Vb(X)$. Let us denote the linear isometries operad by $\mathcal{L}$, which is the canonical multiplicative operad. The definition can be find in [Vog68], where it was introduced by Boardman and Vogt. The multiplicative case is startling easy to do once we have the additive one. Using, $L_+[Top]$ and $L_+[Spt]$ instead of Top and Spt and employing that $\alpha\colon C \to \omega^\infty\Sigma^\infty$ is a map of monads on $L_+[Top]$, the multiplicative case follows from the additive one.

Theorem 2.4.2 (Thm. 9.12 [May09b])**.** *For a $(\mathcal{S},\mathcal{L})$-space X, define $\mathbb{E}X = \mathrm{B}(\Sigma^\infty, C, X)$. Then $\mathbb{E}X$ is a connective $\mathcal{L}$-spectrum and there is a natural diagram in the category of $(\mathcal{S},\mathcal{L})$-spaces*

$$X \xleftarrow{\ \epsilon\ } \mathrm{B}(C,C,X) \xrightarrow{\mathrm{B}(\alpha,id,id)} \mathrm{B}(\Omega^\infty\Sigma^\infty, C, X) \xrightarrow{\ \zeta\ } \Omega^\infty\mathbb{E}X$$

The composite $\nu\colon X \to \Omega^\infty\mathbb{E}X$ is a ring completion. Given an $\mathcal{L}$-spectrum R, there is a composite natural map of $\mathcal{L}$-spectra

$$\epsilon\colon \mathbb{E}\Omega^\infty R \xrightarrow{\mathrm{B}(id,\alpha,id)} \mathrm{B}(\Sigma^\infty, \Omega^\infty\Sigma^\infty, \Omega^\infty R) \xrightarrow{\ \epsilon\ } R$$

and the maps

$$\Omega^\infty\epsilon\colon \Omega^\infty\mathbb{E}\Omega^\infty R \xrightarrow{\Omega^\infty\mathrm{B}(id,\alpha,id)} \Omega^\infty\mathrm{B}(\Sigma^\infty, \Omega^\infty\Sigma^\infty, \Omega^\infty R) \xrightarrow{\ \Omega^\infty\epsilon\ } \Omega^\infty R$$

are maps of $(\mathcal{S},\mathcal{L})$-spaces.

Therefore the adjoint pair $(\mathbb{E}, \Omega^\infty)$ induces an equivalence between the homotopy categories of ringlike *E_∞-ring spaces* and connective *E_∞-ring spectra.*

Corollary 2.4.3. *There exits a connective E_∞-ring $\mathbb{S}^1$-spectrum $Vb(X)$ for any value of X in the presheaf described in subsection 1.3.1. This produces our desired presheaf* $\mathbf{KH}\colon (Sm\,|_S)^\circ \longrightarrow E_\infty$*-rings.*

Proof. Using the ring completion of Theorem 2.4.2 for the *E_∞-ring space* $Vb(X)$ that we have obtained before and taking an equivalent connective spectrum we get the desired presheaf of spectra for affine schemes. Once we have defined the presheaf for affine schemes the only thing missing is to extend it to general schemes by Theorem 1.3.1. $\square$

Remark. Again we are using implicitly subsection 1.3.1. Other wise the presheaf should be $\mathbf{KH}\colon\ (Aff\ |_S)^\circ \to\ E_\infty$-rings.

Remark. This spectrum turns to be connective, this is a difference with the other models for hermitian K-theory in 1.3.3, but it is not our definitive spectrum since it is a $\mathbb{S}^1$-spectrum. We will solve this problem in the next section by the delooping construction.

2.5 Delooping construction.

In this last section we are going to explain the delooping construction. We will apply such construction to our E_∞-ring $\mathbb{S}^1$-spectrum $\mathbf{KQ}_{\mathbb{S}^1}$. In the last sections we have employed $\mathbf{KQ}_{\mathbb{S}^1}$ as the presheaves of E_∞-ring $\mathbb{S}^1$-spectra that we have constructed in Sections 2.3 ($\mathbf{KQ}$) and 2.4 ($\mathbf{KH}$) interchangeably. This construction can be perfectly applied to $\mathbf{KH}$, but we will focus our attention in the presheaf $\mathbf{KQ}$ and we will mean this presheaf when writing $\mathbf{KQ}_{\mathbb{S}^1}$. This is a version of the delooping construction done in [Ost10] for KGL. This construction can be done in a more general context (see the general delooping construction, pg. 114). We will try to keep as general as possible during this section and we will start to make restrictions to our case when necessary.

The machinery developed in the sections 2.3 and 2.4 have like outputs $\mathbb{S}^1$-spectra, more concretely they produce an E_∞-ring in the category $\mathrm{Spt}_{\mathbb{S}^1}$. The delooping explained here has like input an E_∞-ring $\mathbb{S}^1$-spectrum and as output a commutative algebra in the category of $\mathbb{P}^1$-spectra, a ring $\mathbb{P}^1$-spectrum. This is in fact a strict ring spectrum, so this implies that we get an E_∞-ring $\mathbb{P}^1$-spectrum. In the concrete case of the hermitian K-theory we will take as input the spectrum $\mathbf{KQ}_{\mathbb{S}^1}$ obtaining the spectrum $\mathbf{KQ}_{\mathbb{P}^1}$ which is our final $\mathbb{P}^1$-spectrum representing the hermitian K-theory in the motivic case.

In [Ost10] the delooping construction is just sketched, without so much detail. For example, the model structures appearing along the construction and the correction of the E_∞-algebra structure to a commutative algebra structure are omitted. Here we are going to explain the possible model structures that one can employ and the correction of an E_∞-algebra structure to a commutative algebra, since we need such structure to take modules over $\mathbf{KQ}_{\mathbb{S}^1}$ during the delooping. In [Ost10] the following theorem was established.

Theorem 2.5.1 (Thm. 6.1 [Ost10]). *For a fixed scheme S as in 1.1 and a presheaf*

$$\mathcal{K}\colon\ (Sm\,|_S)^\circ \ \longrightarrow\ E_\infty\text{-}rings\ \mathrm{Spt}$$

such that it is equivalent to the presheaf of K-theory K as homotopy commutative $\mathbb{S}^1$-spectra. Then they are equivalents as $E_\infty\mathbb{S}^1$-spectra. In particular, if $X \in Sm\,|_S$, there is an equivalence of E_∞ ring spectra $K(X) = \mathcal{K}(X)$.

In the proof of this theorem in [Ost10] there is what we call here delooping construction. It use the existence of a map $\Sigma^\infty_{\mathbb{S}^1}\mathbb{P}^1 \to \mathcal{K}$ equivalent to Bott element in the spectrum K, such that its $\mathcal{K}$-linear extension, $\Sigma^\infty_{\mathbb{S}^1}\mathbb{P}^1 \wedge \mathcal{K} \to \mathcal{K}$, gives rise by adjoint to an equivalence $\Omega_{\mathbb{P}^1}\mathcal{K} \simeq \mathcal{K}$. With such a map it is constructed a new $\mathbb{P}^1$-spectrum equivalent to $\mathcal{K}$ which inherit a natural commutative algebra structure in $\mathbb{P}^1$-spectra and thus an E_∞-ring structure. And then, using the non-existence of phantom maps one can prove the uniqueness result of the theorem.

In our case, we are not going to compare with an abstract presheaf as $\mathcal{K}$. We are going to use a version of this construction for $\mathbf{KQ}_{\mathbb{S}^1}$, but we will use it to get a commutative ring $\mathbb{P}^1$-spectrum $\mathbf{KQ}_{\mathbb{P}^1}$. We will also give an uniqueness result, but different to the one in [Ost10]. We will get an uniqueness result by comparison with Panin-Walter spectrum $\mathbf{BO}$. It is still not a whole uniqueness result for the E_∞ structure, but it proves the uniqueness as commutative monoid in the stable motivic category $\mathsf{SH}(S)$. Now, we are going to present the general statement for the delooping construction.

Delooping construction 1. Let $(\mathcal{C}, \wedge, \mathrm{Hom})$ be a closed symmetric monoidal category. This category plays the role of presheaves of the $\mathbb{S}^1$-spectra. We choose an object $D \in \mathcal{C}$. This object corresponds to $\Sigma^\infty_{\mathbb{S}^1}(\mathbb{P}^1, \infty)$ in the K-theory case done in [Ost10] for $\mathcal{K}$, or to $\Sigma^\infty_{\mathbb{S}^1}((\mathbb{HP}^1)^{\wedge 2}, x_0)$ in our hermitian case for $\mathbf{KQ}_{\mathbb{S}^1}$. These are the objects in respect of which we want to stabilize. In the motivic setting, we take an object from $\mathsf{Spc}_\bullet(S)$ and we put it in the category of motivic $\mathbb{S}^1$-spectra by taking the suspension $\Sigma^\infty_{\mathbb{S}^1}$.

Let us also assume that there exists commutative algebra $\mathcal{K} \in C.Alg\,(\mathcal{C})$ and a map $D \to \mathcal{K}$ in $\mathcal{C}$ such that the adjoint of its $\mathcal{K}$-linear extension produces an isomorphism $\mathcal{K} \xrightarrow{\sim} \mathrm{Hom}\,(D, \mathcal{K})$. In our cases, these commutative algebra objects corresponds to the E_∞-algebra objects $\mathcal{K}$ and $\mathbf{KQ}_{\mathbb{S}^1}$ (which can be thought as commutative algebras after rectification), and the desired isomorphisms are given by maps equivalents to the Bott elements $\beta\colon (\mathbb{HP}^1)^{\wedge 2} \to \mathbf{KQ}_{\mathbb{S}^1}$ (subsection 1.3.4) for $\mathbf{KQ}_{\mathbb{S}^1}$ and $\beta\colon \mathbb{P}^1 \to \mathcal{K}$ for $\mathcal{K}$.

If $\mathcal{D} := D - \mathrm{Spt}^\Sigma$ in $\mathcal{C}$, then the delooping construction produces $\underline{\mathcal{K}} \in E_\infty\,(\mathcal{D})$ such that $\Omega^\infty_{\mathcal{D}}\underline{\mathcal{K}} \simeq \mathcal{K}$.

This general construction will be referred along all the section. As we move forward, we will put more requirements to this general construction, but it is felt that this general statement can be established as general as it is. In any case, all these requirements are verify by the case of $\mathbf{KQ}_{\mathbb{S}^1}$, giving rise to the desired delooping. In contrast with the uniqueness result got in [Ost10] we are going to compare with spectrum $\mathbf{BO}$ and get the following result.

Theorem 2.5.2 (Theorem 2.5.9)**.** *The E_∞-ring $\mathbb{P}^1$-spectrum $\mathbf{KQ}_{\mathbb{P}^1}$ is unique as homotopy commutative monoid in the category $\mathsf{SH}(S)$, provided the following condition is verify. Let $m \in \mathrm{Hom}_{\mathsf{SH}(S)}(\mathbf{KQ}_{\mathbb{P}^1} \wedge \mathbf{KQ}_{\mathbb{P}^1}, \mathbf{KQ}_{\mathbb{P}^1})$ be the induced morphism in $\mathsf{SH}(S)$ by the multiplicative structure of the E_∞-ring spectrum $\mathbf{KQ}_{\mathbb{P}^1}$. This morphism agrees with the one induced by the spectrum $\mathbf{BO}$ and it is the unique defining a pairing which when restrict to the Grothendieck-Witt groups coincides with the tensor product pairing*

$$\mathsf{KO}_0^{[2p]}(X_+) \times \mathsf{KO}_0^{[2q]}(Y_+) \to \mathsf{KO}_0^{[2p+2q]}(X_+ \wedge Y_+)$$

for $X, Y \in \mathrm{Sm}\,|_S$.

Remark. The uniqueness result in our case is different since in the proof given in [Ost10] they compare with an abstract spectrum $\mathcal{K}$ and they use the fact that the spectrum KGL has not phantom maps. A phantom map is map between objects in a stable homotopy category such that the induced map between cohomology functors on the full subcategory of finite objects is the zero map. We have always the trivial maps, but none more for KGL, as one can see in [Ost08]. Then to prove that the difference of both multiplicative structures $\delta \in KLG^{00}(\mathsf{KGL} \wedge \mathsf{KGL})$ vanishes it is enough to see that δ maps to 0 when it is pulled back along any map $T \xrightarrow{g} \mathsf{KGL} \wedge \mathsf{KGL}$, with T compact. This allows us to prove the uniqueness respect to other presheaf equivalent as homotopy commutative monoid in $\mathrm{Spt}_{\mathbb{S}^1}$ for KGL but not for $\mathbf{KQ}_{\mathbb{S}^1}$, since we do not know about the existence of phantom maps different from the trivials.

In the subsection 2.5.1 we will see the rectification from E_∞-algebras to commutative algebras, since we want to take modules over $\mathbf{KQ}_{\mathbb{S}^1}$ during the delooping construction. In this part of the section we will restrict our category $\mathcal{C}$ from the general delooping construction (pg. 114) to $\mathrm{Spt}_P(\mathcal{C}')$, the category of P-spectra in $\mathcal{C}'$, where P is an object of $\mathcal{C}'$. The requirement of being symmetric closed monoidal category will be for $\mathcal{C}'$ instead of $\mathcal{C}$. Our concrete case will be $\mathcal{C}' = s\mathbf{PSh}_\bullet(Sm\,|_S)_{Nis}$ and $P = \mathbb{S}^1$. We will prove the existence of a rectification for E_∞-algebras in a general $\mathbb{F}$-spectra category, for $\mathbb{F}$ an object of $\mathcal{C}'$. This result can be applied for $\mathbb{F}$ a pointed motivic space, such as $(\mathbb{P}^1, \infty)$ or $(\mathbb{HP}^1, x_0)^{\wedge 2}$, spectra categories in which we use to work for K-theory and Hermitian K-theory. We will consider the commutative monoid $R_n = (\mathbb{F})^n$ in the category of symmetric sequences in $\mathcal{C}'$, $\Sigma\mathcal{C}'$, to state Lemma 10. This means, we are going to require $D = \Sigma^{\infty}_{\mathbb{S}^1}\mathbb{F}$ for $\mathbb{F} \in \mathsf{Spc}_{\bullet}(S)$ in the general delooping construction (pg. 114) for case of working with motivic spectra.

In the subsection 2.5.2 we will see the categories appearing in the delooping construction. From the several models for the category of spectra we choose to work with symmetric spectra. We will describe the categories appearing in the delooping, as well as the construction and properties of the symmetric spectra. This does not suppose new restrictions to the general construction since we have already assume in subsection 2.5.1 that we are working with presheaves of spectra. Right after (subsection 2.5.3) there is a short reminder of the Day convolution, which is the product that we use in the commutative algebras.

The subsection 2.5.4 tackles the model structures that are implicit in the delooping construction. This section is done for the concrete case of the spectrum $\mathbf{KQ}_{\mathbb{S}^1}$. The arguments are the usuals to induce model structures via the adjoints given by the free functor and the forgetful. This kind of induced model structure can be adapted for a concrete category $\mathcal{C}$ in the general delooping construction (pg. 114) In fact, after we assume that $\mathcal{C}$ has a cofibrantly generated model structure the remaining model structures can be done analogously. At the end there is a last part about co-localizations to work with connective spectra. The spectrum $\mathbf{KQ}_{\mathbb{S}^1}$ is a connective spectrum, and we would like to preserve this property. As it will be explained in this section, we could apply any of the truncation theorems for spectra to get an Ω-spectrum but we also would like to say the functor f (see subsection 2.5.5) in the construction is the forgetful, and therefore the resulting spectrum is nothing else that the old spectrum in each entry. So we are going to apply a co-localization to obtain presheaves of connective spectra.

Remark. The model structures appear in two subsections. First, in subsection 2.5.1 where we will define the necessary model structures for the correction of commutative algebras, and second, in 2.5.4 where we will define the model structures that will be used implicitly in the delooping construction done in 2.5.5.

The following subsection, 2.5.5, is the delooping construction. The general delooping construction (pg. 114) was thought for this purpose and we do not need any additional assumptions to get the commutative algebra structure. Anyway the construction will be done in the concrete case of $\mathbf{KQ}_{\mathbb{S}^1}$. There is a small correction in the construction for $\mathbf{KQ}_{\mathbb{S}^1}$, since we want more properties that just the commutative algebra structure. Since the Bott element used for the construction is from $\left(\mathbb{HP}^1\right)^{\wedge 2}$ we get a connective $\left(\mathbb{HP}^1\right)^{\wedge 2}$-spectrum, but the categories of $\mathbb{P}^1$-spectra and $\left(\mathbb{HP}^1\right)^{\wedge 2}$-spectra are Quillen equivalents and moreover they induce an equivalence at the level of E_∞-rings. We will obtain an E_∞-ring $\mathbb{P}^1$-spectrum after moving our spectrum to the category of $\mathbb{P}^1$-spectra. This will be our definitive spectrum $\mathbf{KQ}_{\mathbb{P}^1}$. We will get the following result at the end.

Theorem 2.5.3 (Theorem 2.5.7)**.** *The motivic Hermitian K-theory is represented in $\mathsf{SH}(S)$ by the E_∞-ring $\mathbb{P}^1$-connective spectrum $\mathbf{KQ}_{\mathbb{P}^1}$.*

The last subsection deals with the uniqueness. Clearly our E_∞-ring spectrum $\mathbf{KQ}_{\mathbb{P}^1}$ restricts to a commutative monoid up to homotopy in the category $\mathsf{SH}(S)$. We use the uniqueness result given by Panin and Walter in [PW10b] to see that their spectrum agrees with our spectrum as commutative monoid up to homotopy and then we get the same uniqueness for our spectrum. Since this proof employs pairings, first we will revise the construction of the tensor pairing for $\mathbf{BO}$, and then, we will see that the pairing induced by the tensor product is preserved along the direct sum K-theory (Def. 2.3.6). This sequence of infinity functors produces our spectrum $\mathbf{KQ}_{\mathbb{S}^1}$ and the spectrum obtained after the delooping is just $\mathbf{KQ}_{\mathbb{S}^1}$ in each entry, thus the tensor pairing will be not modified along the construction and it will agree with the Panin-Walter one.

2.5.1 From E_∞ algebras to commutative algebras.

In this subsection we are going to rectify E_∞ algebras to commutative algebras. We need to do this correction since the delooping construction is done for a strict commutative ring and $\mathbf{KQ}_{\mathbb{S}^1}$ is just and E_∞-algebra. We will try to stay the most general as possible. It will be explain a criterion which covers the $\mathbb{P}^1$-spectra case as well as other motivic spectra cases. In fact, in the general case we are going to require $\mathcal{C} = s\mathbf{PSh}_\bullet\,(Sm\,|_S)_{Nis}$ and we will work with a commutative monoid $R_n = (\mathbb{F})^n$, for $\mathbb{F}$ the motivic space respect to which it will be stabilized the category $\mathcal{C}$.

To convert commutative algebras in E_∞-ring spectra they have appear several constructions during the 1990's by dropping any of the Lewis's requirements for a desired category of spectra ([Lew91]). Either the $\mathbb{S}$-modules of EKMM or, the more algebraic, the symmetric spectra of Smith et. al (see subsection 2.5.2). All these constructions forms a symmetric monoidal model category, thereby obtaining a symmetric monoidal structure on the homotopy category. These constructions turn out to be equivalents, but each one of these settings has its advantages. Normally it is easier to work with E_∞ algebras that to work with commutative algebras. The category of algebras over a cofibrant operad come into a model structure if the monoid axiom is satisfied (see [Spi01]), but in our delooping we will need to work with (strictly) commutative monoids in R-modules, for R a commutative monoid in symmetric sequences (see subsection 2.5.2. Then we will also need a rectification between E_∞-algebras and commutative monoids, since our presheaf take values over E_∞-rings and our delooping construction is given in the strict context.

We can use any of these models since in all model categories of spectra, if commutative ring spectra forms a model category, the category of commutative algebras is Quillen equivalent to E_∞-algebras (see for example [May97], or [SS99] for a comparison of the different models). Because of the categories that appear during the delooping construction and the tools that we employ to get such a rectification it looks more convenient to use the symmetric spectra. Any way, we are going to see, the last model structures in E_∞-algebras and rectifications starting in the general setting and finishing at the end of the section with our current case. To induce a model structure on monoids and commutative monoids they have appear divers publications in the last years. Two of the most recents are due to White ([Whi14]), and to Pavlov and Scholbach ([PS14]). In the first one, White works in a more general setting that symmetric spectra, defining a commutative monoid axiom analogously to work of Shipley in [Shi04] for the non-commutative case with the monoid axiom. Shipley defined a model structure on the category of symmetric spectra, now a days called the positive flat model structure, with the compatibility property that the weak equivalence and fibrations forget to weak equivalence and fibrations in the positive R-model structure on R-modules. White construct a model structure on $CAlg(R)$, for R a commutative monoid in the model category $\mathcal{C}$, which also forgets the weak equivalence and fibrations but to the original category $\mathcal{C}$. He also proved a rectification result generalizing the property behind the rectification property in the spectra categories. Another one is the Lurie's result [Lur09], but this one has harder requirements, among them a cofibrant unit, which is very restrictive and, for example, it does not work for the case of the motivic $\mathbb{P}^1$-spectra when it is induced from the projective model structure. We are going to apply the other one for our construction, but this one can be apply for other version of the general delooping construction (pg. 114).

To obtain a rectification between E_∞-algebras and commutative algebras we have to use a positive model structure. The positive model structures were introduced in [SS00], and they break the cofibrancy of the unit, which is one of the Lewis's conditions. Following the direction of the positive model structures Pavlov and Scholbach in [PS14] concentrate their attention in the rectification for spectra in general model categories. They cut up the commutative monoid axiom and the strong version by defining (acyclic) power cofibrations and (acyclic) symmetric cofibrations ([PS14, Def. 4.5]), and also the admissible collection ([PS14, Def. 7.1]) to ensure the existence of a set of cofibrations on symmetric sequence such that every cofibration is a symmetric cofibration. To preserve the monoid axiom under Bousfield localization they use the h-monoidality and introduce an axiom called the cow axiom. The cow axiom is also used to avoid the requirement on the cofibrancy of R for the rectification result.

In [PS14] Pavlov and Scholbach put a model structure on the categories $\Sigma_m \, \mathcal{C} = \mathrm{Fun}\,(\Sigma_m, \mathcal{C})$ for $m \geq 0$, let say the projective, the injective, or the flasque, provided that they are admissible. This model structure is transferred successively to the following categories: $\Sigma\, \mathcal{C}$, Mod_R, for R commutative monoid in $\Sigma\, \mathcal{C}$, the stable case Mod^s_R, via a Bousfield localization, and to $CMon(Mod^s_R)$ by the adjoint given by the forgetful and the free functor. Finally for the operadic rectification they use the homotopy orbits property. There were already rectification results for sSet, e.g, [Har09, Thm. 1.4] by Harper, or in the case of simplicial presheaves with the injective model structure [Hor13, Thm. 3.6] due to Hornbostel. In these last two cases all objects are cofibrant, which is not necessary for the setting-up in [PS14]. This construction allow us to obtain model structures in commutative algebras in the motivic setting starting from different models.

Now we are going to focus the attention in our case. First of all, in what follows we fix the category $\mathcal{C}$ as the category of pointed simplicial presheaves on the Nisnevich site $(Sm\,|_S)$, i.e, $\mathcal{C} = s\mathbf{PSh}_\bullet\,(Sm\,|_S)_{Nis}$. This category is clearly pointed, simplicial and closed symmetric monoidal. To clarify matters we present the list of properties in a model category $\mathcal{C}$ which will be required at some point in what follows.

1. *Pointed*: It has a zero object, i.e, an object which is final and initial at the same time.
2. *Simplicial model category*: A category enriched over simplicial sets ([Hov99, Def. 4.2.18]).
3. *Pretty small*: A model category $\mathcal{C}$ such that it exists other model category $\mathcal{C}'$ on the same underlying category with the same weak equivalences and cofibrations contained in the cofibrations of $\mathcal{C}$, such that the (co)domains of the generating cofibrations are $\aleph_0$-small.
4. *Combinatorial model category*: A category generated from a small set of (acyclic) cofibrations between small objects ([Lur09, Def. A.2.6.1]).
5. *Tractable*: It is a combinatorial model category such that the generating cofibration can be chosen with cofibrant (co)domains.
6. *Left proper*: A model category such that pushouts along cofibration preserves equivalences.
7. *Symmetric monoidal model category*: A model category with a symmetric monoidal structure such that it induces a symmetric monoidal structure at the homotopy level ([Hov99, Def. 4.2.6]).
8. *Monoid axiom*: When in a monoidal model category the maps obtained by transfinite composition of pushouts of tensor product between trivial cofibrations and any object are weak equivalences.

9. *h-monoidal*: It is a symmetric monoidal model category such that $f \otimes C$ for any trivial cofibration f and any object C gives rise to a h-cofibration.

10. *cow axiom*: it is satisfied by a monoidal model category if $f \otimes C$ is always a weak equivalence for g a weak equivalence and a C cofibrant object.

Hereinafter, we use the tools given in [PS14] to lead with model structures for the rectification. As an aside, we could work in the context of $\mathbb{HP}^1$ motivic spectra because the $\mathbb{A}^1$-Nisnevich left Bousfield localization satisfies the h-monadity as well as the crow axiom required before. In fact, since our category $\mathcal{C} = s\mathbf{PSh}_{\bullet}\left(Sm\,|_S\right)_{Nis}$ verifies the required properties, we are going to consider a general commutative monoid R_n in $\mathcal{C}$, which can be $R_n = \left(\mathbb{P}^1\right)^n$, $R_n = \left(\mathbb{S}^1\right)^n$, $R_n = \left(\mathbb{HP}^1\right)^n$, or $R_n = (\mathbb{F})^n$ for another motivic pointed space $\mathbb{F} \in \mathcal{C}$. First we are going to state the result and the corollary for our particular case and then we will prove it.

Lemma 10. Let $\mathbb{F}$ be a motivic pointed space $\mathbb{F} \in \mathsf{Spc}_{\bullet}(S)$, and $\mathrm{Spt}^{\Sigma}_{\mathbb{F}}$ the category of symmetric motivic $\mathbb{F}$-spectra. Then the category of commutative algebras in $\mathrm{Spt}^{\Sigma}_{\mathbb{F}}$ and the category E_∞-rings in $\mathrm{Spt}^{\Sigma}_{\mathbb{F}}$ are Quillen equivalents with the model structures described just below.

Applying the Lemma 10 to the special case of the category of symmetric $\mathbb{S}^1$-spectra we get the following corollary that we may employ for our case.

Corollary 2.5.4. *The commutative algebras in the category of symmetric $\mathbb{S}^1$-spectra and the category of E_∞-rings in symmetric $\mathbb{S}^1$-spectra are Quillen equivalents $E_\infty(\mathrm{Spt}^{\Sigma}_{\mathbb{S}^1}) \simeq C.Alg.(\mathrm{Spt}^{\Sigma}_{\mathbb{S}^1})$.*

Coming back to the construction: Let us put the projective or the injective model structure on $\Sigma_m\,\mathcal{C}$ for $m \geq 1$, which are weakly admissible collections of model structures, and consider the commutative monoid $R_n = \mathbb{F}^n$ in the category of symmetric sequences, $\Sigma\ s\mathbf{PSh}_{\bullet}\left(Sm\,|_S\right)$. Either the injective like the projective models satisfy the requirements given in [PS14, Def. 7.1] to be weakly admissible collections. In fact, we have more, the injective one in $\mathcal{C}$ is combinatorial ([Lur09, Prop. $A.2.8.2$], A.2.8.2) and therefore tractable, since all the objects are cofibrants. Because it is monoidal and all the objects are cofibrant we have that it is h-monoidal and verifies the cow and monoid axioms. The injective model structure is also pretty small, then we can apply [PS14, Lemma 7.5] to see that it is in fact admissible. In the projective one the generating cofibrations are of the form $\left(X \times \partial\Delta^k\right)_+ \to \left(X \times \Delta^k\right)_+$ where $X \in \left(Sm\,|_S\right)$ and $k \geq 1$. These are cofibrant and $\aleph_0$-small objects, so it is pretty small and tractable. [PS14, Lemma 2.5] and the injective one check that it is left proper, h-monoidal and satisfies the cow and monoid axioms. So we can take the injective or the projective one interchangeably.

Now, it will be explained where each one of the properties from 2.5.1 is required. By [PS14, Thm. 8.2] we obtain a positive model structure on $\Sigma\ s\mathbf{PSh}_{\bullet}(Sm\mid_S)$ with weak equivalences (resp. fibrations) the maps which are weak equivalences (resp. fibrations) in $\Sigma_m\ \mathcal{C}$ for all $m \geq 1$. This is possible because of the requirements of [PS14, Not. 8.1] ($\mathcal{C}$ is tractable, pretty small and closed symmetric monoidal model) are satisfied. Now in addition to this requirements when the monoid axiom is verified, by Theorem 9.2, this transfer to a positive model structure on Mod_R (that we denote by Mod_R^+) such that the weak equivalences and fibrations forget to $\Sigma_m\ \mathcal{C}$. Employing [PS14, Thm. 9.5] Mod_R^+ gives rise to a stable positive model structure on Mod_R, that we denote by $Mod_R^{s,+}$. The stabilization, as we will seen in Section 2.5.2, is a Bousfield localization and we put some assumptions in the list, as combinatorial or left proper, to ensure its existence. To employ the Theorem 9.5 we need to require on $\mathcal{C}$ to be left proper, tractable, h-monoidal and the cow axiom, which are satisfied in our case. Under the requirements of the list, which are satisfied in our case, all operads are admissible, which means that the category of modules over an operad admits a model structure transferred along the forgetful from the collections in $\mathcal{C}$. For each operad, like the Barratt-Eccles operad or Comm, the category of algebras over it with values in Mod_R inherit a model structure from $Mod_R^{s,+}$. Moreover, using [PS14, Prop. 11.2] the weak equivalence between E_∞ operads and Comm induce a Quillen equivalence between E_∞ algebras and commutative algebras, as it was stated on Lemma 10.

2.5.2 Categories of the delooping.

In this section we are going to describe the objects that turn up in the construction of the delooping and present the general construction for spectra categories. Once we have chosen to work in the symmetric spectra setting there are two natural points of view for the categories of the delooping. The first is to take modules, symmetric sequence and other structures directly from the category of simplicial presheaves, which is the correct one to work on with the modern tools. The second, evaluating this objects in a scheme, is the one that we start presenting and maybe the most intuitive to begin.

Nowadays, this is not the usual way to think about it. In fact, the usual way to think in some kind of category of spectra is to take a category $\mathcal{C}$ with a endofunctor G and obtain a new category where this functor is a Quillen equivalence. We will discus this pattern and the symmetric version at the end of the section using the concepts that appear in [Hov01]. The origin of spectra arise from spaces and the functor given by smashing with the circle, so first, we write the case for $\mathbb{S}^0$ which is the classical way to describe spectra and symmetric spectra, and then we will make the analogous construction for $\mathbf{KQ}_{\mathbb{S}^1}$.

1. $mod_{\mathbb{S}^0}$: The evaluation of an object of this category in $S' \in (Sm\,|_S)^{\circ}$ gives rise to the following objects:

1. $X(S')_n, \forall n \geq 0, X(S')_n \in \mathsf{sSet}_+$, with structural maps
2. $\sigma_n(S') \colon X(S')_n \wedge \mathbb{S}^1 \;\to\; X(S')_{n+1}$

The smash product appearing here is the normal in simplicial sets. This category is just a classic $\mathbb{S}^1$-spectrum in sSet_*. The maps between objects are sequence of maps such that the diagram with the structure maps commute.

2. $mod^{\Sigma}_{\mathbb{S}^0}$: The evaluation of an object of this category in $S' \in (Sm\,|_S)^{\circ}$ gives rise to the following objects:

1. $R(S')_n, \forall n \geq 0, R(S')_n \in \mathbb{S}^1\text{-Spt}_{\mathsf{sSet}_*}$,
2. with actions$\Sigma_n \circlearrowleft R(S')_n$
3. and the collection $\sigma_n(S') \colon X(S')_n \wedge Y(S')_p \;\to\; Z(S')_{p+n}$ of maps which are $\Sigma_n \times \Sigma_p$ - equivariants.

This is just the symmetric sequences on $\mathbb{S}^1\text{-Spt}_{\mathsf{sSet}}$. There are two ways to think about this category: the first one as a sequences of spectra with actions of the symmetric groups, and the second the functor category given by the functors from Σ to $\mathbb{S}^1\text{-Spt}_{\mathsf{sSet}}$, as it will be discussed below.

Our last category is $C.Alg\left(mod^{\Sigma}_{\mathbb{S}^0}\right)$, the commutative algebras in $mod^{\Sigma}_{\mathbb{S}^0}$, in other words, the commutative monoids, which we will see in Section 2.5.3 that they are concretely the lax monoidal functors.

The sphere spectra $\mathbb{S}$ has the behavior of the integers in the rings (all the rings are $\mathbb{Z}$-algebras). Here all the $\mathbb{S}$-algebras will be rings of spectra. Now we work analogously with $\mathbf{KQ}_{\mathbb{S}^1}$. The idea is to take $\mathbf{KQ}_{\mathbb{S}^1}$ like base as we did for the sphere, as a ring in spectra, adding associativity and unicity to obtain modules, and over such modules we will restrict our attention to the commutative monoids.

To get modules over $\mathbf{KQ}_{\mathbb{S}^1}$ it should be the analogous of ring. Remember that $\mathbf{KQ}_{\mathbb{S}^1}$ is an E_{∞}-ring spectra for every $S' \in Sm\,|_S$. In this case we do not have a real strict ring structure but yes at a homotopy level. That is:

3. $\mathbf{KQ}_{\mathbb{S}^1}$ **:** The evaluation in $S' \in (Sm\ |_S)^{\circ}$ gives rise to the following objects:

1. $\mathbf{KQ}_{\mathbb{S}^1}(S')_n \in \mathsf{sSet}\ \forall n \geq 0$,
2. with actions $\Sigma_n \circlearrowleft \mathbf{KQ}_{\mathbb{S}^1}(S')_n$,
3. $(\Sigma_n \times \Sigma_m)$-equivariant maps $\mu_{n,m}(S')\colon \mathbf{KQ}_{\mathbb{S}^1}(S')_n \wedge \mathbf{KQ}_{\mathbb{S}^1}(S')_m \ \rightarrow\ \mathbf{KQ}_{\mathbb{S}^1}(S')_{n+m}$
4. and unit maps $i_0\colon \mathbb{S}^0 \rightarrow \mathbf{KQ}_{\mathbb{S}^1}(S')_0$ and $i_1\colon \mathbb{S}^1 \rightarrow \mathbf{KQ}_{\mathbb{S}^1}(S')_1$.

In this case, the maps $\mu_{n,m}$, i_0 and i_1 are not strictly the commutative and units maps, but they verify these properties up to coherent higher homotopies.

4. $mod_{\mathbf{KQ}_{\mathbb{S}^1}}$ **:** The evaluation of an object of this category in $S' \in (Sm\ |_S)^{\circ}$ gives rise to the following objects:

1. $M(S')_n \in \mathsf{sSet}_*\ \forall n \geq 0$,
2. with structural maps $\alpha_{n,m}(S')\colon M(S')_n \wedge \mathbf{KQ}_{\mathbb{S}^1,m}(S') \ \rightarrow\ M(S')_{n+m}\ \forall n, m \geq 0$,
3. and $\sigma_n(S')\colon M(S')_n \wedge \mathbb{S}^1 \ \rightarrow\ M(S')_{n+1}\ \ \forall n \geq 0$.

These are the $\mathbf{KQ}_{\mathbb{S}^1}$-modules in $\mathbb{S}^1$-spectra.

Now, we are going to define the symmetric sequences over $mod_{\mathbf{KQ}}$. The symmetric sequences are mainly used as intermediate step to construct symmetric spectra. The idea behind is that they form a simpler and larger category (with more objects and morphisms that symmetric spectra) where it is easy to put a closed symmetric monoidal structure such that it will induce the desired smash product in the modules over a commutative monoid. There is a functor category definition easier that the one shown, but before we need to define the category Σ.

Definition 2.5.1. Let $\Sigma = (\Sigma_n)_n$ be the graded monoid of symmetric groups, which have as objects the one object categories Σ_n with morphisms the n-symmetric group, i.e, Σ_n is the finite set $\overline{n} = \{1, 2, ..., n\}$ for all $n \geq 0$ and the morphisms are the bijections.

A symmetric sequence in a category $\mathcal{C}$ is a Σ_n-representation, in other words, a sequence $(X_n)_{n \geq 0}$ of objects of the category ($mod_{\mathbf{KQ}}$ in our case) with actions $a_n\colon \Sigma_n \rightarrow Aut\,(X_n)$. But it is sometimes better to define this category as $\mathcal{C}^{\Sigma} = Func(\Sigma, \mathcal{C})$. Then we can define the category $mod^{\Sigma}_{\mathbf{KQ}}$ as $mod^{\Sigma}_{\mathbf{KQ}} = Func(\Sigma, mod_{\mathbf{KQ}})$.

Restricting our attention to the commutative monoids of this last category we obtain the category $C.Alg\left(mod^{\Sigma}_{\mathbf{KQ}}\right)$ of commutative algebras.

In what follows, we are going to see the construction of general spectra and symmetric spectra for models categories to reflect better the idea behind these spectra. We start with the original idea of spectra. Let us take a model category $\mathcal{C}$ with all the necessary requirements, normally one requires to be cofibrantly generated and cellular among other properties, and G a left Quillen endofunctor in $\mathcal{C}$ with right adjoint U. Now we want to invert G in a compatible way with the model structure, i.e, to obtain a Quillen equivalence. The category of spectra $Sp(\mathcal{C}, G)$ is the category with objects $X = (X_0, X_1, ...)$ together with maps $\sigma\colon GX_n \to X_{n+1}$ for all n. A map f between X and Y is a collection of maps $f_n\colon X_n \to Y_n$ commuting with the structured maps. This category inherits a projective model structure taking the weak equivalence and fibrations levelwise, and defining the projective cofibrations by left lifting property. This model structure is not the desired one, the prolongation of the functor G remains a left Quillen functor but not a Quillen equivalence. The spectra which are fibrant levelwise and the adjoints of theirs structure maps $\tilde{\sigma}\colon X_n \to UX_{n+1}$ are weak equivalence for all n are called U-spectra. This is the natural generalization of the Ω-spectra. To transform the functor G in a Quillen equivalence we make a Bousfield localization taking the set of morphisms $\mathcal{S}$ in such a way that the $\mathcal{S}$-fibrant objects are the U-spectra. To carry out the Bousfield localization it is necessary to call for a left proper and cellular model category structure in $\mathcal{C}$. This model structure over $Sp(\mathcal{C}, G)$ is the stable model structure, in which G is a Quillen equivalence.

If G is already a Quillen equivalence then the embedding $\mathcal{C} \to Sp(\mathcal{C}, G)$ is also a Quillen equivalence providing us a way to see how universal is this stabilization. This fact allows us to show that $Sp(\mathcal{C}, G)$ is the initial, up to homotopy, stabilization of $\mathcal{C}$ respect to G.

The category $Sp(\mathcal{C}, G)$ has the handicap that if $\mathcal{C}$ is already a symmetric monoidal category and G a monoidal functor $Sp(\mathcal{C}, G)$ is almost never a symmetric monoidal category. To avoid this problem and define a correct smash product it was introduced the symmetric spectra. This applies to the special case when $\mathcal{C}$ is a bicomplete category enriched, tensored, and cotensored over $(\mathcal{D}, \otimes)$, a bicomplete symmetric monoidal model category, and G a $\mathcal{D}$-functor given by tensoring with a cofibrant object $K \in \mathcal{D}$, i.e, $G\colon X \to X \otimes K$. Henceforth, we fix the notation denoting the functor G just as K. We define the category $Sp^{\Sigma}(\mathcal{C}, K)$ as follows.

Definition 2.5.2. The category $Sp^{\Sigma}(\mathcal{C}, K)$ is the category with objects the sequences $X = (X_0, X_1, \dots)$ of objects of $\mathcal{C}$ with an action of the symmetric groups Σ_n in every X_n and such that the iterated structural maps $X_n \otimes K^{\otimes p} \to X_{n+p}$ are $\Sigma_n \times \Sigma_p$-equivariant, where Σ_p acts on $K^{\otimes p}$.

To define the symmetric monoidal structure in $Sp^{\Sigma}(\mathcal{C}, K)$ we use the intermediate category of symmetric sequence and the fact that it is $\mathcal{D}$-enriched. For X and Y in $\mathcal{D}^{\Sigma}$ we can define the product levewise by:

$$(X \otimes_{\Sigma} Y)_n := \coprod_{p+q=n} \Sigma_n \otimes_{\Sigma_p \times \Sigma_q} (X_p \otimes Y_q)$$

If $(\mathcal{D}, \otimes)$ is a closed symmetric monoidal category , so is $\mathcal{D}^{\Sigma}$ with this product. The unit of this monoidal structure is the sequence $(S, 0, 0, ...)$, where S is the unit over $(\mathcal{D}, \otimes)$, and the closed structure is define levelwise by:

$$Hom\,(X, Y)_k = \prod Hom_{\Sigma_n}(X_n, Y_{n+k})$$

where for $X, Y \in \mathcal{C}^{\Sigma_n}$ $Hom_{\Sigma_n}(X, Y)$ is the equalizer of the two obvious maps $Hom\,(X, Y) \to (X \times \Sigma_n, Y)$. Since $\mathcal{C}$ is enriched, tensored and cotensored over $\mathcal{D}$, so is $\mathcal{C}^{\Sigma}$ over $\mathcal{D}^{\Sigma}$. If $\mathcal{C}$ is an enriched category over $\mathcal{D}$, so it is $\mathcal{C}^{\Sigma}$ over $\mathcal{D}^{\Sigma}$.

Now, we are going to take modules over a desired commutative monoid such that the spectra category obtained is a symmetric monoidal category. This monoid should be induced by the object K in a symmetric way. Let $Sym(X) = \coprod_{n=0}^{n=\infty} (X^{\otimes n}/\Sigma_n)$ be the functor that associated to every object the appropriate free commutative algebra. Let us take $Sym(K)$ for $(0, K, 0, ..., 0, ...) \in \mathcal{D}^{\Sigma}$, which gives the symmetric sequence $\left(S, K, K^{\otimes 2}, ..., K^{\otimes n}, ...\right)$. This is a commutative monoid in $\mathcal{C}^{\Sigma}$. We can define now the category $Sp^{\Sigma}(\mathcal{C}, K)$ as follows.

Definition 2.5.3. The category of symmetric spectra $Sp^{\Sigma}(\mathcal{C}, K)$ is the category of modules in $\mathcal{C}^{\Sigma}$ over the commutative monoid $Sym(K)$ in $\mathcal{D}^{\Sigma}$.

This definition agree with the one given before, but this form allows us to define a good smash product. Since $Sym(K)$ is a commutative monoid, the category $Sp^{\Sigma}(\mathcal{C}, K)$ is a bicomplete and closed symmetric monoidal category with $Sym(K)$ as unit. This product $M \wedge N = M \otimes_{Sym(K)} N$ can be given as the colimit of the diagram:

$$M \otimes Sym(K) \otimes N \rightrightarrows M \otimes N$$

Where the arrows in this coequalizer are given by $id \otimes m$ and $m \otimes id$. It is again possible to put a projective model structure on this category. Applying the same localization that for the not symmetric case we have afresh a stable model structure on which tensoring with K is a Quillen equivalence.

2.5.3 The Day convolution.

Let consider $mod^{\Sigma}_{\mathbf{KQ}}$ as functors $\Sigma \xrightarrow{F} mod_{\mathbf{KQ}}$ and represent by $F(\Sigma)$ all the modules where the functor acts.

The Day convolution gives a monoidal structure $(*_{\mathbf{KQ}}\ ,\ mod^{\Sigma}_{\mathbf{KQ}})$, coming from the one on $mod_{\mathbf{KQ}}$ which exists since $\mathbf{KQ}_{\mathbb{S}^1}$ is an E_∞ ring. This product is the left Kan extension of $\otimes_{\mathbf{KQ}} \circ (F \times G)\ :\ \Sigma \times \Sigma \to mod_{\mathbf{KQ}}$ along $\times_\Sigma \colon \Sigma \to \Sigma$ and it is expressed in the coend form as:

$$F * G = \int^{\Sigma_n\ ,\ \Sigma_m\ \in \Sigma} F\,(\Sigma_n) \wedge F\,(\Sigma_m) \wedge Hom_\Sigma\,(-\ ,\ \Sigma_n \times \Sigma_m)$$

Using the coend expression of the Day convolution, the Yoneda embedding, Fubini and the symmetry of the products, we can see that the lax monoidal functors correspond to commutative algebras over $mod^{\Sigma}_{\mathbf{KQ}}$. The category Σ is symmetric monoidal as it is also $(mod_{\mathbf{KQ}}\ ,\ \wedge_{\mathbf{KQ}})$ with the product induced by the typical coequalizer. For two given symmetric monoidal categories the Day convolution in the functors from one to other gives a symmetric monoidal structure, as one can see in [Gla13]. Then the resulting category $mod^{\Sigma}_{\mathbf{KQ}}$ is a symmetric monoidal category, and also $C.Alg(mod^{\Sigma}_{\mathbf{KQ}})$.

2.5.4 Model structure.

In this section we are going to exhibit the categories of the delooping as model categories using lifting arguments, i.e, the following statements are going to be always levelwise, we are going to work in the categories that we obtain once we have evaluate the presheaves $\mathcal{F}\colon\ (Sm\,|_S)^\circ \to \mathrm{Spt}_{\mathbb{S}^1}$ in a scheme of $Sm\,|_S$. So we will start giving a model structure to $\mathrm{Spt}_{\mathbb{S}^1}$, the category of simplicial $\mathbb{S}^1$-spectra. Since it is necessary for us to work with connective spectra along the delooping construction at the end of the section we are going to modify the models structures defined in the different categories using co-localizations. It should be note that these model structures that we are going to define are different from the ones employed for the rectification in subsection 2.5.1. Those ones were just defined for the rectification.

Concretely we are going the work in symmetric spectra that we saw in subsection 2.5.2, which is the most convenient definition of spectra for our delooping. In the general category of symmetric spectra (as in Section 2.5.2) one can choose several level model categories. According to the desired properties the model structure can be projective, injective or flat, depending on the kind of cofibrations and fibrations that we want to work with. In all these models they can be used stable weak equivalences instead of the level ones, obtaining stable model structures. And in each one of these models we can choose which levels are used to define the sets of morphisms, given rise to the absolute or the positive model structure. The positives model structures were develop to have a good model structure to induce a model structure on the commutative monoids. The positive model structures are given by taking into consideration just the homotopy information up to 0. In the case of the positive level model structure the objects in level 0 have no homotopical significance.

We are going to define the model structures stepwise, starting with $Spt^{\Sigma}_{\mathbb{S}^1}$ until arriving to the model structure over the commutative monoids in the category of $\mathbf{KQ}_{\mathbb{S}^1}$-modules for the commutative ring $\mathbf{KQ}_{\mathbb{S}^1}$. Let remember the definitions of morphisms that we are going to use along this subsection.

Definition 2.5.4.

1. A morphism $X \xrightarrow{f} Y$ in $Spt^{\Sigma}_{\mathbb{S}^1}$ is a *level weak equivalence* (resp. *level cofibration*, resp. *level fibration*) if $X_n \xrightarrow{f_n} Y_n$ is a weak equivalence (resp. cofibration, resp. fibration) for all $n \geq 0$.

2. A morphism $X \xrightarrow{f} Y$ in $Spt^{\Sigma}_{\mathbb{S}^1}$ is a *positive level fibration* if $X_n \xrightarrow{f_n} Y_n$ is a fibration for all $n > 0$.

3. A morphism $X \xrightarrow{f} Y$ in $G\ Spt_{\mathbb{S}^1}$, for a group G, is a *weak G-equivalence* (resp. *weak G-fibration*) if the underlying map in $Spt_{\mathbb{S}^1}$ is a weak equivalence.

4. A morphism $X \xrightarrow{f} Y$ in $G\ Spt_{\mathbb{S}^1}$, for a group G, is a *free G-cofibration* if it has the left lifting property respect to all the weak G-trivial fibrations.

5. A morphism $X \xrightarrow{f} Y$ in $Spt^{\Sigma}_{\mathbb{S}^1}$ is a *projective cofibration* if the latching morphism $\nu_n(f) : X_n \bigcup_{L_n X} L_n Y \to Y_n$ is a free Σ_n-cofibration for all $n \geq 0$.

6. A morphism $X \xrightarrow{f} Y$ in $Spt^{\Sigma}_{\mathbb{S}^1}$ is a *positive stable fibration* if it is a positive level fibration and the square

$$\begin{tikzcd} X \arrow[r, "\lambda_X"] \arrow[d, "f"] & \Omega(shX) \arrow[d, "\Omega(shf)"] \\ Y \arrow[r, "\lambda_Y"] & \Omega(shY) \end{tikzcd}$$

where shX denote the shifted spectra and λ_X is the adjoint of $\lambda'\colon \mathbb{S}^1 \wedge X \to X$, is homotopy cartesian after forgetting the symmetric group action for $n \geq 0$.

7. A morphism $X \xrightarrow{f} Y$ in $Spt^{\Sigma}_{\mathbb{S}^1}$ is a *positive projective cofibration* if it is a projective cofibration and at the level 0 we have that $X_0 \xrightarrow{f_0} Y_0$ is an isomorphism.

With these morphisms we can define easily a model structure over symmetric spectra. Let us take the positive projective stable model structure, as defined in [Sch07, Thm. 4.11, Chap. *III*]. This gives rise to a model structure such that we can induce other model structure on the modules over a commutative monoid, as $\mathbf{KQ}_{\mathbb{S}^1}$ by 2.5.1. In this model structure the equivalences are the stable weak equivalences, the cofibrations are the positive projective cofibrations, and the fibrations the positive stable fibrations. Moreover, this model structure is proper, topological, cofibrantly generated and monoidal respect to the smash product.

Lemma 11. The category of modules over $\mathbf{KQ}_{\mathbb{S}^1}$ in the category of symmetric spectra, that we denote by $mod_{\mathbf{KQ}_{\mathbb{S}^1}}$, has a proper, simplicial and cofibrantly generated model structure where the weak equivalences are the morphisms which are stable equivalences at the level of symmetric spectra, and the fibrations are the morphisms which are positive stable fibrations. Moreover, such a category is monoidal with respect to the smash product over $\mathbf{KQ}_{\mathbb{S}^1}$.

Proof. We use [Sch07, Thm. 1.3, Chap. *IV*] where it is defined the positive projective stable model structure on modules over a commutative ring spectrum. Moreover, this model structure is cofibrantly generated, as it has been proof in [Shi04]. □

Moreover, $mod_{\mathbf{KQ}_{\mathbb{S}^1}}$ has symmetric monoidal structure since $\mathbf{KQ}_{\mathbb{S}^1}$ is a ring spectrum.

The next step is to define a model structure over the symmetric sequence over $mod_{\mathbf{KQ}_{\mathbb{S}^1}}$. This category is naturally a symmetric monoidal category by the Day convolution product defined in subsection 2.5.3.

Lemma 12. The category $mod^{\Sigma}_{\mathbf{KQ}_{\mathbb{S}^1}}$ admits a cofibrantly generated model structure with the level weak equivalences, the level fibrations and cofibrations the morphisms which can be expressed as the retract of a transfinite composition of pushouts of generating cofibrations.

Proof. Since this is a diagram category as we have seen in Section 2.5.2 we can apply [Hir03, Thm. 11.6.1]. □

Lemma 13. The category $C.Alg(mod^{\Sigma}_{\mathbf{KQ}_{\mathbb{S}^1}})$ admits a cofibrantly generated model structure with weak equivalence (resp. fibration) in $C.Alg(mod^{\Sigma}_{\mathbf{KQ}_{\mathbb{S}^1}})$ if it is a weak equivalence (resp. fibration) as a morphism in $mod^{\Sigma}_{\mathbf{KQ}_{\mathbb{S}^1}}$ and the cofibrations are defined as the morphisms in $C.Alg(mod^{\Sigma}_{\mathbf{KQ}_{\mathbb{S}^1}})$ which have the left lifting property respect to the trivial fibrations.

Proof. Let us denote by I and J the sets of generating cofibrations and generating trivial cofibrations in $mod^{\Sigma}_{\mathbf{KQ}_{\mathbb{S}^1}}$. We can define the model structure on $C.Alg(mod^{\Sigma}_{\mathbf{KQ}_{\mathbb{S}^1}})$ by the forgetful functor. The left functor of this adjoint is the free functor, denoted by F. Since we are in a symmetric monoidal category $F(X)$ is the coproduct of all tensor powers over X. The right functor of the adjoint between these two model categories preserves weak equivalences and fibrations by definition, therefore it is a Quillen adjoint. We use [Hir03, Thm. 11.3.2], which is the transfer principle of Dan Kan, to define a right-induced model structure on $C.Alg(mod^{\Sigma}_{\mathbf{KQ}_{\mathbb{S}^1}})$. In this theorem, the sets of morphisms defined just before in $C.Alg(mod^{\Sigma}_{\mathbf{KQ}_{\mathbb{S}^1}})$ give rise to a model structure which is cofibrantly generated, with $F(I)$ and $F(J)$ the sets of generating cofibrations and generating trivial cofibrations respectively. Moreover, since it is a Quillen adjoint the equivalences are just the morphisms which forget to weak equivalences via U. □

All the model structures defined until here are for general spectra. The spectrum $\mathbf{KQ}_{\mathbb{S}^1}$ is a connective spectrum and we would like to preserve such property. And not just that, for the delooping construction we finish with a functor (the left adjoint in 2.5.5) which must be a forgetful functor. This is not possible if we are not considering connective spectra.

To work with connective spectra we are going to use co-localizations, i.e, right Bousfield localizations. In the right Bousfield localization we localize respect to a set of objects instead of a set of morphisms as in the usual Bousfield localizations. And instead of the left adjoint functor of the inclusion functor we have a right adjoint respect to the inclusion. In a general context, we start with a set of objects S in a model category $\mathcal{C}$ and after applying the right Bousfield localization R_S we get $R_S\mathcal{C}$ a new model structure in the same category such that it has the same fibrations, S is contained in the class of cofibrant objects and the weak equivalences are the weak equivalences by S (given by the homotopy function objects).

Instead of co-localizing the positive stable model structure on symmetric spectra we can co-localize directly in the category $mod_{\mathbf{KQ}_{\mathbb{S}^1}}$, which is the category where we really want to co-localize. In [WY16] it was stabilized a criterion such that if one can co-localize in category automatically this induces a co-localization in the category of algebras respect to a monad. By [Bar10, Thm. 5.13] we can co-localize in the category of spectra to obtain the category of connective spectra. Since $\mathbf{KQ}_{\mathbb{S}^1}$ is a connective ring spectrum we can apply [WY16, Thm. 5.4] and we obtain that this co-localization induces a co-localization at the level of modules over $\mathbf{KQ}_{\mathbb{S}^1}$. In fact, it is the same taking first the co-localization and then the modules in the inverse way. So, it exists a model structure in $mod_{\geq 0, \mathbf{KQ}_{\mathbb{S}^1}}$, the category of modules over $\mathbf{KQ}_{\mathbb{S}^1}$ in symmetric connective spectra.

Corollary 2.5.5. *The category modules over $\mathbf{KQ}_{\mathbb{S}^1}$ in connective spectra, $mod_{\geq 0, \mathbf{KQ}_{\mathbb{S}^1}}$, has a model structure.*

The rest of the induced model structure in the categories $mod^{\Sigma}_{\geq 0, \mathbf{KQ}_{\mathbb{S}^1}}$ and $C.Alg(mod^{\Sigma}_{\geq 0, \mathbf{KQ}_{\mathbb{S}^1}})$ can be done in an analogous way that for usual spectra.

2.5.5 Delooping.

In this section, we are going to develop the construction appearing in [Ost10] for usual K-theory but in the hermitian setting. This construction can be more general, as it was said in the general delooping construction (pg. 114). In Section 2.3 we have applied the direct sum K-theory to the category with duality of the algebraic vector bundles and we got a spectrum that we called $\mathbf{KQ}_{\mathbb{S}^1}$. We also constructed the spectrum $\mathbf{KH}$ in Section 2.4, but we are going to apply this construction to $\mathbf{KQ}_{\mathbb{S}^1}$. In fact, there is no difference between applying the delooping to one or another. The spectrum $\mathbf{KQ}_{\mathbb{S}^1}$ has the structure of an E_∞-ring but it is a $\mathbb{S}^1$-spectrum not a $\mathbb{P}^1$-spectrum. The delooping construction will produce a new $\mathbb{P}^1$-spectrum $\mathbf{KQ}_{\mathbb{P}^1}$ with a structure of commutative algebra. The usefulness of the delooping construction is that it produce a new spectrum, in desired the category of spectra, with a natural commutative algebra structure and in consequence an E_∞-ring structure.

Along the section we will denote by $mod_{\mathbb{S}^0}$ the category of $\mathbb{S}^1$-spectra, with the idea that the sphere spectrum $\mathbb{S}^0$ is the unit in such a category, and analogously for $mod_{\mathbf{KQ}_{\mathbb{S}^1}}$, the category of $\mathbf{KQ}_{\mathbb{S}^1}$ modules in $mod_{\mathbb{S}^0}$. This is an E_∞-ring, not an strict commutative ring. Then, we use the subsection 2.5.1 to correct to an strict commutative ring that will be also denoted by $\mathbf{KQ}_{\mathbb{S}^1}$.

Remark. During the construction there are used different model structures in the same categories. The model structure defined in subsection 2.5.1 was done only for giving a strict ring structure to $\mathbf{KQ}_{\mathbb{S}^1}$, while the model structures of subsection 2.5.4 are used along the delooping construction.

Remark. Along the section we are going to use implicitly connective instead of general spectra. Then, the resulting spectrum will be a connective spectrum, and the functor f at the end of the construction a forgetful functor.

To apply the delooping we are going to use the Bott element defined in Definition 1.3.9. Remember that we had a Bott element $\beta\colon \mathbb{HP}^1 \wedge \mathbb{HP}^1 \to \mathbf{KQ}_{\mathbb{S}^1}$. Let us move $\mathbf{KQ}_{\mathbb{S}^1}$ to $mod_{\mathbf{KQ}_{\mathbb{S}^1}}$ via the Quillen left functor $- \wedge \mathbf{KQ}_{\mathbb{S}^1} \colon mod_{\mathbf{KQ}_{\mathbb{S}^1}} \to mod_{\mathbb{S}^0}$. Since this functor preserves cofibrations and trivial cofibrations we can use the unit $\mathbb{1} \in Mod_{\mathbb{S}^0}$, which is cofibrant, to prove that $\mathbf{KQ}_{\mathbb{S}^1}$ is cofibrant as object of $Mod_{\mathbf{KQ}_{\mathbb{S}^1}}$. Taking the fibrant replacement $\mathbf{KQ}_{\mathbb{S}^1} \hookrightarrow R\mathbf{KQ}_{\mathbb{S}^1}$ in $Mod_{\mathbf{KQ}_{\mathbb{S}^1}}$ it exists a map $\Sigma^{\infty}_{\mathbb{S}^1}\left(\mathbb{HP}^1\right)^{\wedge 2} \to R\mathbf{KQ}_{\mathbb{S}^1}$ induced by β, and then we can take the corresponding $\mathbf{KQ}_{\mathbb{S}^1}$-linear extension $Q\Sigma^{\infty}_{\mathbb{S}^1}\left(\mathbb{HP}^1\right)^{\wedge 2} \wedge \mathbf{KQ}_{\mathbb{S}^1} \to R\mathbf{KQ}_{\mathbb{S}^1}$, in which we will apply the functor $Sym_{\mathbf{KQ}_{\mathbb{S}^1}}$.

Remark. Since the notation is already complicated, we are going to still denote the by $\beta\colon \mathbb{HP}^1 \wedge \mathbb{HP}^1 \to \mathbf{KQ}_{\mathbb{S}^1}$ when we will mean $Q\Sigma^{\infty}_{\mathbb{S}^1}\left(\mathbb{HP}^1\right)^{\wedge 2} \wedge \mathbf{KQ}_{\mathbb{S}^1} \to R\mathbf{KQ}_{\mathbb{S}^1}$.

We want to construct an adjoint

$$Sym_{\mathbf{KQ}_{\mathbb{S}^1}}(-)\colon mod_{\mathbf{KQ}_{\mathbb{S}^1}} \;\leftrightarrows\; CAlg\left(mod^{\Sigma}_{\mathbf{KQ}_{\mathbb{S}^1}}\right)$$

between $\mathbf{KQ}_{\mathbb{S}^1}$-modules in the category the category of symmetric spectra and commutative algebras in the category of symmetric sequences of $\mathbf{KQ}_{\mathbb{S}^1}$-modules.

Defining $Sym_{\mathbf{KQ}_{\mathbb{S}^1}}(-)$ as the functor that assign to each object the sequence given by smashing with the successive powers of $\mathbf{KQ}_{\mathbb{S}^1}$, and as right adjoint the forgetful which pick up the first entry of the sequence, we get the desired adjunction. If we apply the functor $Sym_{\mathbf{KQ}_{\mathbb{S}^1}}(-)$ to $\beta\colon \Sigma^{\infty}_{\mathbb{S}^0}\mathbb{T}^{\wedge 4} \wedge \mathbf{KQ}_{\mathbb{S}^1} \to \mathbf{KQ}_{\mathbb{S}^1}$, we obtain:

$$\begin{aligned} Sym_{\mathbf{KQ}_{\mathbb{S}^1}}(\beta) \;:\; Sym_{\mathbf{KQ}_{\mathbb{S}^1}}\left(\Sigma^{\infty}_{\mathbb{S}^0}\mathbb{T}^{\wedge 4} \wedge \mathbf{KQ}_{\mathbb{S}^1}\right) &\longrightarrow Sym_{\mathbf{KQ}_{\mathbb{S}^1}}\left(\mathbf{KQ}_{\mathbb{S}^1}\right) \\ \left(\Sigma^{\infty}_{\mathbb{S}^0}\mathbb{T}^{\wedge 4} \wedge \mathbf{KQ}_{\mathbb{S}^1}, ...\right) &\longrightarrow \left(\mathbf{KQ}_{\mathbb{S}^1}, \mathbf{KQ}_{\mathbb{S}^1}, ...\right) \end{aligned}$$

Then $Sym_{\mathbf{KQ}_{\mathbb{S}^1}}\left(\mathbf{KQ}_{\mathbb{S}^1}\right)$ has a structure of commutative monoid over the $Sym_{\mathbf{KQ}_{\mathbb{S}^1}}\left(\Sigma^{\infty}_{\mathbb{S}^0}\mathbb{T}^{\wedge 4} \wedge \mathbf{KQ}_{\mathbb{S}^1}\right)$-modules on symmetric $\mathbf{KQ}_{\mathbb{S}^1} - Spt$. We can work analogously with the spectrum $\mathbb{S}^0$ and get a structure of $Sym_{\mathbb{S}^0}\left(\Sigma^{\infty}_{\mathbb{S}^0}\mathbb{T}^{\wedge 4}\right)$-module on it.

$$\begin{tikzcd} mod_{\mathbf{KQ}_{\mathbb{S}^1}} \arrow[r, shift left] \arrow[d, shift left, "U"] & CAlg\left(mod^{\Sigma}_{\mathbf{KQ}_{\mathbb{S}^1}}\right) \arrow[l, shift left] \arrow[d, shift left, "mod_{\mathbb{S}^0}(U)"] \\ mod_{\mathbb{S}^0} \arrow[u, shift left, "-\wedge\mathbf{KQ}_{\mathbb{S}^1}"] \arrow[r, shift left] & CAlg\left(mod^{\Sigma}_{\mathbb{S}^0}\right) \arrow[l, shift left] \arrow[u, shift left, "Sym_{\mathbf{KQ}_{\mathbb{S}^1}}(-\wedge\mathbf{KQ}_{\mathbb{S}^1})"] \end{tikzcd}$$

This induces an adjoint:

$$Sym_{\mathbf{KQ}_{\mathbb{S}^1}}\left(\Sigma^{\infty}_{\mathbb{S}^0}\mathbb{T}^{\wedge 4}\wedge \mathbf{KQ}_{\mathbb{S}^1}\right) - mod^{\Sigma}_{\mathbf{KQ}_{\mathbb{S}^1}} \underset{\longleftarrow}{\xrightarrow{f}} Sym_{\mathbb{S}^0}\left(\Sigma^{\infty}_{\mathbb{S}^0}\mathbb{T}^{\wedge 4}\right) - mod^{\Sigma}_{\mathbb{S}^0}$$

The functor f is just the forgetful functor, even if one still have the suspension respect to $\mathbf{KQ}_{\mathbb{S}^1}$. The resulting object after apply the functor f is the $\left(\mathbb{HP}^1\right)^{\wedge 2}$-spectrum $:= f\left(Symm_{\mathbf{KQ}_{\mathbb{S}^1}}\left(\mathbf{KQ}_{\mathbb{S}^1}\right)\right) = \left(\mathbf{KQ}_{\mathbb{S}^1}, \mathbf{KQ}_{\mathbb{S}^1}, ...\right)$, that we are going to denote by $\mathbf{KQ}_{(\mathbb{HP}^1)^{\wedge 2}}$. Since this spectrum is in the category of commutative algebras we get a ring structure on it, in fact, a strict commutative ring structure.

Corollary 2.5.6. *The delooping construction applied to the spectrum $\mathbf{KQ}_{\mathbb{S}^1}$ produces a strict commutative ring $\left(\mathbb{HP}^1\right)^{\wedge 2}$-connective spectrum $\mathbf{KQ}_{(\mathbb{HP}^1)^{\wedge 2}}$.*

We want now to get a $\mathbb{P}^1$-spectrum from the spectrum $\mathbf{KQ}_{(\mathbb{HP}^1)^{\wedge 2}}$. In [PW10c], it is proved that the motivic homotopy categories of $\mathbb{T}$-spectra and $\mathbb{HP}^1$-spectra are equivalents. In fact, we can argue analogously with $\mathbb{P}^1$-spectra, $\mathbb{T}$-spectra, $\mathbb{HP}^1$-spectra or $\left(\mathbb{HP}^1\right)^{\wedge 2}$-spectra. Since the spectrum that we want is a $\mathbb{P}^1$-spectrum, and moreover an E_∞-structure on it, we are going to see that we can obtain a $\mathbb{P}^1$-spectrum $\mathbf{KQ}_{\mathbb{P}^1}$ from $\mathbf{KQ}_{(\mathbb{HP}^1)^{\wedge 2}}$ preserving the structure. We can define a category of bispectra, let us denote it by $\mathrm{Spt}(\left(\mathbb{HP}^1\right)^{\wedge 2}, \mathbb{P}^1)$, as we did for $\mathbb{S}^1_s$ and $\mathbb{S}^1_t$ in Definition 1.1.13 but with the motivic spaces $\left(\mathbb{HP}^1\right)^{\wedge 2}$ and $\mathbb{P}^1$. Let denote $\mathrm{Spt}_{(\mathbb{HP}^1)^{\wedge 2}}$ and $\mathrm{Spt}_{\mathbb{P}^1}$ the categories of motivic $\left(\mathbb{HP}^1\right)^{\wedge 2}$-spectra and motivic $\mathbb{P}^1$-spectra respectively. There are two natural inclusions $i_1\colon \mathrm{Spt}_{(\mathbb{HP}^1)^{\wedge 2}} \hookrightarrow \mathrm{Spt}(\left(\mathbb{HP}^1\right)^{\wedge 2}, \mathbb{P}^1)$ and $i_2\colon \mathrm{Spt}_{\mathbb{P}^1} \hookrightarrow \mathrm{Spt}(\left(\mathbb{HP}^1\right)^{\wedge 2}, \mathbb{P}^1)$. Considering the respective forgetful functors, u_1 and u_2, we get the adjoints (i_1, u_1) and (i_1, u_1). Since $\left(\mathbb{HP}^1\right)^{\wedge 2} \simeq \left(\mathbb{P}^1\right)^{\wedge 4}$ we get $\left(\mathbb{HP}^1\right)^{\wedge 2}$-spectra by tensoring. Then, the adjoints (i_1, u_1) and (i_1, u_1) turn to be Quillen equivalences. We can find a more explicit proof of this in [PW10b]. What we are doing is tensoring with something which was already invertible, so we get that it also induces adjunctions at the level of E_∞-rings. After applying these adjoints we denote the resulting E_∞-ring $\mathbb{P}^1$-spectrum by $\mathbf{KQ}_{\mathbb{P}^1}$, getting the following result.

Theorem 2.5.7. *There is an E_∞-ring connective $\mathbb{P}^1$-spectrum $\mathbf{KQ}_{\mathbb{P}^1}$ representing the motivic Hermitian K-theory.*

Remark. We have done the whole delooping construction for $(\mathbb{HP}^1)^{\wedge 2}$ and the presheaf $\mathbf{KQ}_{\mathbb{S}^1}$ but with the tools developed along this section we could do it for a general presheaf of spectra $\mathcal{K}$, and a general motivic pointed space $\mathbb{F} \in \mathsf{Spc}_\bullet(S)$. This would give us a commutative algebra in the category of presheaves of $\mathrm{Spt}^{\Sigma}_{\mathbb{F}}$. Moreover, this case is also covered by the rectification Section 2.5.1, and by Lemma 10 we would get an E_∞-algebra in the category of presheaves of $\mathrm{Spt}^{\Sigma}_{\mathbb{F}}$.

Remark. Note that the $\mathbb{P}^1$-spectrum $\mathbf{KQ}_{\mathbb{P}^1}$ is equivalent to the $\mathbb{S}^1$-spectrum $\mathbf{KQ}_{\mathbb{S}^1}$ as motivic symmetric $\mathbb{S}^1$-spectra since they are both connective spectra with the same spaces and bonding maps, they have the same motivic homotopy type as $\mathbb{S}^1$-spectra.

2.5.6 Uniqueness.

In this section we are going to prove an uniqueness result for the multiplicative structure in $\mathbf{KQ}_{\mathbb{P}^1}$. We would like to get a an uniqueness result as in [Ost10] but it stills not know the existence of phantom maps in the Hermitian K-theory, so we can not use them to prove such uniqueness. To tackle the problem we are going to compare it with the Panin Walter spectrum using an uniqueness result from [PW10b]. In fact, we are going to see that the spectra $\mathbf{BO}$ and $\mathbf{KQ}_{\mathbb{P}^1}$ are equivalents as commutative monoids up homotopy. In [PW10b], Panin and Walter proved the uniqueness of its commutative monoid structure of $\mathbf{BO}$ in $\mathsf{SH}(k)$ provide a restriction condition is verified. Since $\mathbf{KQ}_{\mathbb{P}^1}$ has a structure of E_∞-ring it also induces a commutative monoid in $\mathsf{SH}(S)$.

Remark. Along this section when speaking about the hermitian K-theory spectrum we are going to assume that 2 is invertible in the base field.

The restriction condition concerns to the induced pairing on the Grothendieck-Witt groups. The spectra $\mathbf{BO}$ and $\mathbf{KQ}_{\mathbb{P}^1}$ are both $\mathbb{P}^1$-spectra and they can be seen as $\mathbb{HP}^1$-spectra. In this from, they agree by construction for even spaces when restricting to Grothendieck-Witt groups. We want to see that our spectrum $\mathbf{KQ}_{\mathbb{P}^1}$ verifies the apart (b) of [PW10b, Thm. 13.4], and then its commutative monoid structure also verifies such uniqueness. Let us remember the theorem and then clarify the concepts involved on it.

Theorem 2.5.8 (Thm. 13.4, [PW10b]). *Suppose that $\mathsf{KO}_1(S)$ and $\mathsf{KSp}_1(S)$ are finite groups, e.g, $S = Spec\,(\mathbb{Z}[1/2])$. Let $m \in \mathrm{Hom}_{\mathsf{SH}(S)}(\mathbf{BO} \wedge \mathbf{BO}, \mathbf{BO})$ be the morphism defined in [PW10b, Lemma 11.3] and let $e \in \mathrm{Hom}_{\mathsf{SH}(S)}(pt_+, \mathbf{BO}) = \mathbf{BO}^{0,0}(pt_+)$ be the element corresponding to $1 \in \mathrm{GW}^+(pt) = \mathsf{KO}_0^{[0]}(pt)$.*

1. *Then $(\mathbf{BO}, m, e)$ is a commutative monoid in $\mathsf{SH}(S)$.*
2. *The map m is the unique element of $\mathrm{Hom}_{\mathsf{SH}(S)}(\mathbf{BO} \wedge \mathbf{BO}, \mathbf{BO})$ defining a pairing which, when restricted to pairing*

$$\mathbf{BO}^{4p,2p}(X_+) \times \mathbf{BO}^{4q,2q}(Y_+) \to \mathbf{BO}^{4p+4q,2p+2q}(X_+ \wedge Y_+)$$

with $X, Y \in \mathrm{Sm}\,|_S$ coincides with the tensor product pairing

$$\mathsf{KO}_0^{[2p]}(X_+) \times \mathsf{KO}_0^{[2q]}(Y_+) \to \mathsf{KO}_0^{[2p+2q]}(X_+ \wedge Y_+)$$

on the Grothendieck-Witt groups.

Remark.

1. We should not care about the finiteness requirement at the start of the theorem since we have assume that 2 is invertible at the start of the section. Otherwise, we could consider our result for the case $S = Spec\,(\mathbb{Z}[1/2])$ and therefore using base change we could extend it to a scheme from $\mathrm{Sm}\,|_S$.

2. Our spectrum $\mathbf{KQ}_{\mathbb{P}^1}$ has an E_∞-ring structure and induces commutative monoid structure in $\mathsf{SH}(S)$. To see that this commutative monoid structure agrees with the one given in the theorem above we just need to care about the second point.

3. By the periodicity of the spectrum **BO** to check the statement for $p = q = 2$ is the same to prove it for all even p and q, and the same for the odd case.

4. Our spectrum $\mathbf{KQ}_{\mathbb{P}^1}$ is an $(\mathbb{HP}^1)^{\wedge 2}$-spectrum so we can just check the second point, i.e, for $p = q = 2$.

5. To supply the missing products we will extend the result for $p = q = 1$ by taking loops.

In [PW10b] Panin and Walter extend the definition of pairings for Witt groups given by Gille and Nenashev in [GN03] to Grothendieck-Witt groups. The multiplicative structure used in the theorem comes from applying this pairings to the Schlichting construction. We are going to revise the construction of such a pairing and we will see that in the particular case of the point (b) of Theorem 2.5.8 the pairing defining such a product comes from the same product that our product in the spectrum **KQ** and in consequence our product reduces to the product in the second point of the theorem.

In [Shc10b], Schlichting defined the hermitian K-theory using the Waldhausen style for exact categories with weak equivalences and duality. More concretely, for complicial exact categories with weak equivalences and duality (see [Sch12] for the definition). The Schlichting construction does not give an spectrum but a space. In our case, for $S = Spec\,(\mathbb{Z}[1/2])$, the negative groups are given by the Balmer construction for triangulated categories with duality since these groups agree in the homotopy category $\mathcal{H}o\,(\mathcal{C}, W) = \mathcal{C}[w^{-1}]$, for $\mathcal{C}\,(*, \nu)$ (see § 1.2.3). Let us denote the obtained space by $\mathsf{KO}^{[n]}(\mathcal{C}, *, \omega, \nu)[n]$ for a exact category with duality $(\mathcal{C}, *, \omega, \nu)$ (see § 1.2.3), where the duality is shifted by n. The homotopy groups are indicate by the subscript i, $\mathsf{KO}_i^{[n]}(\mathcal{C}, *, \omega, \nu)[n] = \pi_i \mathsf{KO}^{[n]}(\mathcal{C}, *, \omega, \nu)[n]$ for $i \geq 0$.

The multiplicative structure m in $\mathbf{BO}$ comes from the partial multiplicative structure in KO, which in turn is the induced from the Schlichting localization theorem for exact categories with duality and weak equivalences in the special case of algebraic vector bundles. But the pairing restricting to Grothendieck-Witt groups can be induced from the tensor product in a explicit way. Let us denote by m' our multiplicative structure in $\mathbf{KQ}$, which comes from the tensor product. It induces an element, let us also denote it by m', in $\mathrm{Hom}_{\mathsf{SH}(S)}(\mathbf{BO} \wedge \mathbf{BO}, \mathbf{BO})$ defining a pairing as in Theorem 2.5.8. This multiplicative structure also comes from the tensor product in the bundles and defines the same pairing for the Grothendieck-Witt groups KO_0, at least for $p = q = 2$, as we will see. To define it we need first to know what is a pairing between such categories.

Definition 2.5.5. [PW10b, § 4] A *pairing of complicial exact categories with weak equivalences and duality*, in short we will say just *pairing*, is an additive bifunctor

$$(\ominus, t_1, t_2, \lambda): \ (\mathcal{C}_1, \omega_1, *_1, \nu_1) \times (\mathcal{C}_2, \omega_2, *_2, \nu_2) \to (\mathcal{C}_3, \omega_3, *_3, \nu_3)$$

which commutes with the shifts via the isomorphisms $t_{1,X,Y}: X[1] \ominus Y \simeq (X \ominus Y)[1]$ and $t_{2,X,Y}: X \ominus Y[1] \simeq (X \ominus Y)[1]$ plus weak equivalences $\lambda_{X,Y}: X^{*_1} \ominus Y^{*_2} \to (X \ominus Y)^{*_3}$, and such that for any $X \in \mathcal{C}_1$ and $X \in \mathcal{C}_2$ the functors $X \ominus -$ and $- \ominus Y$ are exact preserving weak equivalences functors. Moreover, it must verify the properties from [GN03, Def. 1.2 and Def. 1.11].

In what follows along this section (unless stated otherwise) S will be a regular Noetherian separated scheme of finite Krull dimension with $1/2 \in \Gamma(S, \mathcal{O}_S)$ and X and S-scheme. We restrict our construction to this case to use Theorem 2.5.8 and then by Functoriality of the base change we will get the general result.

The complicial exact structure with duality and weak equivalences that we are interested on is

$$(Ch(Vb(X)), \omega_X, \mathrm{Hom}_{\mathcal{O}_X}(-, \mathcal{O}_X), \nu_X)$$

where $Ch(Vb(X))$ is the additive category of bounded complex of big algebraic vector bundles on $X \in Sm|_S$ (see [FS02]). The duality functor and natural transformation are the usual ones in such a category. The coflations are the degreewise-split short exact sequences, and the weak equivalences, denoted by ω_X, correspond to the quasi-isomorphisms.

We have a pairing given by the tensor product of algebraic vector bundles. In fact, it agrees with the above definition for $(\mathcal{C}_1, \omega_1, *_1, \nu_1) = (\mathcal{C}_2, \omega_2, *_2, \nu_2) = (\mathcal{C}_3, \omega_3, *_3, \nu_3) = (Ch(Vb(X)), \omega_X, \mathrm{Hom}_{\mathcal{O}_X}(-, \mathcal{O}_X), \nu_X)$ and $\ominus = \otimes$. For a fixed X we denote the associated spaces from the Schlichting construction as $\mathsf{KO}^n(X)$, i.e,

$$\mathsf{KO}^{[n]}(X) = \mathsf{KO}^{[n]}\left(Ch(Vb(X)), \omega_X, \mathrm{Hom}_{\mathcal{O}_X}\left(-, \mathcal{O}_X\right), \nu_X\right)$$

We also have to introduce a restriction version of these spaces (see [PW10b, § 5]). Suppose $U \subset X$ is an open subscheme of X, and Z the closed subscheme $X - U$. We define a new set of weak equivalences ω_U, consisting of the chain complex maps whose restriction to U is a quasi-isomorphism. Let $Ch(Vb(X))^{\omega_U}$ be the full additive subcategory of complexes which are acyclic on U. We get the Hermitian spaces

$$\mathsf{KO}^{[n]}(X, U) = \mathsf{KO}^{[n]}\left(Ch(Vb(X))^{\omega_U}, \omega_X, \mathrm{Hom}_{\mathcal{O}_X}\left(-, \mathcal{O}_X\right), \nu_X\right)$$

$$\mathsf{KO}^{[n]}(U) = \mathsf{KO}^{[n]}\left(Ch(Vb(X)), \omega_U, \mathrm{Hom}_{\mathcal{O}_X}\left(-, \mathcal{O}_X\right), \nu_X\right).$$

The hermitian groups are defined by taking homotopy groups

$$\mathsf{KO}_i^{[n]}(X, U) = \begin{cases} \pi_i \mathsf{KO}^{[n]}(X, U) & fori \geq 0 \\ W^{n-i}\left(D^b Vb(X) on Z\right) & fori < 0 \end{cases}$$

In our case we are just interested in the Grothendieck-Witt groups

$$\mathsf{KO}_0^{[n]}(X) = \pi_0 \mathsf{KO}^{[n]}(X)$$

Consider the vector bundles (V, ϕ) and (W, ψ) with symmetric forms of degrees r and s (see Section 1.2.3) in $Ch(Vb(X))$ and $Ch(VB(Y))$, respectively. Let $(\otimes, t_1, t_2, \lambda) : Ch(Vb(X)) \times Ch(VB(Y)) \to Ch(VB(Z))$ be the pairing given by the tensor product, which is a pairing in the sense of 2.5.5. These hermitian objects generate the duality-preserving exact functors and thus maps in the Hermitian K-theory spaces, as it was proved in [GN03] by Gille and Nenashev. We get

$$\left(- \otimes (W, \psi)\right)_* : \mathsf{KO}^{[n]}(X) \to \mathsf{KO}^{[n]}(Z)[s]$$

$$\left((V, \phi) \otimes -\right)_* : \mathsf{KO}^{[n]}(Y) \to \mathsf{KO}^{[n]}(Z)[r]$$

These maps depend only of the classes of $[(V, \phi)]$ and $[(W, \psi)]$ ([PW10b, Prop. 4.2]) so we get the pairings

$$\mathsf{KO}_i^{[n]}(X) \times \mathsf{KO}_0^{[s]}(Y) \to \mathsf{KO}_i^{[n+s]}(Z)$$

$$\mathsf{KO}_0^{[r]}(X) \times \mathsf{KO}_j^{[m]}(Y) \to \mathsf{KO}_j^{[r+m]}(Z)$$

Since the spectrum $\mathbf{BO}$ and the spectrum $\mathbf{KQ}_{\mathbb{P}^1}$ agree as $(\mathbb{HP}^1)^{\wedge 2}$-spectra, we are going to care first about the even case. The tensor product pairing give rise, by restricting to the Grothendieck-Witt groups and the pointed case $Z_+ = X_+ \times Y_+$, to a pairing on the spaces of the form

$$\mathsf{KO}_0^{[2p]}(X_+) \times \mathsf{KO}_0^{[2q]}(Y_+) \to \mathsf{KO}_0^{[2p+2q]}(X_+ \wedge Y_+)$$

for evens p and q. More concretely, we have

$$\mathsf{KO}_0^{[4]}(X_+) \times \mathsf{KO}_0^{[4]}(Y_+) \to \mathsf{KO}_0^{[8]}(X_+ \wedge Y_+)$$

This is the pairing induced by the tensor product on KO and the same one should be induced on $\mathbf{KQ}_{\mathbb{P}^1}$.

Now, we are going to check the equivalent pairing on the construction of $\mathbf{KQ}_{\mathbb{P}^1}$. We will start by seeing that the tensor product produces a pairing and that it is preserved in $\mathbf{KQ}_{\mathbb{S}^1}$. It follows directly that it is the pairing on $\mathbf{KQ}_{\mathbb{P}^1}$ since the delooping is just the spectrum $\mathbf{KQ}_{\mathbb{S}^1}$ in each entry. To get the same pairing at the end we need to see that the multiplicative structure, which is preserved along the sequence of functors defined in Definition 2.3.6, preserves such a pairing. So we will put the pairing according to the multiplicative structure and we will see that it is preserved.

First we have to define what is a pairing in such case. The pairings in the Panin and Walter construction are done by taking tensor respect to an symmetric object of degree r. The resulting pairing lies on the spaces $\mathsf{KO}^{[n]}$ and then it is induced to the spectrum $\mathbf{BO}$. In our case we need to care just about the objects of degree 0 since the product that we want to see that is in line is on the Grothendieck-Witt groups. The shifted duality is used to define the higher Grothendieck-Witt groups but we do not need it for the classic groups. So we need to define the pairings in the usual hermitian objects. Since our recognition principle is for infinity preadditive categories $\mathsf{Cat}_\infty^{\text{preadd}}$, we are going to define the pairing in such infinity categories.

Definition 2.5.6. A *pairing on a preadditive infinity category with duality* $\mathcal{C} \in (\mathsf{Cat}_\infty^{\text{preadd}})^{hC_2}$ *is a map* $\mathcal{C} \otimes \mathcal{C} \to \mathcal{C}$.

Remark. The product that we want to see is the tensor product from the preadditive category with duality $\mathcal{C} = (Ch(Vb(X)), \omega_X, \mathrm{Hom}_{\mathcal{O}_X}(-, \mathcal{O}_X), \nu_X)$. As we have seen in subsection 2.3.7 we take the hermitian category $\mathcal{C}_h$ with the orthogonal sum and the tensor product, $(\mathcal{C}_h, \perp, \otimes)$. This is naturally an object of $(\mathsf{Cat}_\infty^{\text{preadd}})^{hC_2}$, and the tensor product $\otimes$ is a pairing according to the last definition.

Remark. We want to see that this pairing that we have just defined is the pairing from Definition 2.5.5. If we reduce the definition of pairing of exact categories to the case $(\mathcal{C}_i, \omega_i, *_i, \nu_i)$ equals for $i = 1, 2, 3$, we can see clearly that the definition of pairing for infinity categories with duality encapsulate the pairing of exact categories by forgetting the exact structure. One only need to realize that forgetting the exact structure everything reduce to the product by an hermitian object and then it is an special case of Definition 2.5.6.

To develop the recognition principle we have used the composition of two localizations $l \circ \beta$ (see subsection

2.3.6) which gives a symmetric monoidal functor that respects the C_2-action and then it induces a functor

$$(\mathsf{Cat}_\infty^{\text{preadd}})^{hC_2} \to (\mathsf{SymMonCat}_\infty)^{hC_2}$$

in Pr^R. We want to see how is it respected the pairing along this functor.

Let see which is the unit in $\mathsf{Cat}_\infty^{\text{preadd}}$, and therefore in $(\mathsf{Cat}_\infty^{\text{preadd}})^{hC_2}$. The unit in Cat_∞ is the point groupoid $pt = \mathbb{1}_{\mathsf{Cat}_\infty}$. Let consider the left adjoint of the inclusion $i'\colon \mathsf{Cat}_\infty^{\Sigma} \subset \mathsf{Cat}_\infty$ and the left adjoint of the inclusion $i\colon \mathsf{Cat}_\infty^{\text{preadd}} \subset \mathsf{Cat}_\infty^{\Sigma}$. This gives the sequences of infinity functors

$$\mathsf{Cat}_\infty \xrightarrow{l'} \mathsf{Cat}_\infty^{\Sigma} \xrightarrow{l} \mathsf{Cat}_\infty^{\text{preadd}}$$

which is a symmetric monoidal infinity functor. So the unit is preserved along this composition of functors and we get $l \circ l'(pt) = \mathbb{1}_{\mathsf{Cat}_\infty^{\text{preadd}}}$.

We can think about an hermitian object as a map from the point, the unit, to the hermitian category (in the infinity context an object in $(\mathsf{Cat}_\infty)^{hC_2}$). We are going to express the hermitian objects of a preadditive infinity category with duality $\mathcal{C}$ as a map $\mathbb{1}_{(\mathsf{Cat}_\infty^{\text{preadd}})^{hC_2}} \to \mathcal{C}$. Let us consider a infinity category with duality $\tilde{\mathcal{C}} \in \mathsf{Cat}_\infty^{hC_2}$. Let consider the space of morphisms from the point to $\tilde{\mathcal{C}}$, $\mathcal{C} = \mathrm{Hom}(*, \tilde{\mathcal{C}})$, these are the hermitian objects of $\tilde{\mathcal{C}}$. This is a groupoid, in fact, going back along the inclusion $\mathsf{Spc}^{hC_2} \subset \mathsf{Cat}_\infty^{hC_2}$ we get $\mathcal{C} \in \mathsf{Spc}^{hC_2}$. Since the action in the infinity category Spc is trivial we had that $\mathsf{Spc}^{hC_2} \simeq \mathsf{Spc}[C_2]$. So the hermitian objects of an infinity category with duality $\mathcal{C}$ is a space with a C_2-action, $\mathcal{C} \in \mathsf{Spc}[C_2]$.

Let consider the adjoint

$$\mathsf{Spc} \ \leftrightarrows \ \mathsf{Spc}[C_2]$$

given by C_2-homotopy fixed points functor $(-)^{hC_2}$ and its adjoint, the trivial action $Triv$. The functor $Triv$ sends the point in Spc to the point in $\mathsf{Spc}[C_2]$. So a map $* \to \mathcal{C}^{hC_2}$ in Spc is equivalent to a map $* \to \mathcal{C}^{hC_2}$ in $\mathsf{Spc}[C_2]$.

We are interested in the case $(\mathsf{Cat}_\infty^{\text{preadd}})^{hC_2}$, since our category of hermitian objects is inside of this case and we want to apply the functors from the recognition principle (see subsection 2.3.6) to the pairing. Based on what has been said, we can express an hermitian object for $\mathcal{C} \in (\mathsf{Cat}_\infty^{\text{preadd}})^{hC_2}$ as a map $\mathbb{1}_{(\mathsf{Cat}_\infty^{\text{preadd}})^{hC_2}} \to \mathcal{C}$, which is nothing else that the unit map.

Given a pairing $\mathcal{C} \otimes \mathcal{C} \xrightarrow{\oplus} \mathcal{C}$ on $\mathcal{C} \in (\mathsf{Cat}_\infty^{\text{preadd}})^{hC_2}$ we can precompose it by the unit map tensored with the identity

$$\mathcal{C} \simeq \mathcal{C} \otimes \mathbb{1} \xrightarrow{id \otimes x} \mathcal{C} \otimes \mathcal{C} \xrightarrow{\oplus} \mathcal{C}$$

so it is a duality preserving functor and therefore it is preserved along

$$E_\infty^{\text{preadd}}(\mathsf{Cat}_\infty)^{hC_2} \to E_\infty(\mathsf{SymMonCat}_\infty)^{hC_2}.$$

Once we have seen that the pairing is preserved until the infinity category $E_\infty(\mathsf{Cat}_\infty)^{hC_2}$, we can apply the sequence of functors of the direct sum Hermitian K-theory functor

$$\mathsf{SymMonCat}_\infty^{hC_2} \xrightarrow{(-)^\sim} \mathrm{Mon}_{E_\infty}(\mathsf{Spc})[C_2] \xrightarrow{(-)^{hC_2}} \mathrm{Mon}_{E_\infty}(\mathsf{Spc}) \to \mathrm{Grp}_{E_\infty}(\mathsf{Spc}) \to \mathrm{Sp}.$$

The first one is just to take isomorphisms so it does not affect to our pairing. The second is a right adjoint and the two last functors are lefts adjoints, so they preserve our pairing. Therefore, the resulting pairing on the spectrum $\mathbf{KQ}_{\mathbb{S}^1}$ after apply this direct sum K-theory has the same pairing that Walter and Panin used in Theorem 2.5.8. The delooping construction applied to $\mathbf{KQ}_{\mathbb{S}^1}$ produces $\mathbf{KQ}_{\mathbb{P}^1}$ but this is none other than the spectrum $\mathbf{KQ}$ in the entrances, so clearly we have the same pairings on $\mathbf{KQ}_{\mathbb{P}^1}$ and $\mathbf{BO}$. Then for $p = q = 2$ we have the same pairing for $\mathbf{BO}$ and $\mathbf{KQ}_{\mathbb{P}^1}$.

Once we have the same pairing for $p = q = 2$ we can extend it taking loops. Now, we are going to see that if we have a commutative monoid E in $\mathsf{SH}(S)$ which is an $\Omega_{\mathbb{P}^1}$-spectrum the product in the even spaces defines the product in the odd spaces. We know that $\mathbf{KQ}_{\mathbb{P}^1}$ and $\mathbf{BO}$ verify these conditions, thus the result that we got for the even spaces will be enough.

Let us denote by $E = (E_0, E_1, \dots)$ a spectrum with the requirements mentioned just before. We have a product $\mu_{n,m} \colon E_n \wedge E_m \to E_{n+m}$ that we know for the even spaces. Taking adjoints to $\mu_{n,m}$ for $n \geq i$ and $m \geq j$, we get the following diagram

$$\begin{tikzcd}
\Omega^i_{\mathbb{P}^1} E_n \wedge \Omega^j_{\mathbb{P}^1} E_m \arrow[r] \arrow[dd, "\wr"] & \Omega^{i+j}_{\mathbb{P}^1}(E_n \wedge E_m) \arrow[d] \\
& \Omega^{i+j}_{\mathbb{P}^1} E_{n+m} \\
E_{n-i} \wedge E_{m-j} \arrow[r, "\mu"] & E_{n+m-i-j} \arrow[u, "\wr"']
\end{tikzcd}$$

Let us consider that n and m are even. We can take i and j such that the lower map $E_{n-i} \wedge E_{m-j} \to E_{n+m-i-j}$ corresponds to the odd levels. The upper map is the one induced by the product in the even levels. We want to see that the odd levels of the pairing are defined by the even levels of E so we have to see that the last diagram commutes. Let us remember that the spectrum E is a ring spectrum so we have an unitary map $\mathbf{1} \to E_0$ such that smashing with $\mathbb{P}^1$ we get $\iota\colon \mathbb{P}^1 \to E_1$ and the bonding maps are given by $E_n \wedge \mathbb{P}^1 \xrightarrow{\mathrm{id}\wedge\iota} E_n \wedge E_1 \xrightarrow{\mu_{n,1}} E_{n+1}$. Let us take adjoints again, but this time with the suspension functor, to both maps $E_{n-i} \wedge E_{m-j} \to \Omega^{i+j}_{\mathbb{P}^1} E_{n+m}$.

$$\begin{tikzcd}
E_1^{\wedge i} \wedge E_{n-i} \wedge E_1^{\wedge j} \wedge E_{m-j} \arrow[rrdd] & & \\
\Sigma^i_{\mathbb{P}^1} E_{n-i} \wedge \Sigma^j_{\mathbb{P}^1} E_{m-j} \arrow[u] & & \\
\Sigma^{i+j}_{\mathbb{P}^1} (E_{n-i} \wedge E_{m-j}) \arrow[u, "\wr"] \arrow[d] & & E_{n+m} \\
\Sigma^{i+j}_{\mathbb{P}^1} E_{n+m-i-j} \arrow[rr] & & E_1^{\wedge(i+j)} \wedge E_{n+m-i-j} \arrow[u]
\end{tikzcd}$$

One can see clearly that this diagram commutes so also the diagram above. We can apply this argument to the spectra $\mathbf{BO}$ and $\mathbf{KQ}_{\mathbb{P}^1}$, so since they agree for $p = q = 2$ they also agree for $p = q = 1$ giving rise to the following theorem.

Theorem 2.5.9. *The E_∞-ring $\mathbb{P}^1$-spectrum $\mathbf{KQ}_{\mathbb{P}^1}$ is unique as homotopy commutative monoid in the category $\mathsf{SH}(S)$, where S is a scheme with 2 invertible, provided the following condition is verify. Let $m \in \mathrm{Hom}_{\mathsf{SH}(S)}(\mathbf{KQ}_{\mathbb{P}^1} \wedge \mathbf{KQ}_{\mathbb{P}^1}, \mathbf{KQ}_{\mathbb{P}^1})$ be the induced morphism in $\mathsf{SH}(S)$ by the multiplicative structure of the E_∞-ring spectrum $\mathbf{KQ}_{\mathbb{P}^1}$. This morphism agrees with one induced by the one induced by the spectrum $\mathbf{BO}$ and it is the unique defining a pairing which when restrict to the Grothendieck-Witt groups coincides with the tensor product pairing*

$$\mathsf{KO}_0^{[2p]}(X_+) \times \mathsf{KO}_0^{[2q]}(Y_+) \to \mathsf{KO}_0^{[2p+2q]}(X_+ \wedge Y_+)$$

for $X, Y \in \mathrm{Sm}\,|_S$.

Proof. Based on what has been said, we get the same restriction pairing for all p and q. $\square$

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